NICPB
KBFI

CONNECTING SCIENCES

Slow-roll inflation in Palatini $F(R)$ gravity

Antonio Racioppi

National Institute of Chemical Physics and Biophysics, Tallinn, Estonia

ICHEP 2022, July 7th, 2022

based on JHEP 06 (2022) 106
with
C. Dioguardi (Taltech & NICPB) & E. Tomberg (NICPB)

Euroopa Liit
Euroopa Sotsiaalfond

Eesti tuleviku heaks

• Motivation •

- Inflation with R^2 term in the Palatini formalism, Enckell et al. 1810.05536

$$S_J = \int \mathrm{d}^4x\sqrt{-g_J}\left[\frac{1}{2}\left(R+\alpha R^2\right) - \frac{1}{2}g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \qquad \boxed{M_P = 1}$$

- Main results:
  - ◇ universally flat Einstein-frame scalar potential

$$U = \frac{V}{1+8\alpha V} = \frac{U^0}{1+8\alpha U^0}$$

  $\ldots^0$ means the same quantity but for $\alpha = 0$

  - ◇ simple inflationary preditions:

$$\begin{aligned} N_e &= N_e^0 \\ A_s &= A_s^0 \\ n_s &= n_s^0 \\ r &= \frac{r^0}{1+8\alpha U^0} = \frac{r_0}{1+12\pi^2\alpha A_s r^0} \end{aligned}$$

  - ◇ $\alpha \gg 1 \Rightarrow r \to 0$ regardless of the initial V
- Q's: What makes $F(R) = R + \alpha R^2$ special? Is there any $F(R) \Rightarrow r \to 0$?

• Palatini $F(R) + \mathcal{L}(\phi)$ •

- We start with the following action in the Palatini formulation

$$S_J = \int d^4x\sqrt{-g_J}\left[\frac{1}{2}F(R(\Gamma)) + \mathcal{L}(\phi)\right]$$

$$\mathcal{L}(\phi) = -\frac{1}{2}k(\phi)g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

- we rewrite the $F(R)$ term using the auxiliary field ζ, obtaining

$$S_J = \int d^4x\sqrt{-g_J}\left[\frac{1}{2}\left(F(\zeta) + F'(\zeta)\left(R(\Gamma) - \zeta\right)\right) + \mathcal{L}(\phi)\right]$$

- we move to the Einstein frame

$$S_E = \int d^4x\sqrt{-g_E}\left[\frac{R}{2} - \frac{1}{2}g_E^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi,\zeta)\right]$$

$$U(\chi,\zeta) = \frac{V(\phi(\chi))}{F'(\zeta)^2} - \frac{F(\zeta)}{2F'(\zeta)^2} + \frac{\zeta}{2F'(\zeta)}$$

$$\frac{\partial\chi}{\partial\phi} = \sqrt{\frac{k(\phi)}{F'(\zeta)}} \quad \text{(canonically normalized scalar)}$$

- N.B. ζ stays auxiliary! Not dynamical like in metric gravity

EoM for ζ

- The full EoM for ζ is

$$2F(\zeta) - \zeta F'(\zeta) - 2k(\phi)\,\partial^\mu\phi\partial_\mu\phi\, F'(\zeta) - 4V(\phi) = 0$$

- The standard procedure would be now to solve the EoM and determine $\zeta(\phi, \partial^\mu\phi\partial_\mu\phi)$ and insert it back into the action.
- However not always solvable for any $F(R) \Rightarrow$ SR approximation
- The EoM for ζ, assuming SR i.e. $k(\phi) g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \ll V(\phi)$ is

$$G(\zeta) = V(\phi)$$

with

$$G(\zeta) = \frac{1}{4}\left[2F(\zeta) - \zeta F'(\zeta)\right]$$

- Still not always solvable for any $F(R)$
- On the other hand it is still possible to perform inflationary computations. The trick is to use the auxiliary field ζ as a computational variable and $G = V$ as a constraint.

- first of all $V \to G$ in U:

$$\begin{aligned} U(\chi,\zeta) &= \frac{V(\phi(\chi))}{F'(\zeta)^2} - \frac{F(\zeta)}{2F'(\zeta)^2} + \frac{\zeta}{2F'(\zeta)} \\ &= \dots \\ &= \boxed{\frac{1}{4}\frac{\zeta}{F'(\zeta)} = U(\zeta)} \end{aligned}$$

- valid for any $F(R)$ and $V(\phi)$
- what changes is the actual *solution* for ζ
- also valid for the pure $F(R)$ case ($\mathcal{L}(\phi) = 0$)
- We already knew that $F(R) = R + \alpha R^2$ gives a flat U for $\alpha \gg 1$ Enckell et al. 1810.05536
- Now we have shown that no other $F(R)$ can give such a result.

