Emergence of resummation scales in the evolution of α_s and PDFs

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RGEs in QCD

- Generic RGE in QCD:

$R = (\alpha_s, PDF, TMD)$

$$\frac{d \ln R}{d \ln \mu}(\mu, \alpha_s(\mu)) = \Gamma(\alpha_s(\mu))$$

Γ = appropriate anomalous dimension

$$\Gamma(\alpha_s(\mu)) = \sum_{n=0}^{k} \Gamma_n \alpha_s^{n+1}(\mu)$$

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k = highest-order known in β-expansion

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- Knowledge of β_k allows to resum $N^K LL$ tower of $\ln(\mu/\mu_0)$
- Question: is it possible to find an analytic expression for each tower?

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Closed-form analytic solutions $(\lambda = \beta_0 \alpha_s(\mu_0) \ln(\mu/\mu_0))$

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beyond LL: *approximate* (rely on PT, RGE not satisfied)

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$$\alpha_s^{NLL}(\mu) = \alpha_s(\mu_0)\left(\frac{1}{1-\lambda}\right) - \alpha_s^2(\mu_0)\left(\frac{\beta_1 \ln(1-\lambda)}{\beta_0(1-\lambda)^2}\right)$$

$$\alpha_s^{NNLL}(\mu) = \alpha_s(\mu_0)\left(\frac{1}{1-\lambda}\right) - \alpha_s^2(\mu_0)\left(\frac{\beta_1 \ln(1-\lambda)}{\beta_0(1-\lambda)^2}\right) + \alpha_s^3(\mu_0)\left(\frac{\beta_0\beta_2\lambda - \beta_1^2(\lambda + \ln(1-\lambda) - \beta_0^2 \ln^2(1-\lambda)}{\beta_0^2(1-\lambda)^3}\right)$$

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$g_1(\lambda)$

$$\alpha_s^{LL}(\mu) = \alpha_s(\mu_0)\left(\frac{1}{1-\lambda}\right)$$

$g_2(\lambda)$

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$$\alpha_s(\mu) = \alpha_s(\mu_0)\, g_1(\lambda) + \alpha_s^2(\mu_0)\, g_2(\lambda) + \alpha_s^3(\mu_0)\, g_3(\lambda) + \cdots$$

(purposely reminiscent of threshold/q_T resummation)

g-functions

- $\mathrm{N}^k\mathrm{LL}$ evolution of α_s:

$$a_s^{\mathrm{N}^k\mathrm{LL}}(\mu) = a_s(\mu_0) \sum_{l=0}^{k} a_s^l(\mu_0) g_{l+1}^{(\beta)}(\lambda)$$

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- Introduce a “resummation scale” $\mu_{res} = \kappa\mu$

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$$g_1^{(\beta)}(\bar\lambda) = \frac{1}{1-\bar\lambda}$$

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➡ κ generates subleading corrections to analytic solution

➡ allows estimate of missing higher orders

Scale variation

- Analytic solution: κ-variation in

$$\alpha_s^{NLL}(\mu) = \alpha_s(\mu_0)\, g_1(\overline{\lambda}\,) + \alpha_s^2(\mu_0)\, g_2(\overline{\lambda}\,, \ln \kappa)$$

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- Numerical solution: displace μ by a factor ξ

$$\overline{\beta}(\mu) = \alpha_s(\xi\mu)\beta_0[1 + a_s(\xi\mu)(\frac{\beta_1}{\beta_0} - 2\beta_0 \ln \xi)] + \mathcal{O}(\alpha_s^3)$$

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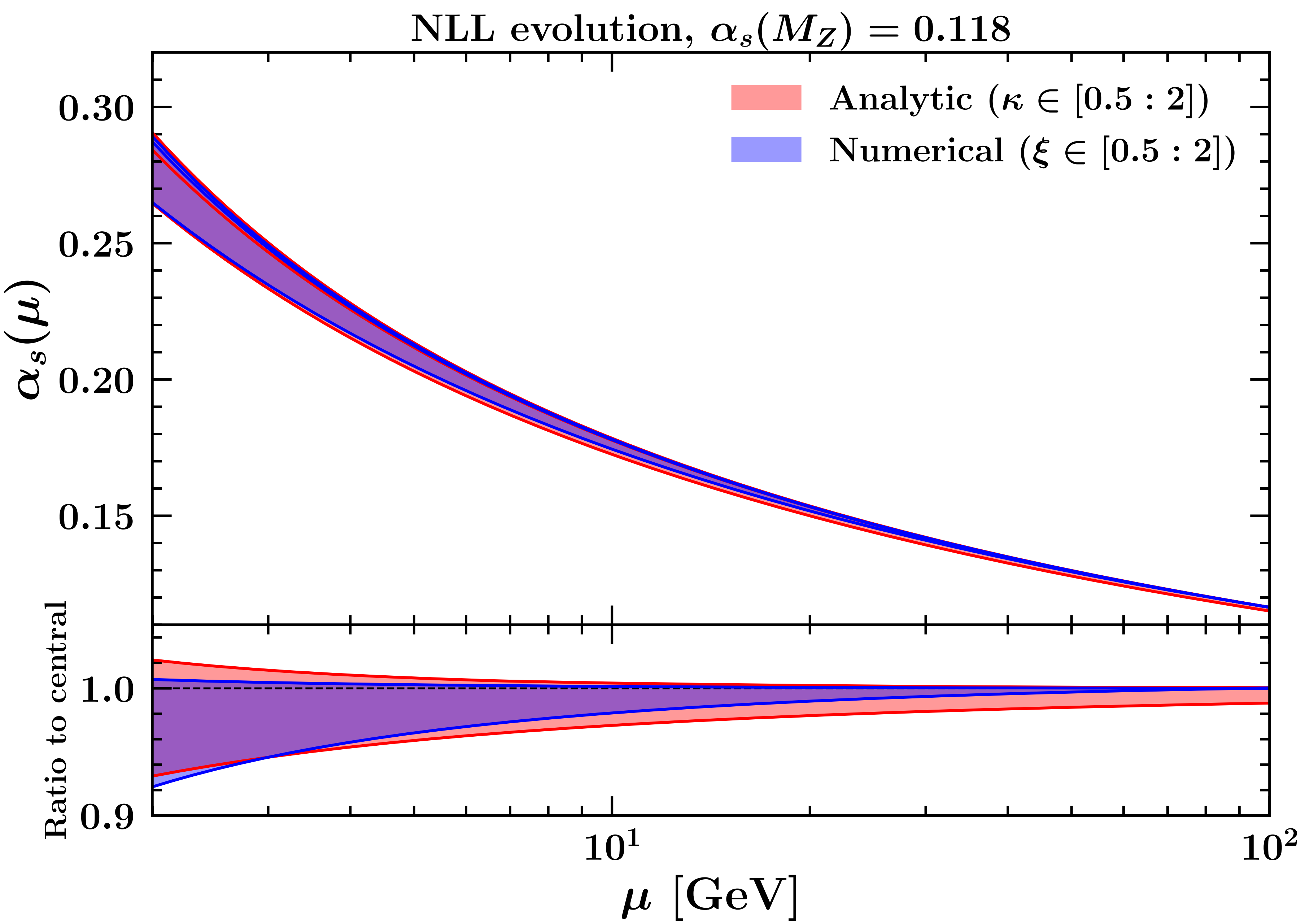
New $\overline{\beta}$-function (different from β by subleading corrections):

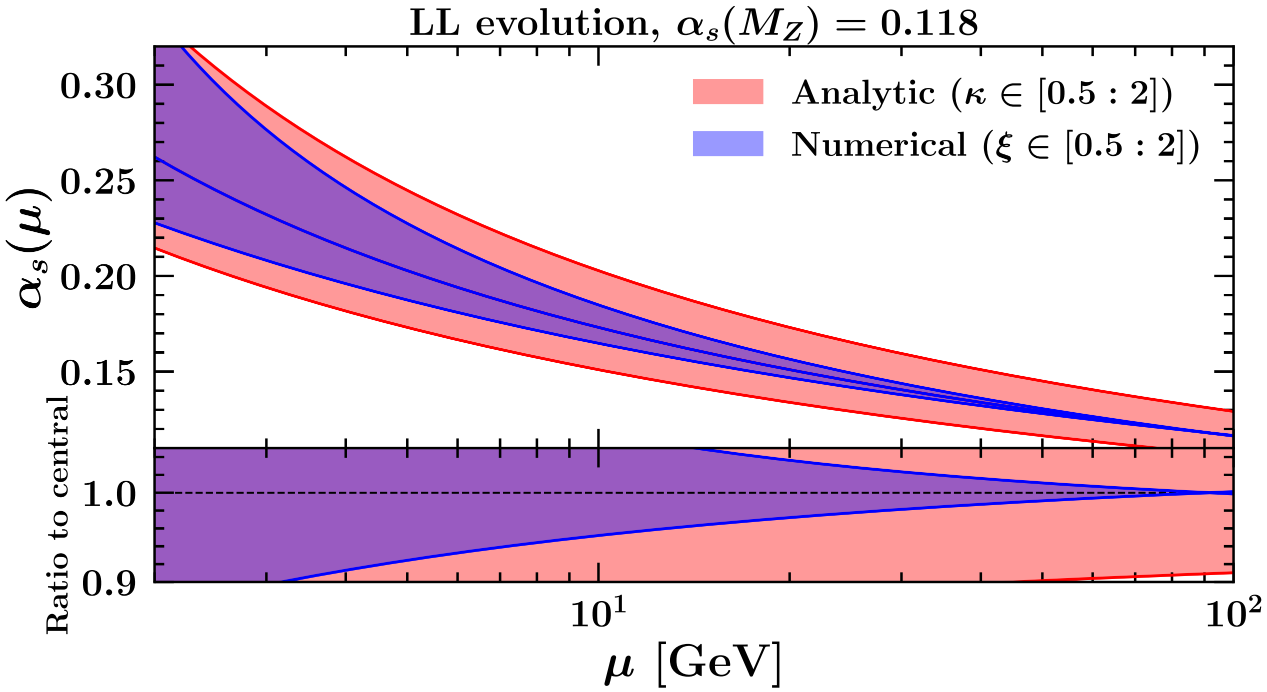
different solution —> estimate of higher-order corrections