• Calculus •

- SR computations → we need derivatives of U
- we start with the 1st derivative:

$$\frac{\partial}{\partial \chi} U(\zeta) = \boxed{\frac{\partial \zeta}{\partial \chi}} \frac{\partial}{\partial \zeta} U(\zeta) \quad \leftarrow \text{we need this!}$$

- $G(\zeta) = V(\phi) \;\Rightarrow\; \phi = V^{-1}(G)$, the inverse function of $V(\phi)$

$$g(\zeta) = \frac{\partial \zeta}{\partial \chi} = \frac{\partial \zeta}{\partial \phi}\frac{\partial \phi}{\partial \chi} = \ldots$$

$$= \boxed{\sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}}\,\frac{1}{\frac{\partial G}{\partial \zeta}\frac{\partial V^{-1}}{\partial G}}}$$

- This allows us to easily express higher derivatives:

$$\frac{\partial^2}{\partial \chi^2} U(\zeta) = g(\zeta)\frac{\partial}{\partial \zeta}\left(g(\zeta)\frac{\partial U}{\partial \zeta}\right) = gg'U' + g^2 U'' \;, \; \ldots$$

where primes denote derivatives w.r.t. ζ.

- we have a method for computing SR parameters

• Inflationary observables •

- SR parameters

$$\epsilon(\zeta) = \frac{1}{2}\left(\frac{\partial U/\partial\chi}{U}\right)^2 = \frac{1}{2}g^2\left(\frac{U'}{U}\right)^2$$

$$\eta(\zeta) = \frac{\partial^2 U/\partial\chi^2}{U} = \frac{gg'U' + g^2 U''}{U}$$

- observables

$$N_e = \int_{\chi_f}^{\chi_N} \frac{U}{\partial U/\partial\chi}\mathrm{d}\chi = \int_{\zeta_f}^{\zeta_N} \frac{U}{g^2\,U'}\mathrm{d}\zeta$$

$$r(\zeta) = 16\epsilon(\zeta) = 8g^2\left(\frac{U'}{U}\right)^2$$

$$n_s(\zeta) = 1 + 2\eta(\zeta) - 6\epsilon(\zeta) = 1 + \frac{2g}{U^2}\left(g'U'U + gU''U - 3gU'^2\right)$$

$$A_s(\zeta) = \frac{U}{24\pi^2\epsilon(\zeta)} = \frac{U^3}{12\pi^2 g^2 U'^2}$$

- N.B. U contains info from $F(R)$, while V-info is in g

• Examples •

- We checked $R + \alpha R^2 \to$ OK! $\to$ backup slides

- We study the test scenario:

$$F(R) = R + \alpha R^n\,, \quad k(\phi) = 1\,, \quad V(\phi) = \frac{m^2}{2}\phi^2\,.$$

Solutions of the $G = V$ problem:

1. $n < 2 \to$ all good!
2. $n > 2 \to$ only effective description

$G > 0$ problem

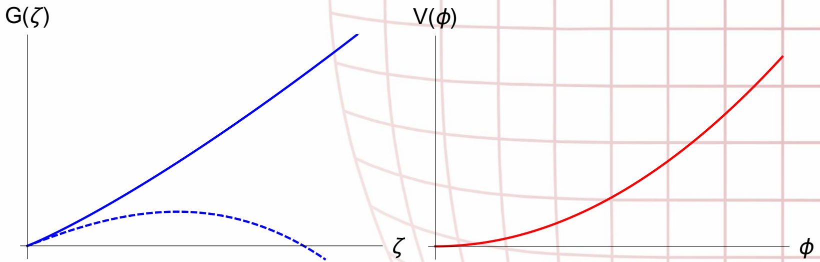


Figure: $G(\zeta)$ (left) and $V(\phi) = \phi^2$ (right) for $F(R) = R + R^n$ with $n = 3/2$ (continuous) and $n = 5/2$ (dashed).

n=3/2 For each ϕ, even though we cannot compute it analytically, we can always find a ζ so that $G(\zeta) = V(\phi)$

n=5/2 For each ϕ, we cannot always find a ζ so that $G(\zeta) = V(\phi)$
$\rightarrow$ effective description: inflation must happen before $\zeta_{\max}$