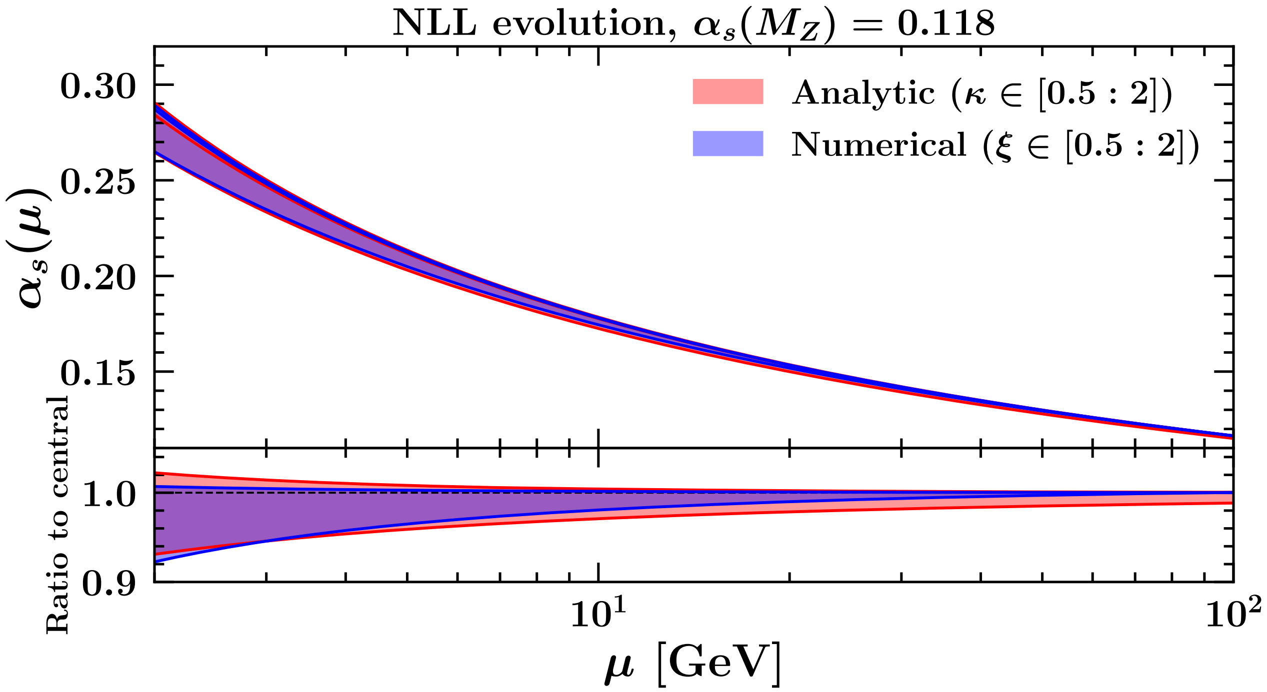
Scale variation α_s @ NLL



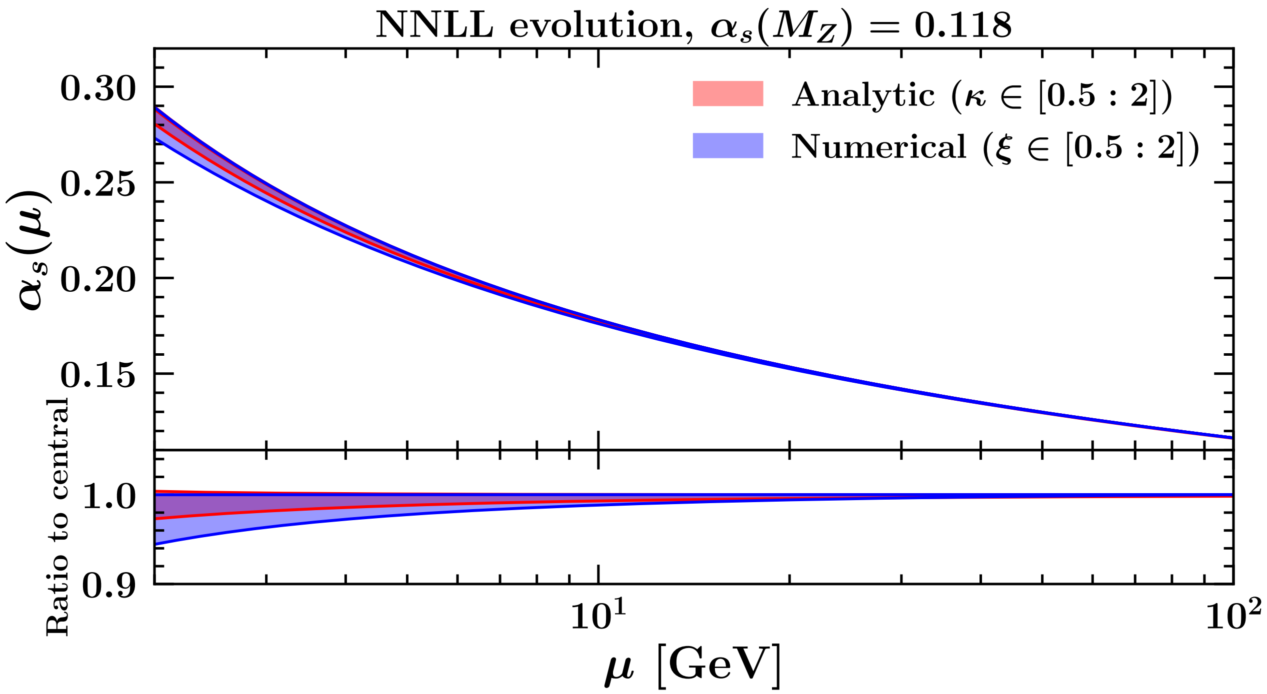
- Uncertainty bands comparable and strongly asymmetric
- Numerical band shrinking to zero at Z mass, where coupling is exactly known

α_s evolution @ LL, NLL, NNLL





- Reduction of uncertainty bands from LL to NNLL
- Analytic uncertainty band shrinking to (almost) zero at NNLL



PDFs

k = highest-order known in γ-expansion

$$\frac{d\ln f}{d\ln\mu} = \gamma(\alpha_S(\mu)) = \sum_{n=0}^{k} \gamma_n \alpha_S^{n+1}(\mu)$$

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$$f^{\mathrm{N}^k\mathrm{LL}}(\mu) = g_0^{(\gamma),\mathrm{N}^k\mathrm{LL}}(\bar{\lambda})\exp\left[\sum_{l=0}^{k} a_s^l(\mu_0)g_{l+1}^{(\gamma)}(\bar{\lambda})\right]f(\mu_0)$$

$$g_0^{(\bar{\gamma}),\mathrm{NLL}}(\bar{\lambda}) = 1 + a_s(\mu_0)\frac{1}{\beta_0}\left(\gamma_1 - \frac{\beta_1}{\beta_0}\gamma_0\right)\frac{\bar{\lambda}}{1-\bar{\lambda}}$$

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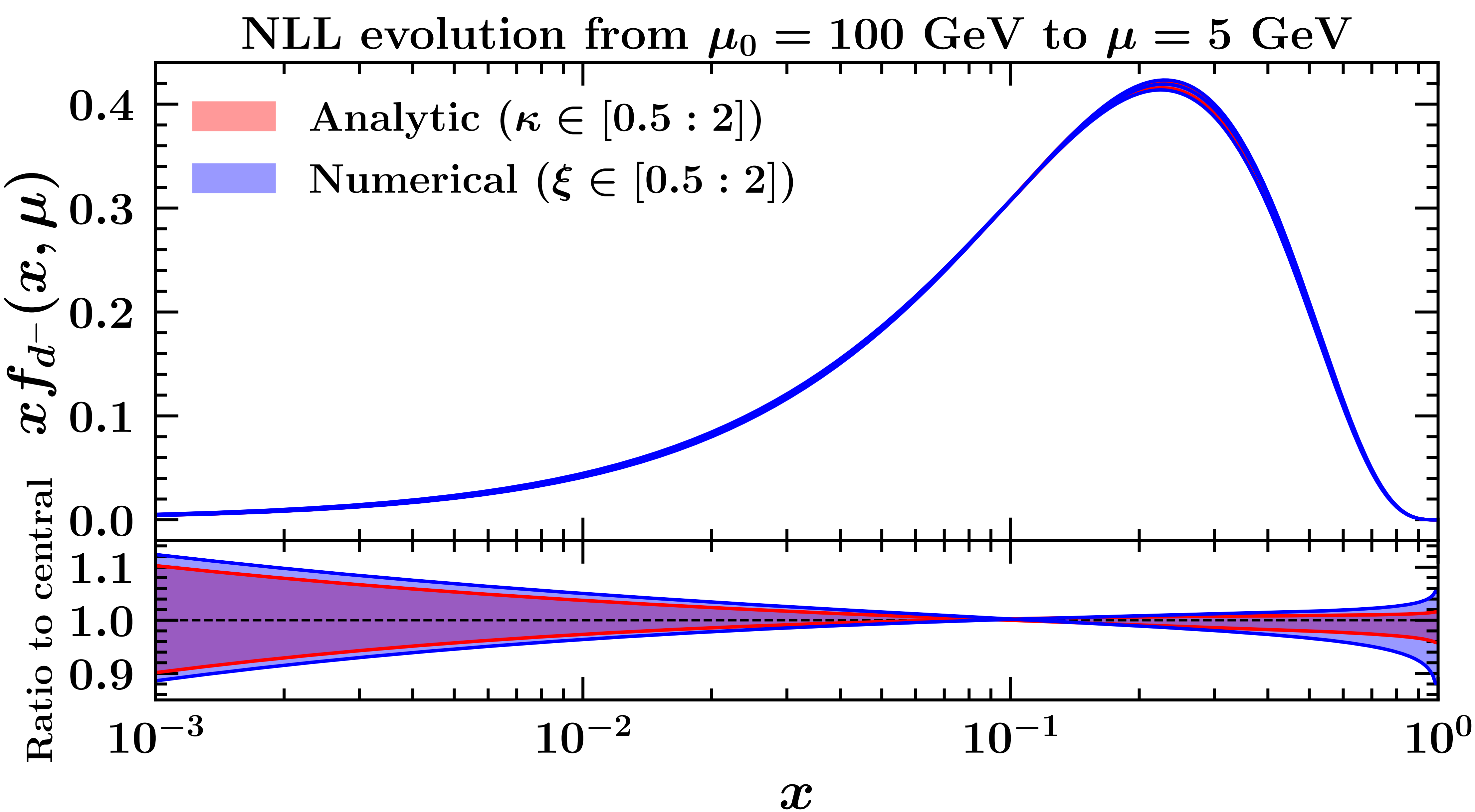
- For the numerical solution: shift μ by ξ —> new anomalous dimension

$$\bar{\gamma}(\mu) = a_s(\xi\mu)\gamma_0 + a_s^2(\xi\mu)(\gamma_1 - \beta_0\gamma_0\ln\xi)$$

Scale variation PDF

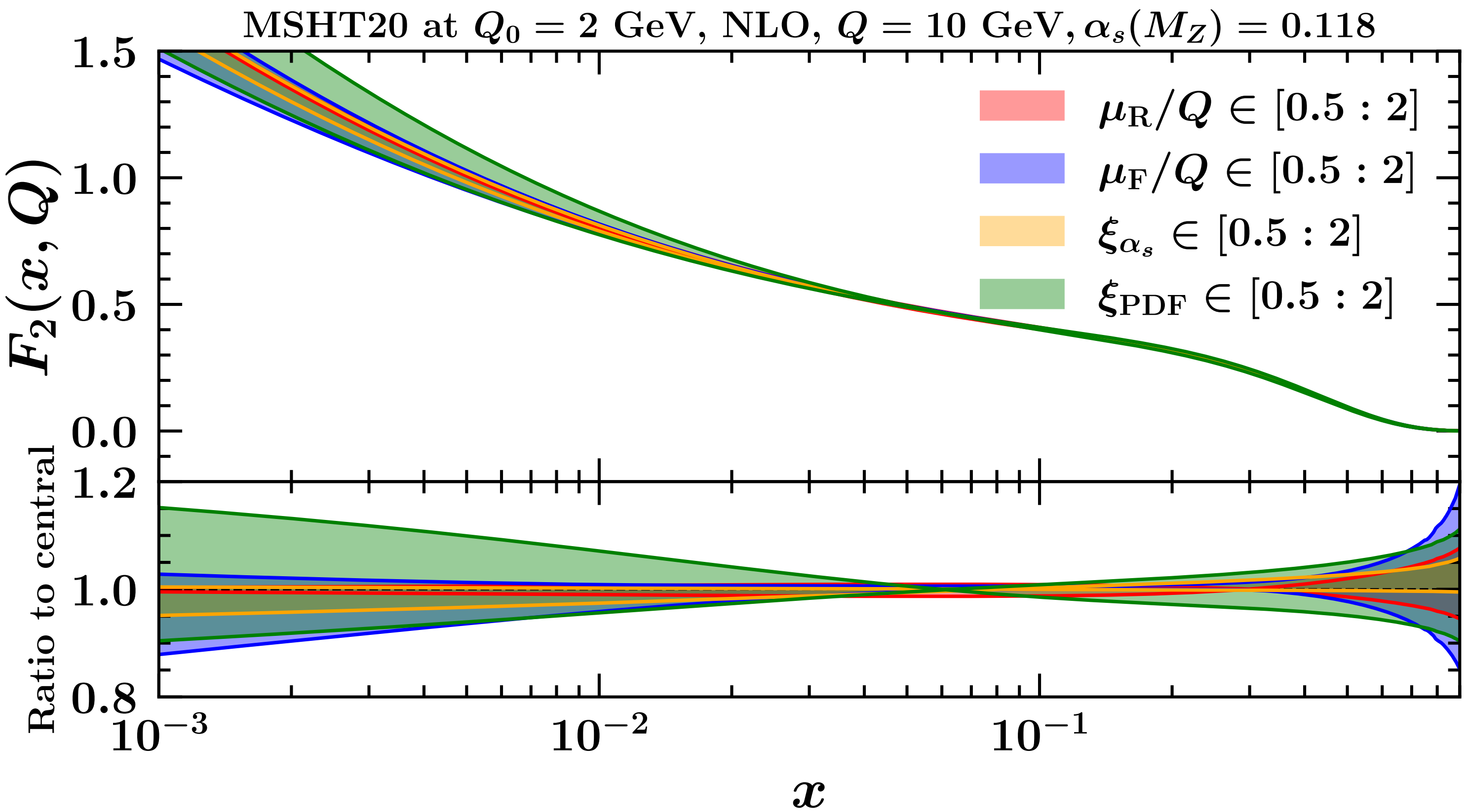
$$f_{NLL}(\mu) = g_0(\bar{\lambda}, \ln\kappa)\exp\left(g_1(\bar{\lambda}) + \alpha_s g_2(\bar{\lambda}, \ln\kappa)\right) f(\mu_0)$$



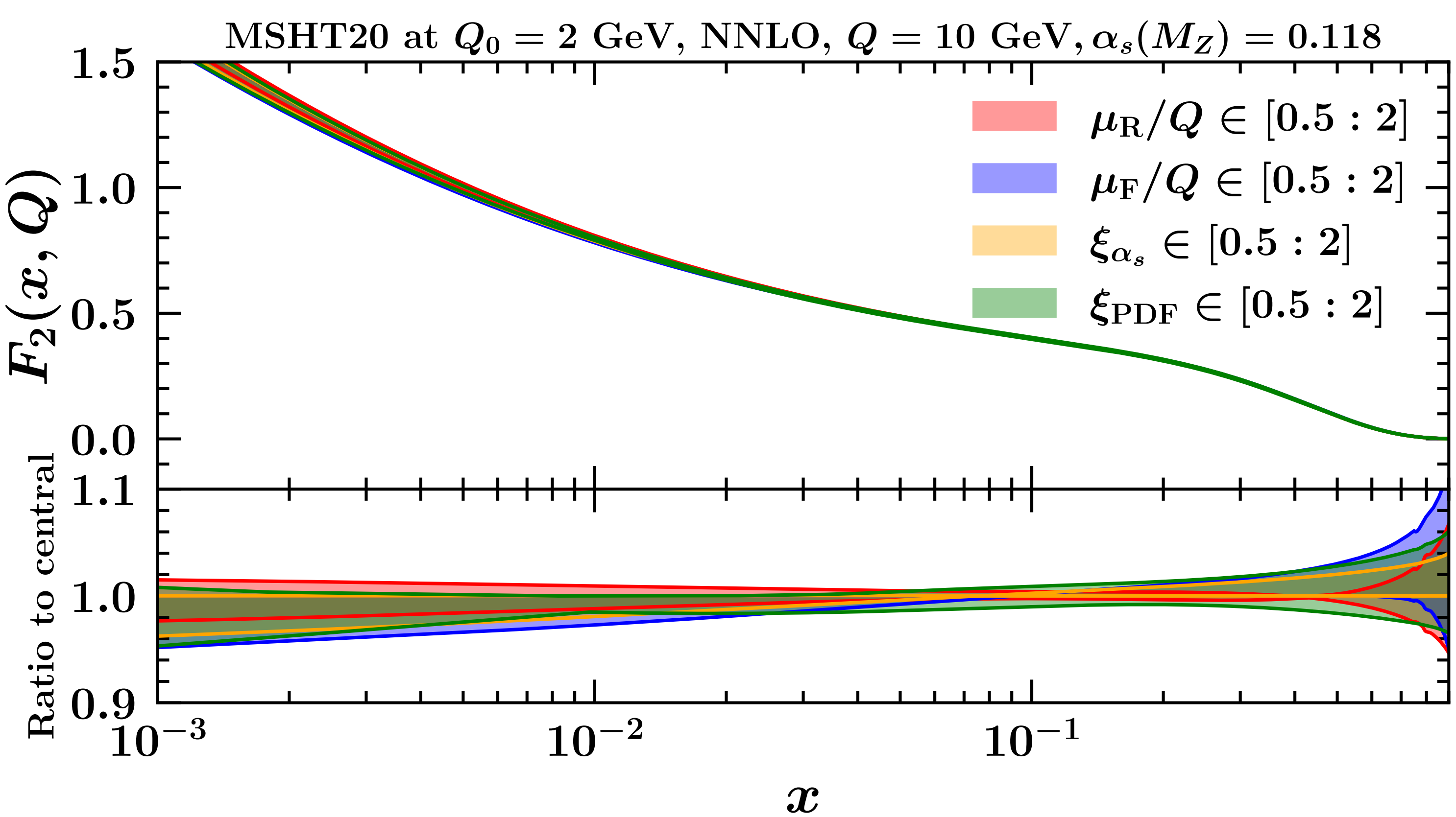
- Uncertainty bands of comparable size

RGE uncertainties for F_2

- $\mu_R, \mu_F, \xi_{\alpha_s}, \xi_{\text{PDF}}$ uncertainty bands for $F_2(x)$ at $Q = 10\,\text{GeV}$



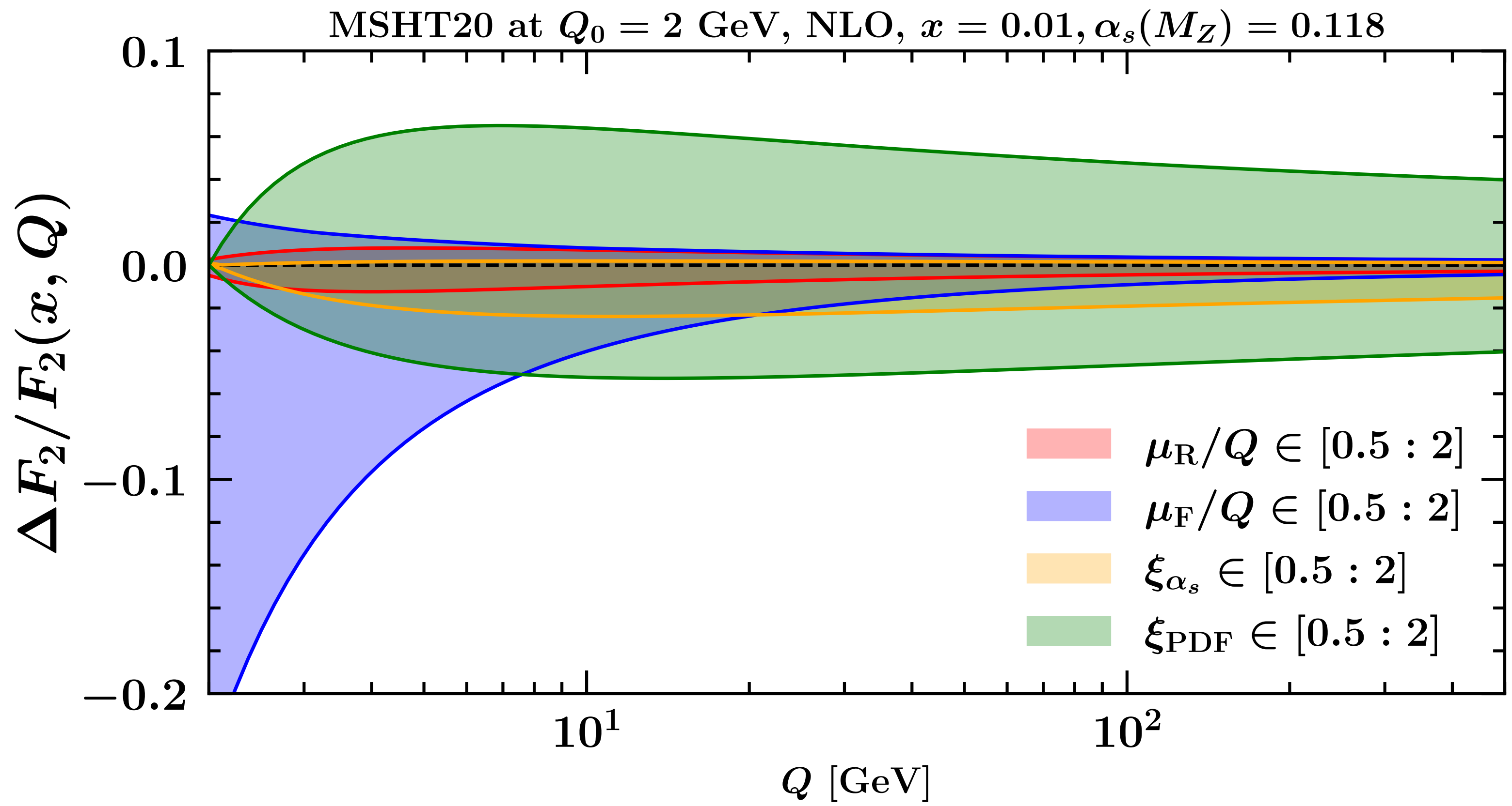
- ξ_{PDF} dominates at low-x
- μ_F dominates at the largest x



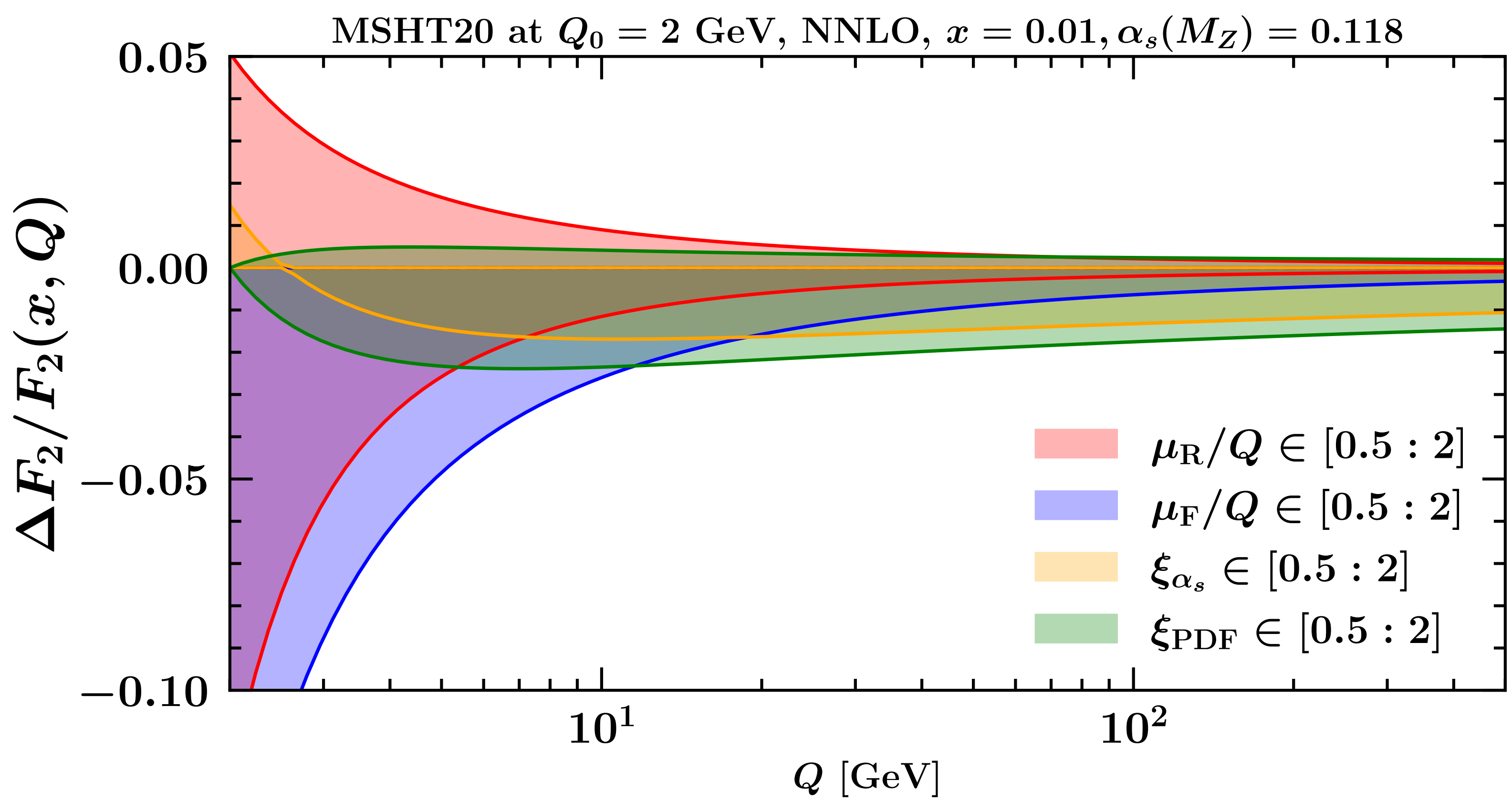
- size of uncertainties reduced at NNLO

RGE uncertainties for $\Delta F_2/F_2$

- $\mu_R, \mu_F, \xi_{\alpha_s}, \xi_{\text{PDF}}$ uncertainty bands for $(\Delta F_2/F_2)(Q)$ at $x = 0.01$