• Analytic solution •

We can compute the phenomenological parameters:

$$N_e = \left[\frac{\zeta\,(n-(n-1))}{8m^2}\,{}_2F_1\left(1, \frac{1}{n-1}; \frac{n}{n-1}; (n-2)\alpha\zeta^{n-1}\right) \right]_{\zeta=\zeta_f}^{\zeta=\zeta_N}$$

$$r(\zeta_N) = \frac{64m^2}{\zeta_N}\frac{1+\alpha(2-n)\zeta_N^{n-1}}{1+\alpha n\zeta_N^{n-1}}$$

$$n_s(\zeta_N) = 1 - \frac{8m^2}{\zeta_N}\frac{2+\alpha(n-2)(n-3)\zeta_N^{n-1}}{1+\alpha(2-n)n\zeta_N^{n-1}}$$

$$A_s(\zeta_N) = \frac{1}{384\pi^2 m^2}\frac{\zeta_N^2}{1+\alpha(2-n)\zeta_N^{n-1}}$$

where we used the hypergeometric function

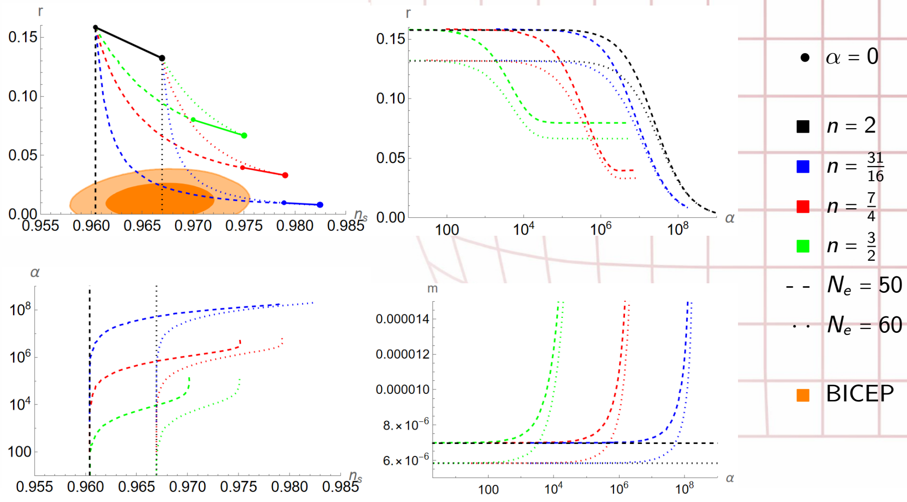
$$ {}_2F_1(a,b,c,z) = \sum_{k=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

with

$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$$

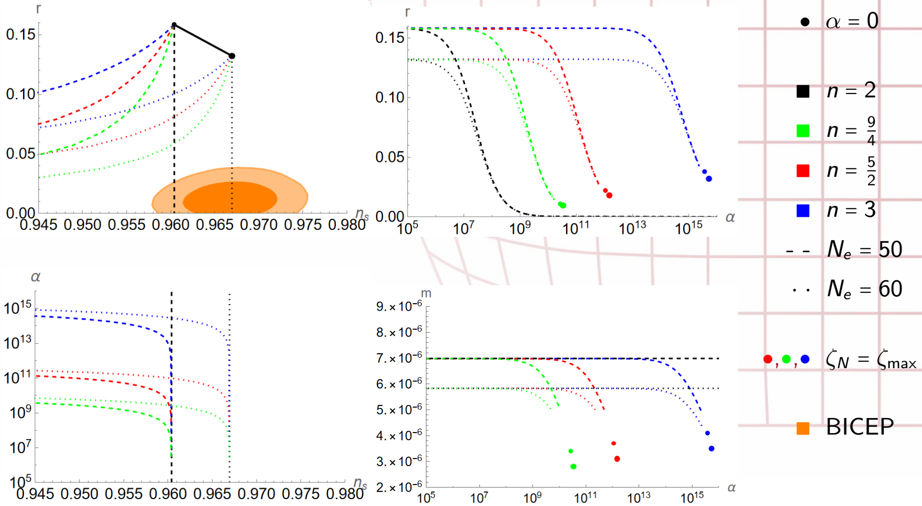
the (rising) Pochhammer symbol

$n < 2$



r
0.15
0.10
0.05
0.00
0.955 0.960 0.965 0.970 0.975 0.980 0.985
n_s
r
0.15
0.10
0.05
0.00
100 10^4 10^6 10^8
α
α
10^8
10^6
10^4
100
0.955 0.960 0.965 0.970 0.975 0.980 0.985
n_s
m
0.000014
0.000012
0.000010
$8.\times10^{-6}$
$6.\times10^{-6}$
100 10^4 10^6 10^8
α
$\alpha = 0$
$n = 2$
$n = \frac{31}{16}$
$n = \frac{7}{4}$
$n = \frac{3}{2}$
$N_e = 50$
$N_e = 60$
BICEP