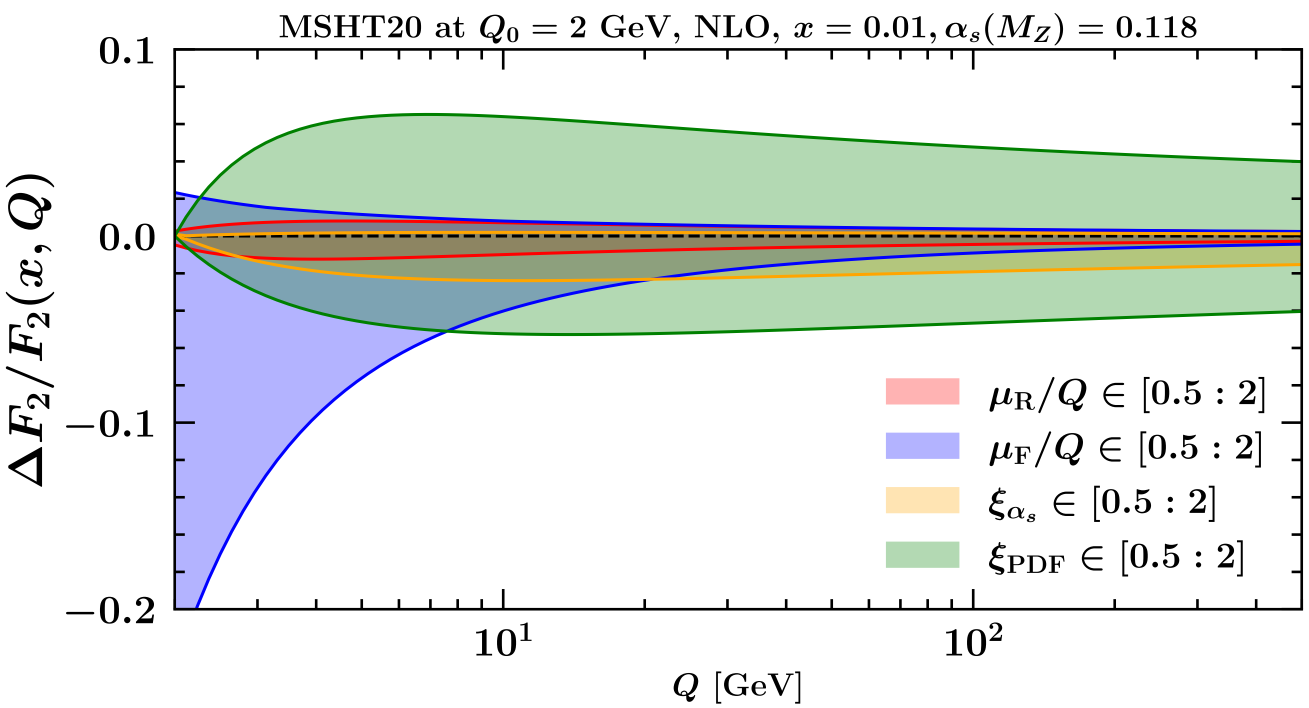


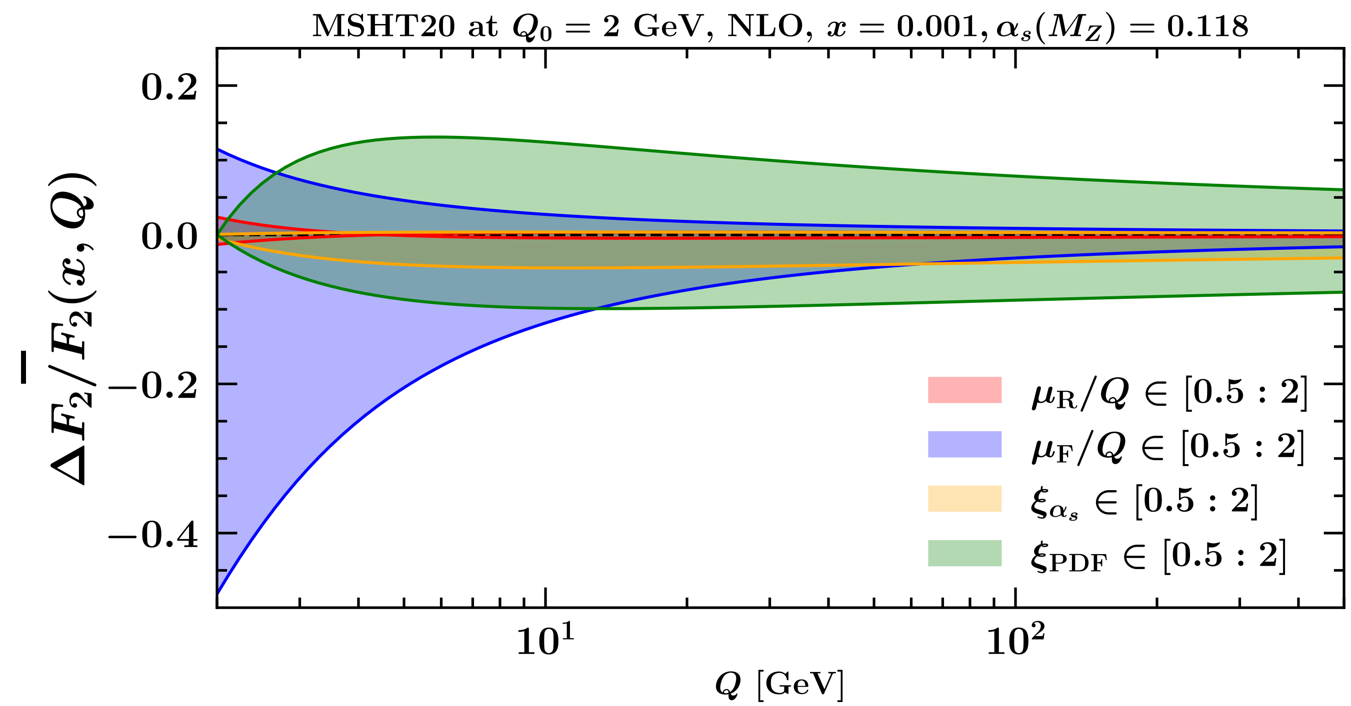
- ξ_{PDF} starts from 0 and saturates very rapidly, staying significant at large Q
- μ_F and μ_R large at small Q, then decrease with Q
- ξ_{α_s} subdominant at low Q, increases with Q

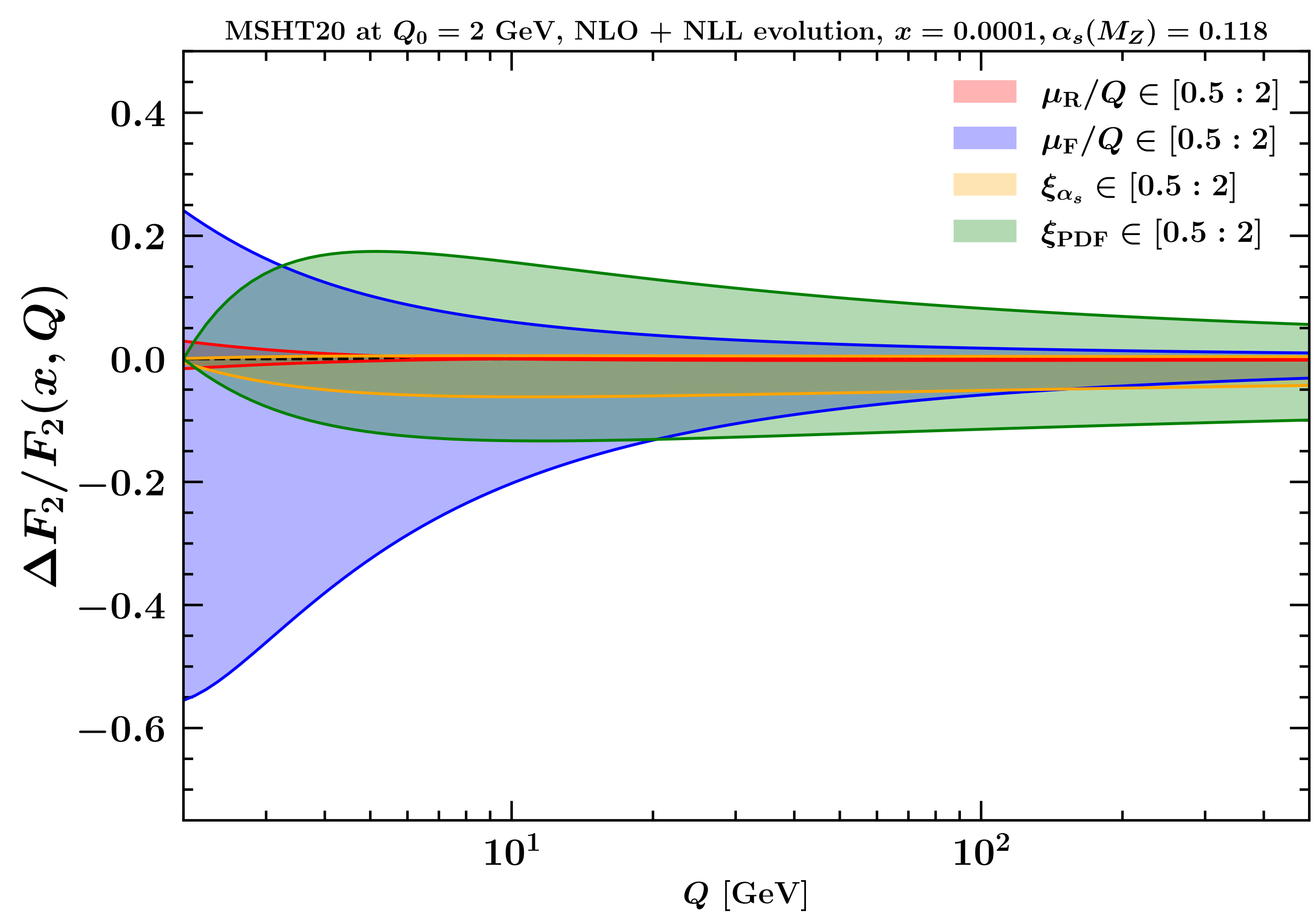


- size of uncertainties reduced at NNLO
- ξ-uncertainties still dominant at large Q