$n > 2$


r
0.15
0.10
0.05
0.00
0.945 0.950 0.955 0.960 0.965 0.970 0.975 0.980
n_s
r
0.15
0.10
0.05
0.00
10^5 10^7 10^9 10^{11} 10^{13} 10^{15}
α
α
10^{15}
10^{13}
10^{11}
10^9
10^7
10^5
0.945 0.950 0.955 0.960 0.965 0.970 0.975 0.980
n_s
m
$9.\times10^{-6}$
$8.\times10^{-6}$
$7.\times10^{-6}$
$6.\times10^{-6}$
$5.\times10^{-6}$
$4.\times10^{-6}$
$3.\times10^{-6}$
$2.\times10^{-6}$
10^5 10^7 10^9 10^{11} 10^{13} 10^{15}
α
$\alpha = 0$
$n = 2$
$n = \frac{9}{4}$
$n = \frac{5}{2}$
$n = 3$
$N_e = 50$
$N_e = 60$
$\zeta_N = \zeta_{max}$
BICEP

• Conclusions •

- We studied single field inflation embedded in Palatini $F(R)$ gravity
- We explained why Palatini $R + R^2$ is so unique
- We found a method to perform inflationary computations even though the EoM of ζ is not solvable.
- We tested the method on a couple of examples
- Unexpected outcome/future outlook:
  - $R + R^n$, $n > 2$ is problematic $\Rightarrow$ hint for a Palatini UV theory of gravity?
  - $n = 2$ quite unstable configuration

Grazie! - Thank you! - Aitäh!

BACKUP SLIDES

• Inflationary computations •

Assuming SR, the inflationary dynamics is described by the SR parameters and the N_e. Assuming that we solved EoM $\zeta(\phi)$ and field redef. $\phi(\chi)$, the SR parameters are

$$\epsilon \equiv \frac{1}{2}\left(\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}\chi}\right)^2 , \quad \eta \equiv \frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}\chi^2} ,$$

and the number of e-folds as

$$N = \int_{\chi_f}^{\chi_N} \mathrm{d}\chi \, U \left(\frac{\mathrm{d}U}{\mathrm{d}\chi}\right)^{-1} ,$$

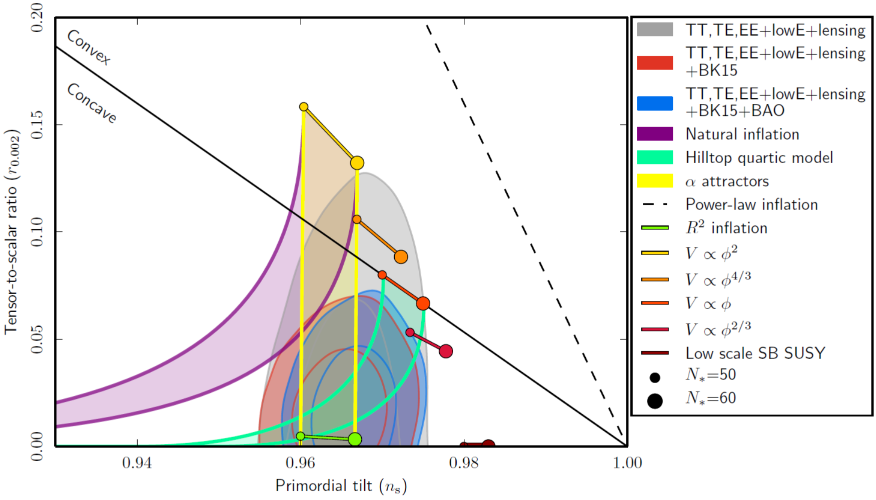
where the field value at the end of inflation, χ_f, is defined via $\epsilon(\chi_f) = 1$. The field value χ_i at the time a given scale left the horizon is given by the corresponding N. To reproduce the correct A_s, the potential has to satisfy

$$\ln\left(10^{10} A_s\right) = 3.044 \pm 0.014 \qquad \text{where} \qquad A_s = \frac{1}{24\pi^2}\frac{U(\chi_N)}{\epsilon(\chi_N)}$$

and the other two main observables, i.e. the spectral index and the tensor-to-scalar ratio are expressed as

$$\begin{aligned} n_s &\simeq 1 + 2\eta - 6\epsilon \\ r &\simeq 16\epsilon \end{aligned}$$

• Planck 2018 data •



- FLAT potentials are strongly FAVORED!!!

• Gravity preliminaries •

The properties of spacetime are essentially described by two objects:

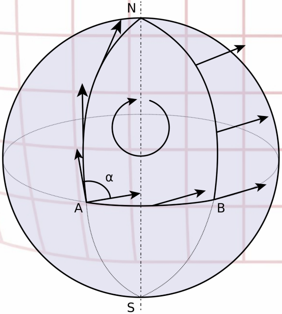
- the affine connection: $\Gamma^{\lambda}_{\alpha\beta}$
- the metric tensor: $g_{\mu\nu}$

• Gravity preliminaries •

The properties of spacetime are essentially described by two objects:

- the affine connection: $\Gamma^{\lambda}_{\alpha\beta}$
- the metric tensor: $g_{\mu\nu}$

$\Gamma^{\lambda}_{\alpha\beta}$ describes the parallel transport of tensor fields along a given curve. If the spacetime is curved, parallel transport around a closed path, after a full cycle, results in a finite mismatch. The curvature is uniquely determined by the Riemann tensor $R^{\mu}_{\ \alpha\nu\beta}(\Gamma)$ whose contraction $R_{\alpha\beta}(\Gamma) \equiv R^{\mu}_{\ \alpha\mu\beta}(\Gamma)$ gives the Ricci tensor[1]



[1] We consider only $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$ i.e. torsion free space-time.

• Gravity preliminaries •

The properties of (torsionless) spacetime are essentially described by:

- the affine connection: $\Gamma^{\lambda}_{\alpha\beta} \to$ parallel transport
- the metric tensor: $g_{\mu\nu} \to$ distance

The connection coefficients and metric tensor are fundamentally independent quantities. They exhibit no *a priori* known relationship. If they are to have any relationship, it must derive from

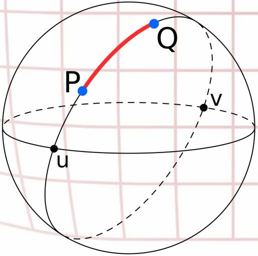
- additional constraints (metric formalism $\nabla_\alpha g_{\mu\nu} = 0$)
- EoM (Palatini formalism)

Gravity preliminaries

The properties of spacetime are essentially described by two objects:

- the affine connection: $\Gamma^{\lambda}_{\alpha\beta}$
- the metric tensor: $g_{\mu\nu}$

$g_{\mu\nu}$ allows us to introduce the notion of distance.



The connection coefficients and metric tensor are fundamentally independent quantities. They exhibit no *a priori* known relationship, and if they are to have any it must derive from additional constraints (metric formalism) or geometrodynamics (Palatini formalism).

• Metric vs Palatini formulation •

Einstein-Hilbert action: $\sqrt{-g^E}\mathcal{L}^E = \sqrt{-g^E}\left[\frac{1}{2}R_E\right]$

Metric	**Palatini**
No torsion $\Rightarrow \Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$	No torsion $\Rightarrow \Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$
$\boxed{\nabla_\alpha g_{\mu\nu} = 0} \Rightarrow \Gamma =$ Levi-Civita $\Gamma = \overline{\Gamma}$	$\boxed{\Gamma\text{'s EoM}} \rightarrow \nabla_\alpha\left[\sqrt{-g}\, g_{\mu\nu}\right] = 0$
$\overline{\Gamma}^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right)$	$\Gamma^\lambda_{\alpha\beta} = \overline{\Gamma}^\lambda_{\alpha\beta}$

- In non-minimally coupled theories, metric and Palatini formalism give different physical theories. (Koivisto & Kurki-Suonio: arXiv:0509422)

• Metric vs Palatini formulation •

Jordan frame:
$$\sqrt{-g^J}\mathcal{L}^J = \sqrt{-g^J}\left[\frac{1}{2}F(R) - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$
$$= \sqrt{-g^J}\left[\frac{1}{2}f(\zeta)R - \frac{1}{2}(\partial\phi)^2 - V(\phi,\zeta)\right]$$

Metric

$$\nabla_\alpha g^J_{\mu\nu} = 0 \Rightarrow \Gamma = \text{Levi-Civita } \Gamma = \bar{\Gamma}$$

$$\bar{\Gamma}^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right)$$

Palatini

$$\Gamma\text{'s EoM: } \nabla_\alpha[f(\zeta)\sqrt{-g^J}g^J_{\mu\nu}] = 0$$

$$\Gamma^\lambda_{\alpha\beta} = \bar{\Gamma}^\lambda_{\alpha\beta} + \delta^\lambda_\alpha\partial_\beta\omega + \delta^\lambda_\beta\partial_\alpha\omega - g_{\alpha\beta}\partial^\lambda\omega$$

$$\omega(\zeta) = \ln\Omega(\zeta)\,, \qquad g^E_{\mu\nu} = \Omega(\zeta)^2\, g_{\mu\nu}\,, \qquad \Omega(\zeta)^2 = f(\zeta) = F'(\zeta)$$

Einstein frame:
$$\sqrt{-g^E}\mathcal{L}^E = \sqrt{-g^E}\left[\frac{1}{2}R_E - K(\phi,\zeta) - U(\phi,\zeta)\right]$$

Metric ($\Gamma_E = \bar{\Gamma}_E$)

$K(\phi,\zeta) \rightarrow$ 2 dyn. fields: ϕ & ζ

Palatini ($\Gamma_E = \bar{\Gamma}_E$)