RGE uncertainties $\Delta F_2/F_2(Q)$ @ NLO

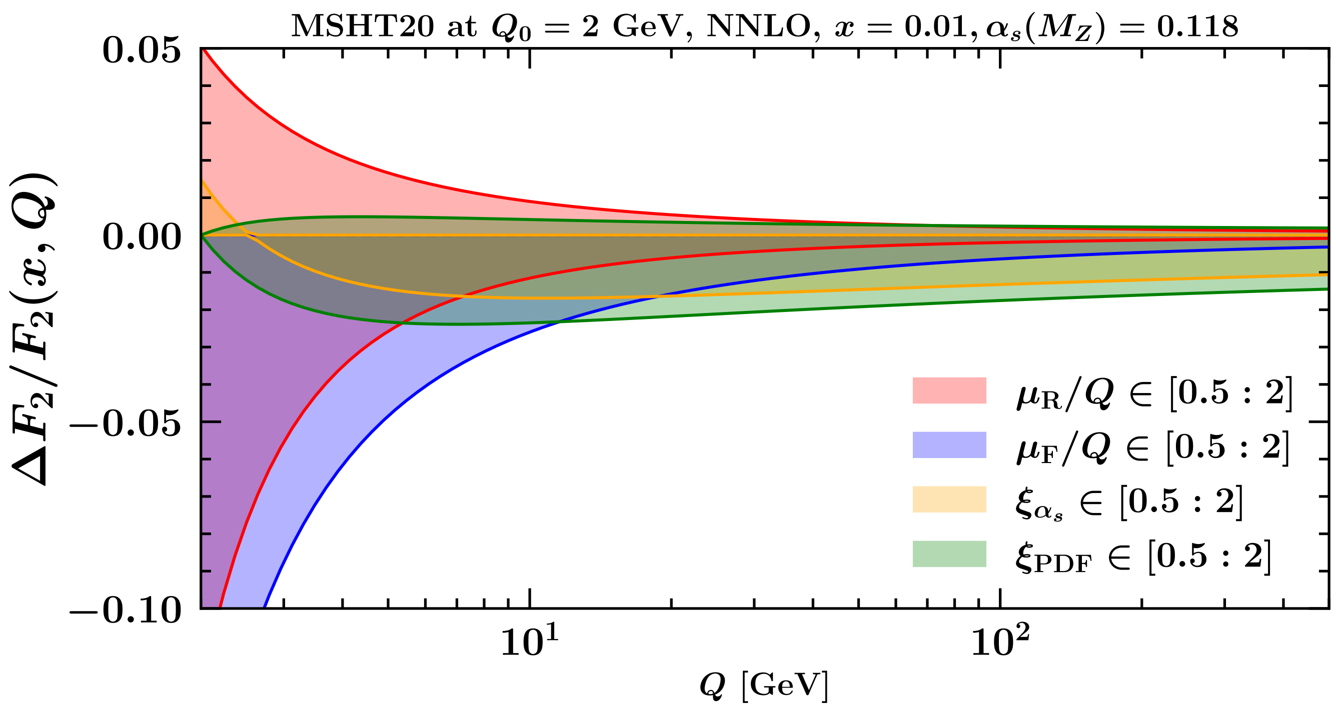


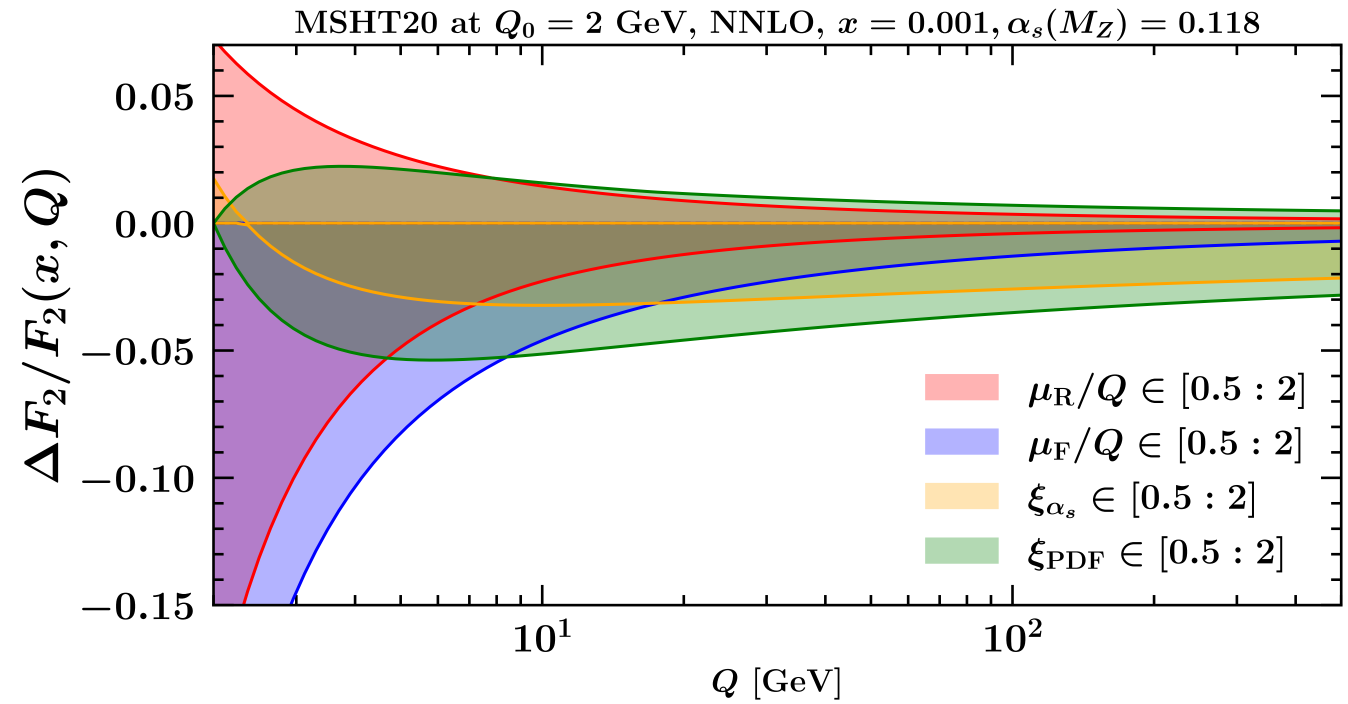


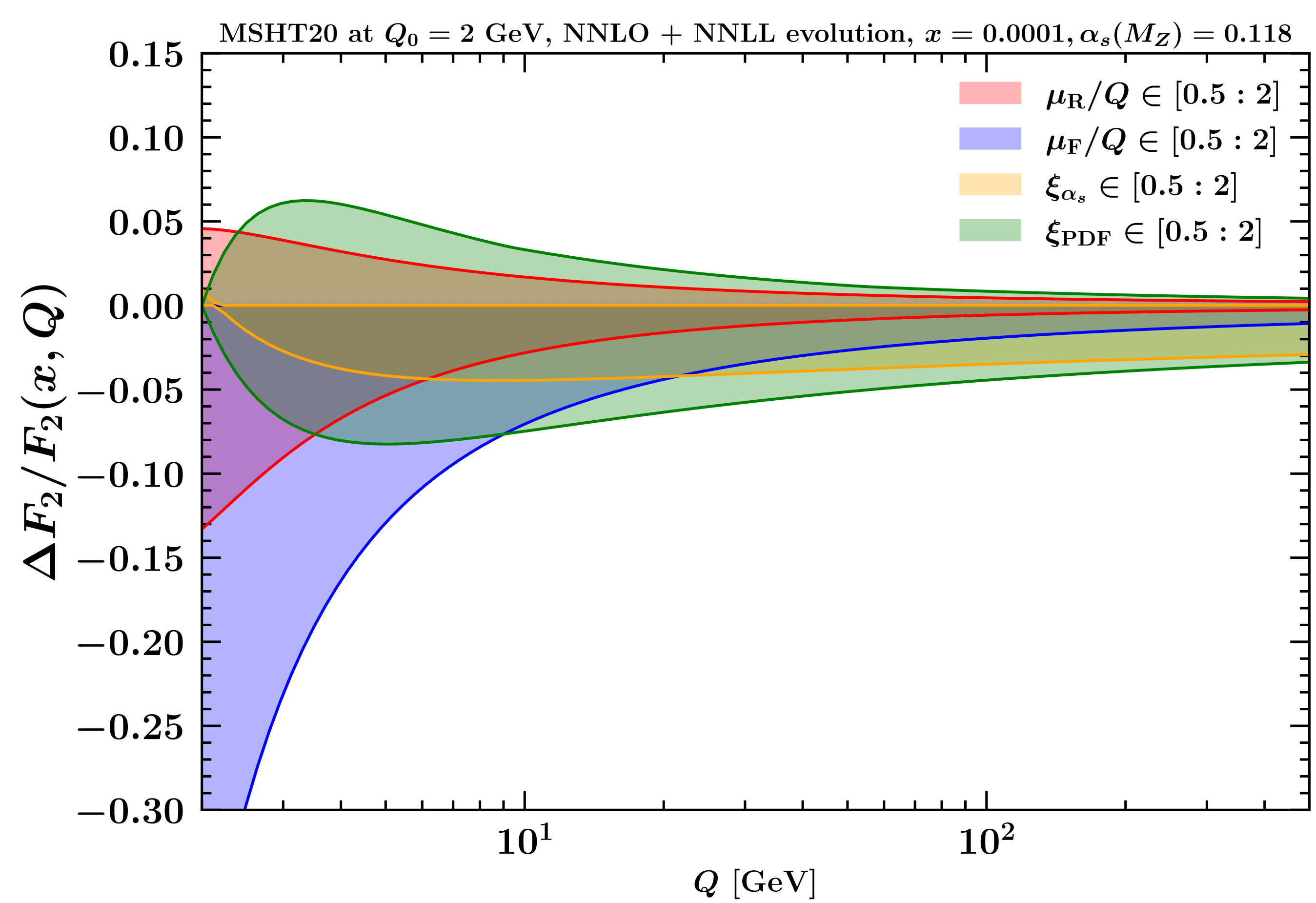


- ξ_{PDF} band grows with decreasing x: relevant at large Q and low x
- ξ-uncertainties are cumulative: increase with the evolution interval —> potentially relevant at high scales

RGE uncertainties $\Delta F_2/F_2(Q)$ @ NNLO







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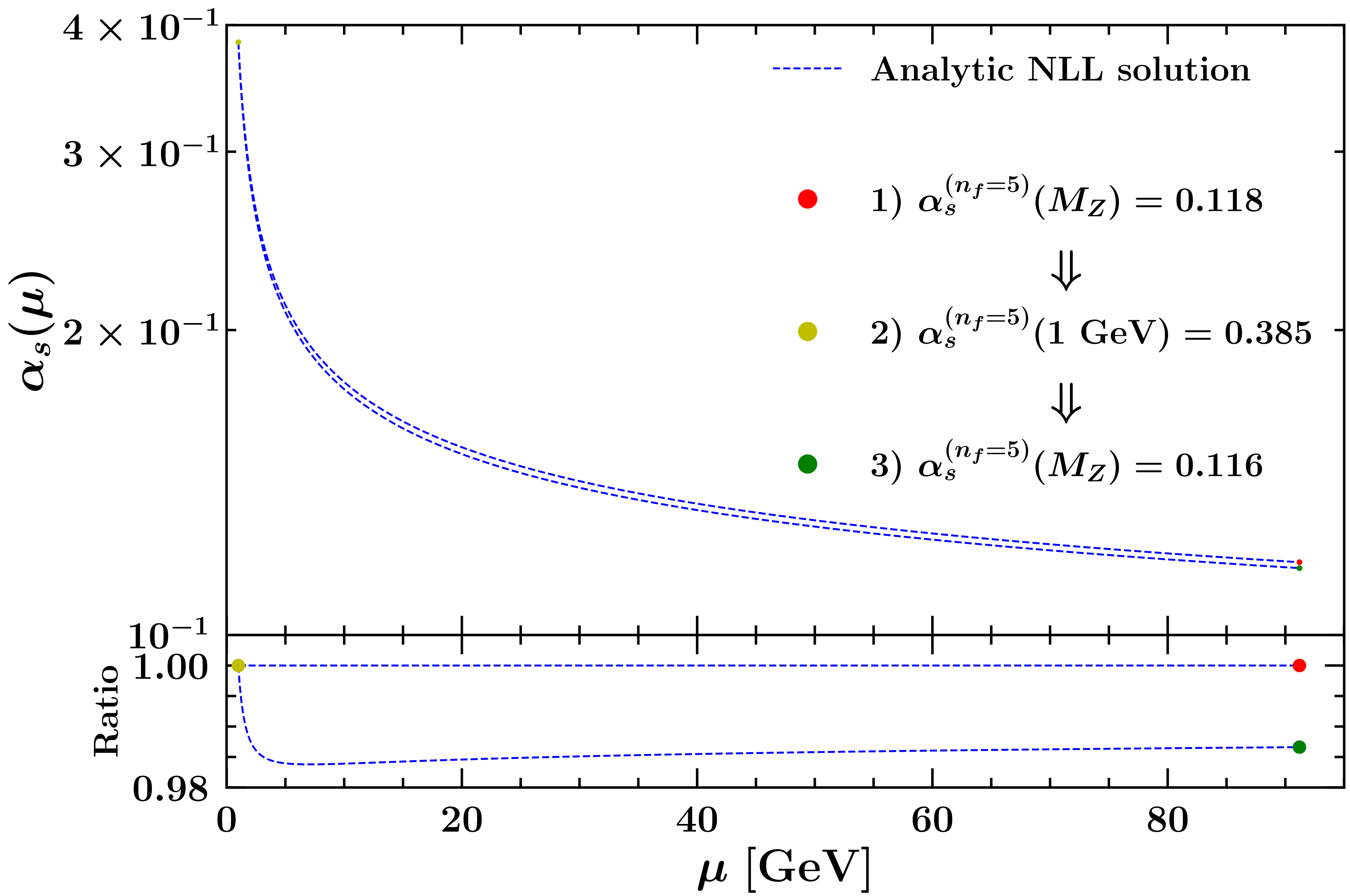
Conclusions

- We have studied the theoretical uncertainties stemming from solutions of RGE
- We extended the "g-functions" formalism commonly used in Sudakov resummation to treat RGE uncertainties on the running coupling and the PDFs
- We quantified the effect of RGE uncertainties on the F_2 structure function and found that they are significant in a kinematic regime relevant for PDF extractions and collider phenomenology
- Outlook:
    - collinear/TMD PDF fit including RGE uncertainties
    - Extension to Sudakov resummation. We aim at identifying an appropriate use of scale variations to estimate theoretical uncertainties in both the analytical and numerical approaches for the DY q_T-spectrum

Backup

Perturbative hysteresis α_s

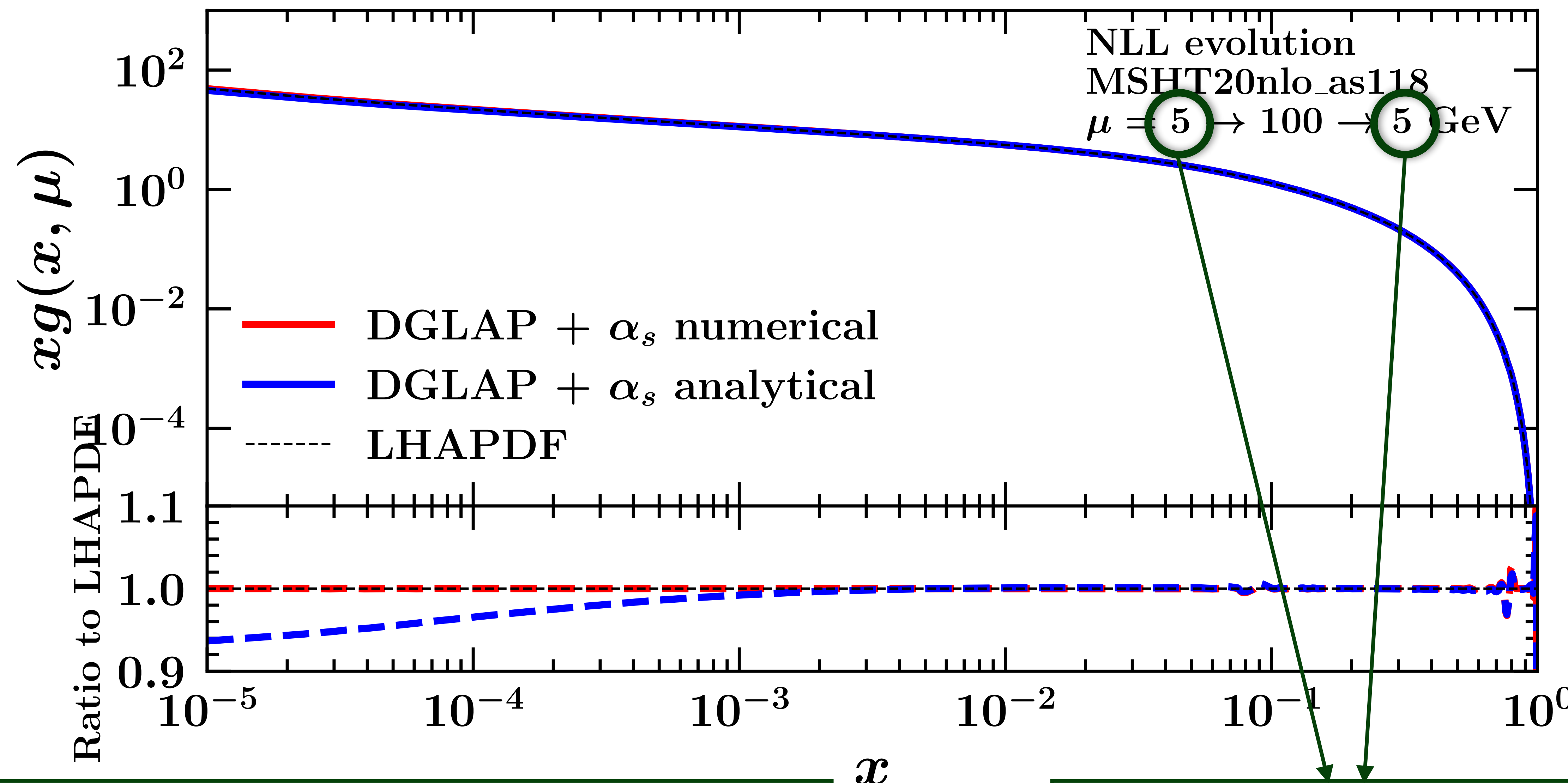
$$\alpha_s^{NLL}(\mu) = \alpha_s(\mu_0)\, g_1(\lambda) + \alpha_s^2(\mu_0)\, g_2(\lambda)$$



Non-conservativity of the analytic evolution: $G(\mu, \mu_0) G(\mu_0, \mu) \neq 1$

Perturbative hysteresis: PDF

$$f_{NLL}(\mu) = g_0(\bar{\lambda}, \ln\kappa)\exp\left(g_1(\bar{\lambda}) + \alpha_s g_2(\bar{\lambda}, \ln\kappa)\right) f(\mu_0)$$

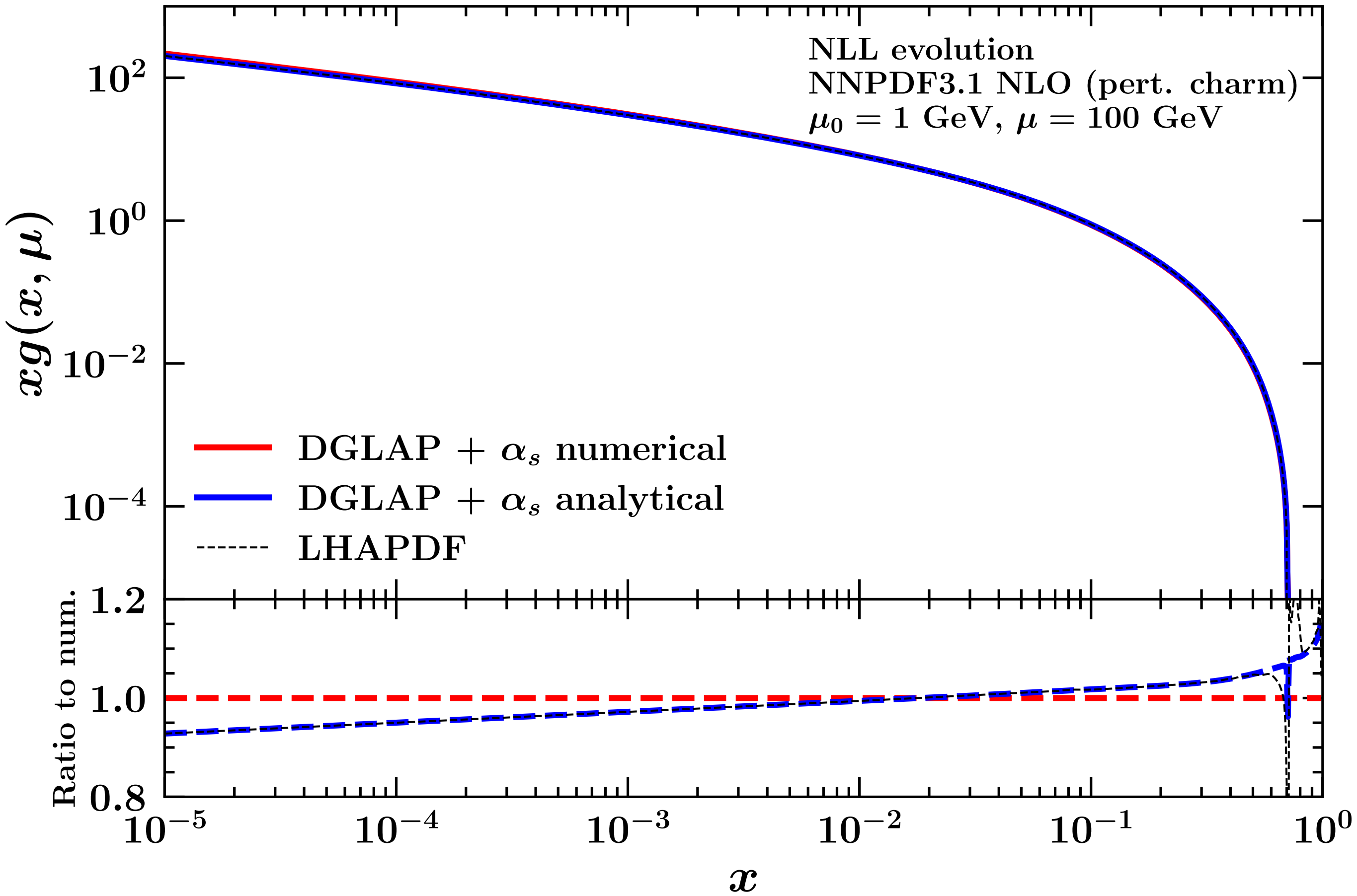


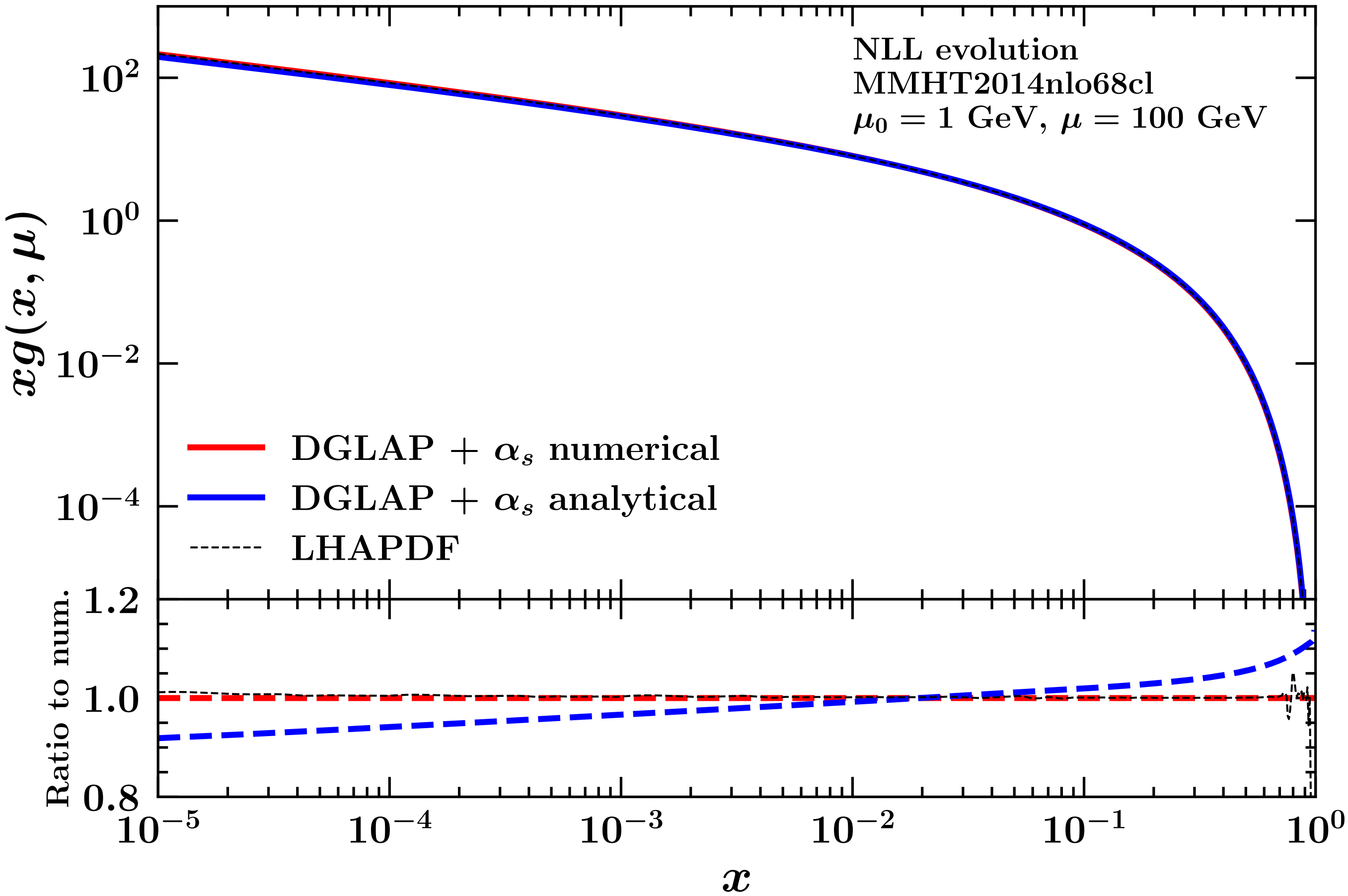
- numerical: no hysteresis
- analytic: a few percent in low-x region