$\boxed{K(\phi,\zeta) = K(\phi) \rightarrow \phi \text{ dyn. \& } \zeta \text{ aux.}}$

Palatini $F(R)$

- we start with Palatini $F(R)$ action alone

$$S_J = \int \mathrm{d}^4x\sqrt{-g_J}\frac{1}{2}F(R(\Gamma))$$

- we rewrite the $F(R)$ term using the auxiliary field ζ

$$S_J = \int \mathrm{d}^4x\sqrt{-g}_J\left[\frac{1}{2}F'(\zeta)R(\Gamma) - V(\zeta)\right]$$

$$V(\zeta) = \frac{-F(\zeta)+\zeta F'(\zeta)}{2} \qquad F' = \frac{\partial F}{\partial \zeta}$$

$\sim$ before ($\zeta = \phi^2$, $F' = f$) but NO kin term for ζ!!!

- we move to the Einstein frame

$$S_E = \int \mathrm{d}^4x\sqrt{-g_E}\left[\frac{R}{2} - U(\zeta)\right]$$

$$U(\zeta) = -\frac{F(\zeta)}{2F'(\zeta)^2} + \frac{\zeta}{2F'(\zeta)}$$

N.B. Still no kinetic term for ζ!!!

- ζ's EoM: $U'(\zeta) = 0 \to$ a bit of albegra:

$$G(\zeta) = \frac{1}{4}\left[2F(\zeta) - \zeta F'(\zeta)\right] = 0$$

$$\zeta = \zeta^* \quad \to \quad F(\zeta^*) = \frac{1}{2}\zeta^* F'(\zeta^*)$$

with $\boxed{F', F'' \neq 0}$

- inserting $\zeta = \zeta^*$ in U we obtain

$$U(\zeta^*) = -\frac{\zeta^* F'(\zeta^*)}{2F'(\zeta^*)^2} + \frac{\zeta^*}{4F'(\zeta^*)} = \frac{1}{4}\frac{\zeta^*}{F'(\zeta^*)}$$

i.e. a CC

- therefore pure Palatini $F(R)$ is equivalent to GR + CC

• Calculus •

- SR computations → we need derivatives of U
- we start with the 1st derivative:

$$\frac{\partial}{\partial\chi}U(\zeta) = \boxed{\frac{\partial\zeta}{\partial\chi}}\frac{\partial}{\partial\zeta}U(\zeta) \quad \leftarrow \text{we need this!}$$

$$= \frac{\partial\phi}{\partial\chi}\frac{\partial}{\partial\phi}U(\zeta) = \sqrt{\frac{F'(\zeta)}{k(\phi)}}\frac{\partial}{\partial\phi}U(\zeta)$$

- $G(\zeta) = V(\phi) \;\Rightarrow\; \phi = V^{-1}(G)$, the inverse function of $V(\phi)$

$$= \sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}}\frac{\partial\zeta}{\partial\phi}\frac{\partial U}{\partial\zeta} =$$

$$= \sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}}\frac{1}{\frac{\partial\phi}{\partial\zeta}}\frac{\partial U}{\partial\zeta} = \sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}}\frac{1}{\frac{\partial V^{-1}}{\partial\zeta}}\frac{\partial U}{\partial\zeta} =$$

$$= \boxed{\sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}}\frac{1}{\frac{\partial G}{\partial\zeta}\frac{\partial V^{-1}}{\partial G}}\frac{\partial U}{\partial\zeta} = \frac{\partial}{\partial\chi}U(\zeta)}$$

- Summarizing:

$$\frac{\partial \zeta}{\partial \chi} = \sqrt{\frac{F'(\zeta)}{k(V^{-1}(G))}\frac{1}{\frac{\partial G}{\partial \zeta}\frac{\partial V^{-1}}{\partial G}}} \equiv g(\zeta), \qquad \frac{\partial}{\partial \chi} f(\zeta) = g(\zeta)\frac{\partial f(\zeta)}{\partial \zeta}$$

N.B. The derivative can be explicitly computed as long as V is invertible.

- This allows us to easily express higher derivatives:

$$\frac{\partial^2}{\partial \chi^2} U(\zeta) = g(\zeta)\frac{\partial}{\partial \zeta}\left(g(\zeta)\frac{\partial U}{\partial \zeta}\right) = gg'U' + g^2 U'' \ , \ \ldots$$

where primes denote derivatives w.r.t. ζ.

- we have a method for computing SR parameters and inflationary observables!