- no heavy-quark threshold crossed in the evolution

Analytic/numerical mismatch

$$f^{NLL}(\mu) = g_0^{NLL}(\bar{\lambda})\ \exp\left(g_1(\bar{\lambda}) + \alpha_s g_2(\bar{\lambda})\right)\ f(\mu_0)$$





- Mismatch can exceed 10% at very low and very high values of x
- Comparison with native evolution (from LHAPDF) shows that MMHT (NNPDF) implements the numerical (analytic) evolution

Solution of RGE for PDFs

$$
\begin{aligned}
f(\mu) &= \exp\left[\int_{\mu_0}^{\mu} d\ln\mu'\,\gamma(a_s(\mu'))\right] f(\mu_0) \\
&= \exp\left[\sum_{n=0}^{k}\gamma^{(n)}\int_{\mu_0}^{\mu} d\ln\mu'\,a_s^{n+1}(\mu')\right] f(\mu_0) \\
&= \prod_{n=0}^{k}\exp\left[\gamma^{(n)} I_n\right]\, f(\mu_0),
\end{aligned}
$$

$$
I_n = \int_{\mu_0}^{\mu} d\ln\mu'\,a_s^{n+1}(\mu') = \int_{a_s(\mu_0)}^{a_s(\mu)} da_s \left(\frac{a_s^n}{\overline{\beta}(a_s)}\right)
$$

$$
\begin{aligned}
I_0 &= \frac{1}{\beta^{(0)}}\int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{a_s + b_1 a_s^2 + b_2 a_s^3} = \frac{1}{\beta^{(0)}}\ln\left(\frac{a_s(\mu)}{a_s(\mu_0)}\right) - \frac{b_1}{\beta^{(0)}}(a_s(\mu)-a_s(\mu_0)) + \frac{b_1^2-b_2}{2\beta^{(0)}}(a_s^2(\mu)-a_s^2(\mu_0)) + \mathcal{O}(\alpha_s^3)\,, \\
I_1 &= \frac{1}{\beta^{(0)}}\int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{1 + b_1 a_s + b_2 a_s^2} = \frac{1}{\beta^{(0)}}(a_s(\mu)-a_s(\mu_0)) - \frac{b_1}{2\beta^{(0)}}(a_s^2(\mu)-a_s^2(\mu_0)) + \mathcal{O}(\alpha_s^3)\,, \\
I_2 &= \frac{1}{\beta^{(0)}}\int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s\,a_s}{1 + b_1 a_s + b_2 a_s^2} = \frac{1}{2\beta^{(0)}}(a_s^2(\mu)-a_s^2(\mu_0)) + \mathcal{O}(\alpha_s^3)\,.
\end{aligned}
$$

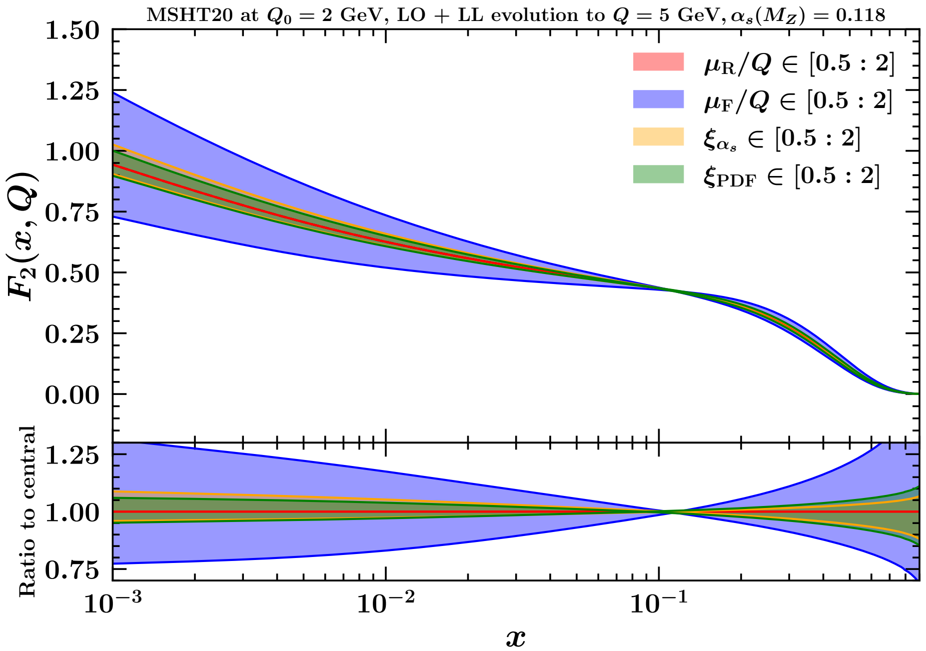
(20)

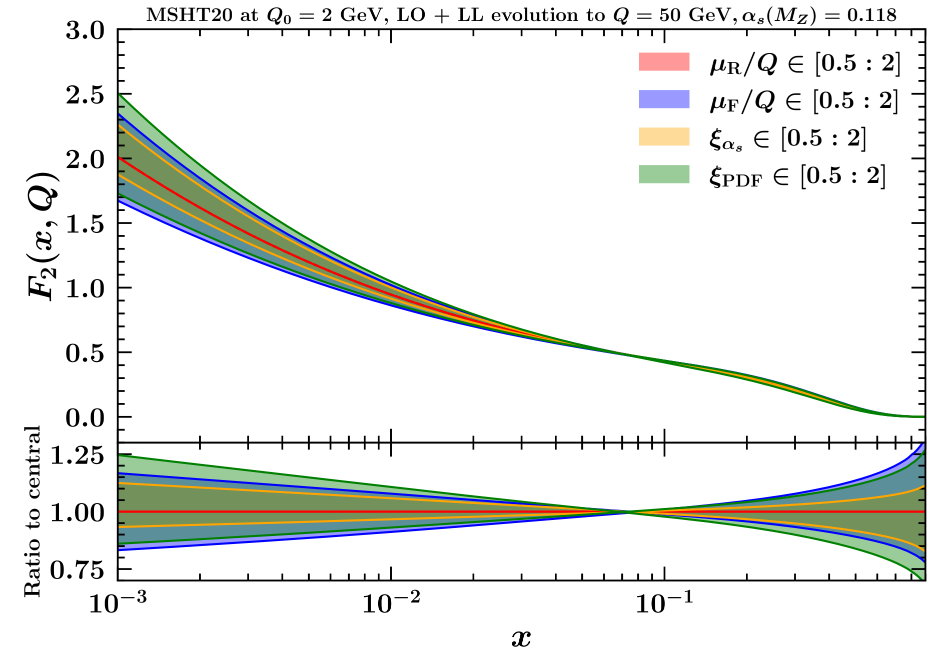
$$
\exp\left[\gamma^{(n)} I_n\right] = 1 + \gamma^{(n)} \times \mathcal{O}(\alpha_s^n)\,, \quad n \geq 1\,.
$$

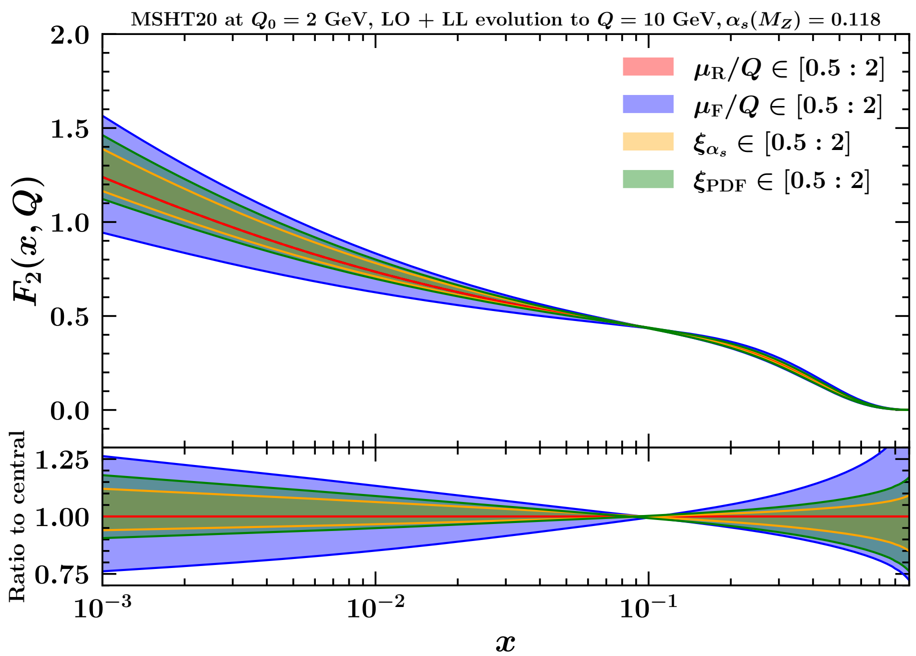
$$
f(\mu) = [1 + \mathcal{O}(\alpha_s)]\left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} f(\mu_0)\,.
$$

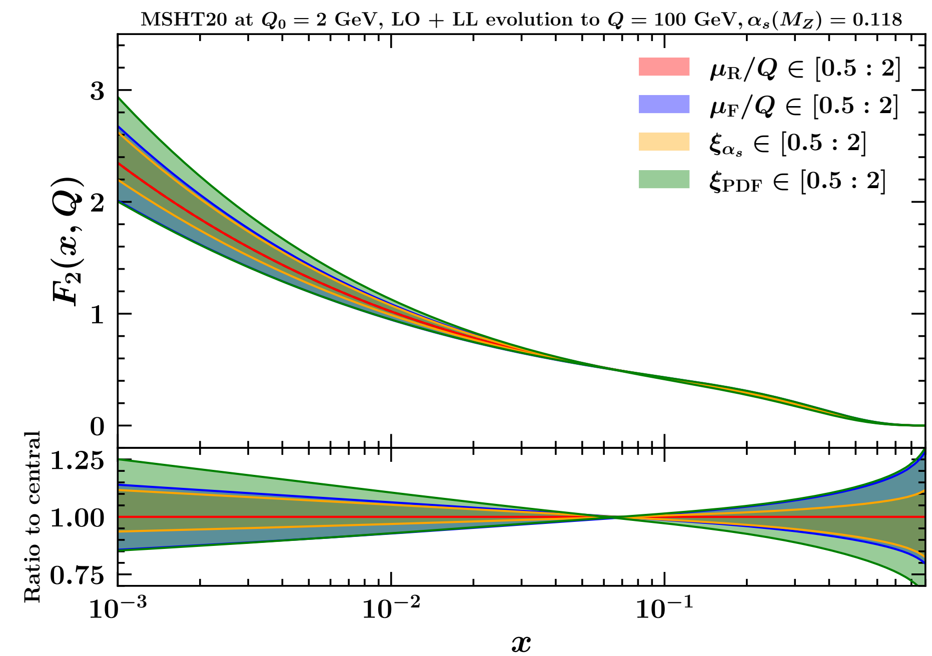
$$
\exp\left[\gamma^{(0)} I_0\right] = \exp\left[\frac{\gamma^{(0)}}{\beta^{(0)}}\ln\left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)\right] + \mathcal{O}(\alpha_s) = \left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} + \mathcal{O}(\alpha_s)\,.
$$