• $G > 0$ problem •

$$\boxed{G(\zeta) = \frac{1}{4}\left[2F(\zeta) - \zeta F'(\zeta)\right] = V(\phi)}$$

- For $\zeta \to +\infty$, if $F(\zeta) \approx \zeta^n \Rightarrow G(\zeta) \approx (2-n)\zeta^n$
  $\Rightarrow$ with $n > 2 \Rightarrow G \to -\infty$
- Problem: $V(\infty) \to \infty \Rightarrow$ no real values for ζ

- Solutions:

$n < 2$ $\Rightarrow G(+\infty) \to +\infty \Rightarrow$ OK!

$n > 2$ $G(0^+) \sim \zeta \Rightarrow$ G is a first a crescent function which reaches a local max and then decreases towards $-\infty$
$\Rightarrow$ we need to ensure that inflation happens within $\zeta = 0$ and ζ =local max. (in order to avoid also the bijectivity problem)
$\Rightarrow$ the model is only an effective description

• Reproduce R^2 results •

- $F(R) = R + \alpha R^2$, $k(\phi) = 1$
- Let's check: $G(\zeta) = V(\phi)$

$$G(\zeta) = \frac{1}{4}\left[2F(\zeta) - \zeta F'(\zeta)\right] = \frac{1}{4}\left[2\zeta + 2\alpha\zeta^2 - \zeta(1 + 2\alpha\zeta)\right]$$

$$= \boxed{\frac{1}{4}\zeta = V(\phi)} = U^0 \quad \leftarrow \text{no } \alpha!!!$$

- Einstein frame scalar potential:

$$U = \frac{1}{4}\frac{\zeta}{F'(\zeta)} = \frac{\zeta}{4 + 8\alpha\zeta} = \frac{U^0}{1 + 8\alpha U^0} \quad \rightarrow \text{OK!}$$

- g function

$$g = \sqrt{F'(\zeta)}\frac{1}{\frac{\partial G}{\partial \zeta}\frac{\partial V^{-1}}{\partial G}} = \sqrt{1 + 2\alpha\zeta}\frac{1}{\frac{\partial V^{-1}}{\partial \zeta}}$$

- inflationary observables

$$r(\zeta) = 8g^2\left(\frac{U'}{U}\right)^2 = \left(\frac{1}{\frac{\partial V^{-1}}{\partial \zeta}}\right)^2 \frac{8}{\zeta^2}\left(\frac{1}{1 + 2\alpha\zeta}\right) = \frac{r^0}{1 + 8\alpha U^0} \rightarrow \text{OK!}$$

analogously we get $N_e = N_e^0$, $A_s = A_s^0$ and $n_s = n_s^0$

• Example 1. $n < 2$ •

We can derive more readable expressions considering the limit $|n-2|\alpha \to \infty$. In such a limit we can approximate the number of e-folds as

$$N_e \sim \frac{n}{8m^2}\zeta_N$$

obtaining

$$\begin{aligned} r(\zeta_N) &\simeq \frac{8(2-n)}{N_e} \\ n_s(\zeta_N) &\simeq 1-\frac{3-n}{N_e} \\ A_s(\zeta_N) &\simeq \frac{2^{2-3n}\left(\frac{n}{N_e}\right)^{n-3}}{3\pi^2(2-n)}\frac{\left(m^2\right)^{2-n}}{\alpha} \end{aligned}$$

N.B. Valid only for $\boxed{n \neq 2}$

• Example 1. $n > 2$ •

- full analytical expression still valid
- $G > 0$ during inflation $\Rightarrow$ no $\alpha \to +\infty$ $\Rightarrow$ given n, α has an upper limit
- inflation happens$[0, \zeta_{\max}]$ $\Rightarrow$ N_e is bounded from above
- we need $\zeta_N < \zeta_{\max}$ at $N_e \in [50, 60]$
- $\alpha \nearrow \quad \Rightarrow \quad \zeta_{\max} \searrow$ and $|\zeta_N - \zeta_{\max}| \searrow$
- rough upper limit for $\alpha = \bar{\alpha}$ when $\zeta_N = \zeta_{\max}$.
  The limit is only rough because η has a pole at $\zeta = \zeta_{\max}$ meaning the loss of validity of the slow-roll approximation. Therefore the actual upper limit $\bar{\alpha}$ takes place for ζ_N not equal, but slightly smaller than $\zeta_{\max}$.
- we can still provide useful estimates by using $\zeta_N = \zeta_{\max}$.

$$m^2_{\bar{\alpha}} \simeq \frac{n(\bar{\alpha}(n-2)n)^{-\frac{2}{n-1}}}{384\pi^2(n-1)A_s}$$

$$N_e \sim (\bar{\alpha}(n-2)n)^{\frac{1}{n-1}} \frac{48\pi^2(n-1)A_s}{n}\left[n + (1-n)\,{}_2F_1\left(1, \frac{1}{n-1}; \frac{n}{n-1}; \frac{1}{n}\right)\right] \to$$

$\to$ we can use it as a definition for $\bar{\alpha}$

$$r_{\bar{\alpha}} \simeq \frac{(n-2)(\bar{\alpha}(n-2)n)^{\frac{1}{1-n}}}{6\pi^2(n-1)A_s}$$

Beyond SR

After some manipulations, the full Einstein frame EoMs read:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{V'(\phi)}{F'(\zeta)k(\phi)} = \frac{\dot{\phi}\dot{\zeta}F''(\zeta)}{F'(\zeta)} - \frac{1}{2}\frac{k'(\phi)}{k(\phi)}\dot{\phi}^2$$

$$3H^2 = \frac{1}{2}\frac{\dot{\phi}^2}{F'(\zeta)}k(\phi) + U(\phi,\zeta)$$

$$-\frac{1}{2}\dot{\phi}^2 F'(\zeta)k(\phi) + 2V(\phi) - 2G(\zeta) = 0$$

These can be used to also derive

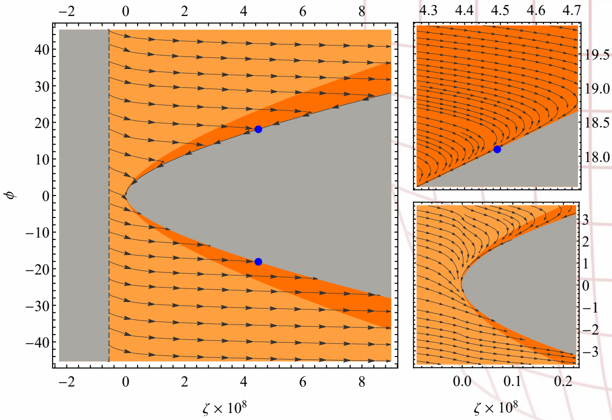
$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{F'(\zeta)}k(\phi)$$

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} = \frac{12V(\phi) - 6F(\zeta) + 3\zeta F'(\zeta)}{6V(\phi) - 3F(\zeta) + 2\zeta F'(\zeta)}$$

$$\dot{\zeta} = \frac{3H\dot{\phi}^2 F'(\zeta)k(\phi) + 3V'(\phi)\dot{\phi}}{2G'(\zeta) + \frac{3}{2}\dot{\phi}^2 F''(\zeta)k(\phi)}$$

N.B. Even though ζ is only auxiliary, it has an implicit time dependence via ϕ

Beyond SR. $n < 2$



- $n = 3/2$, $\alpha = 8710$, $m = 1.15 \cdot 10^{-5}$

  $\sim$ for any $1 < n < 2$
- $H > 0$, $\dot{\phi} < 0$ branch

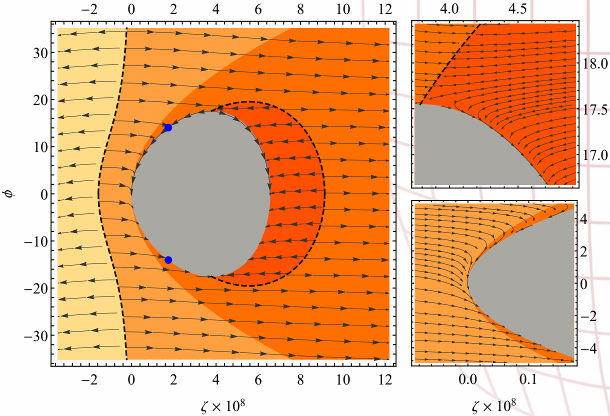
■ $F' < 0$ or no ζ's EoM

■ dark $\rightarrow \epsilon_H < 1$

● $N_e = 50$, $n_s = 0.967$, $r = 0.096$

- $\dot{\phi} > 0 \rightarrow$ mirror with respect to x-axis
  $H < 0 \rightarrow$ switch the direction of the flow. not reached smoothly when 1 ϕ
- $\phi > 0$: trajectory sharp turns into SR slow-roll
- $\phi < 0$: trajectory enters SR in $\dot{\phi} > 0$ branch

• Beyond SR. $n > 2$ •



- $n = 3$, $\alpha = 2.32 \cdot 10^{14}$, $m = 6.40 \cdot 10^{-6}$

  $\sim$ for any $n > 2$

- $H > 0$, $\dot{\phi} < 0$ branch

■ no ζ's EoM

■ dark $\rightarrow$ $\epsilon_H < 1$

● $N_e = 50$, $n_s = 0.952$, $r = 0.116$

- - $\dot{\zeta} \rightarrow \infty$: trajectories diverging from there. cannot be crossed
left: $G' < 0$ $\rightarrow$ in SR only if ϕ not too big
right: $F'' < 0$ $\rightarrow$ *crazy* isolated inflationary region on the left

- SR is not an attractor of EoM
- illness of the $n > 2$ case