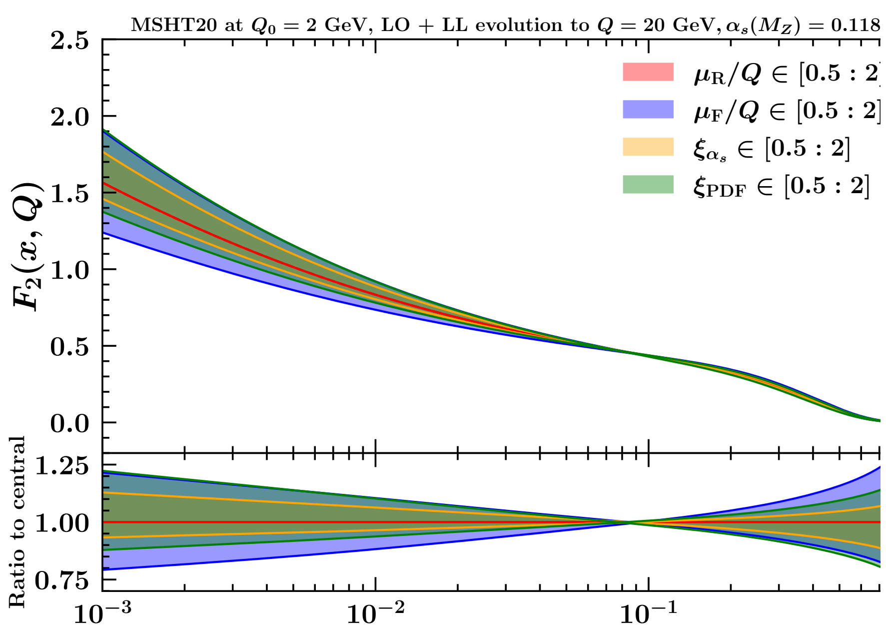
RGE uncertainties $F_2(x)$ @ LO

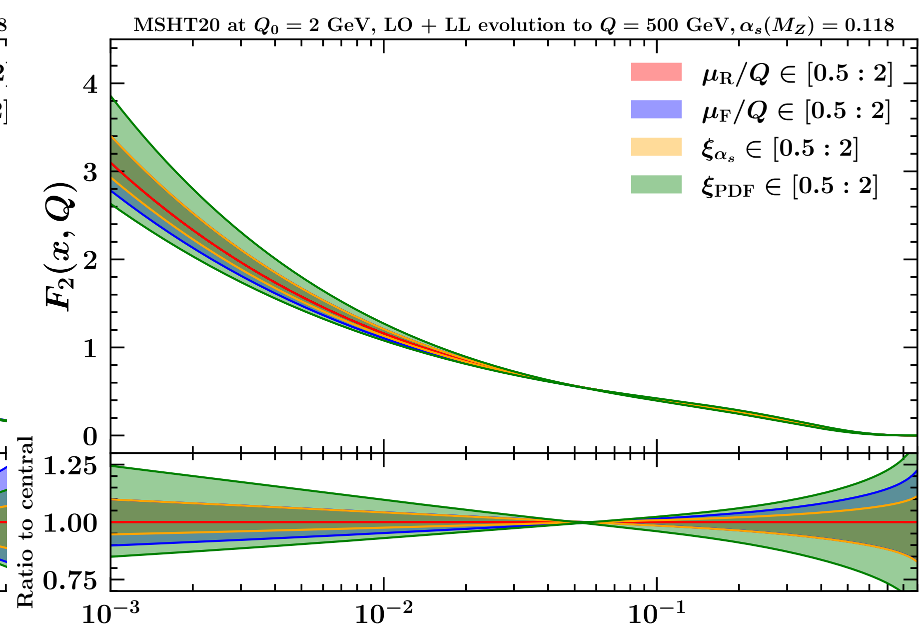


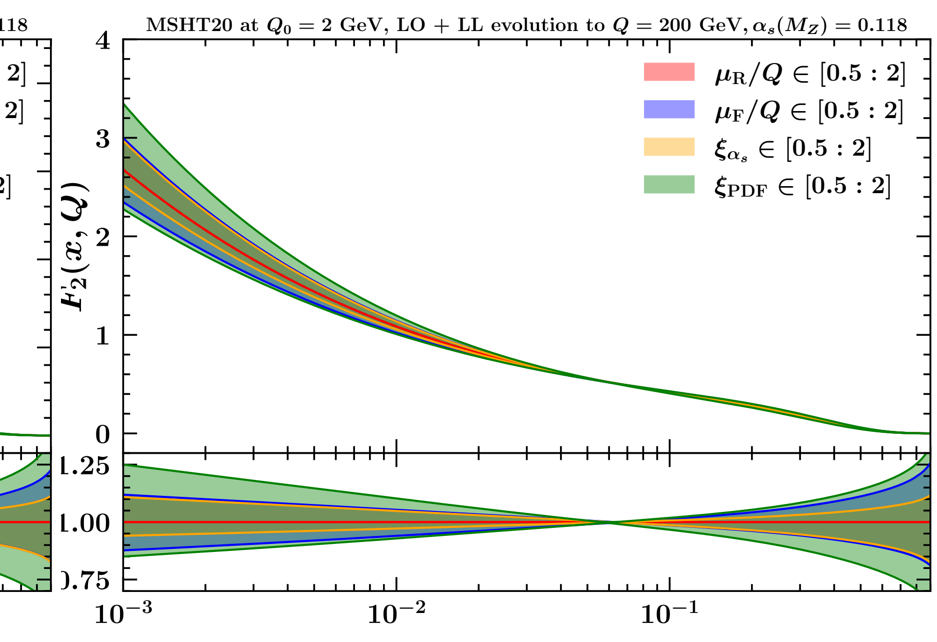






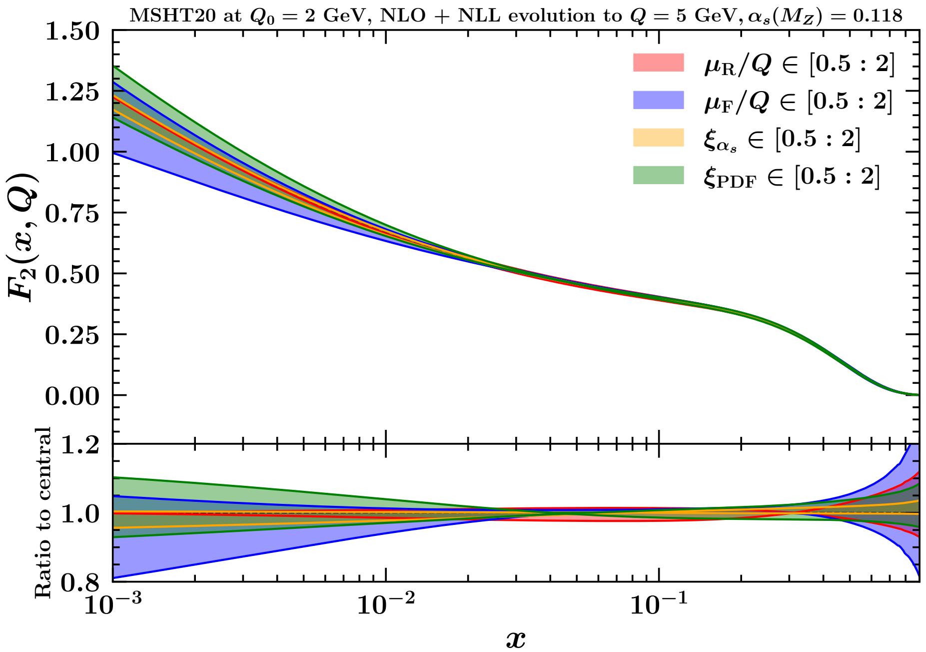


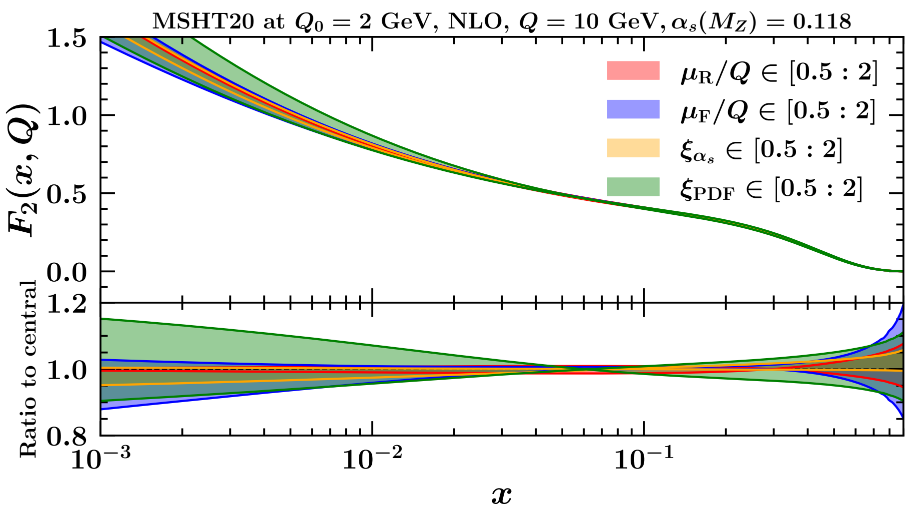


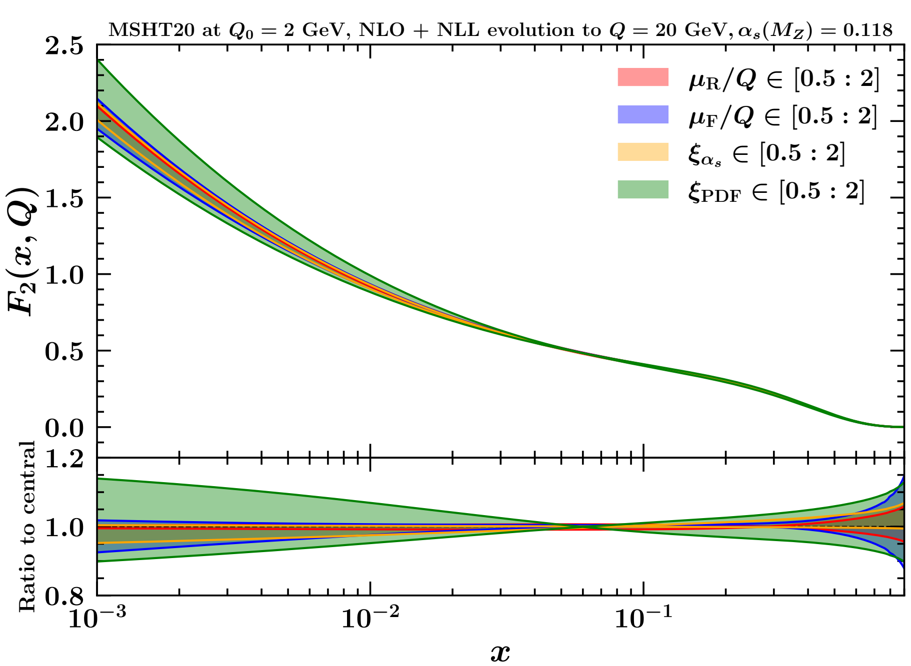


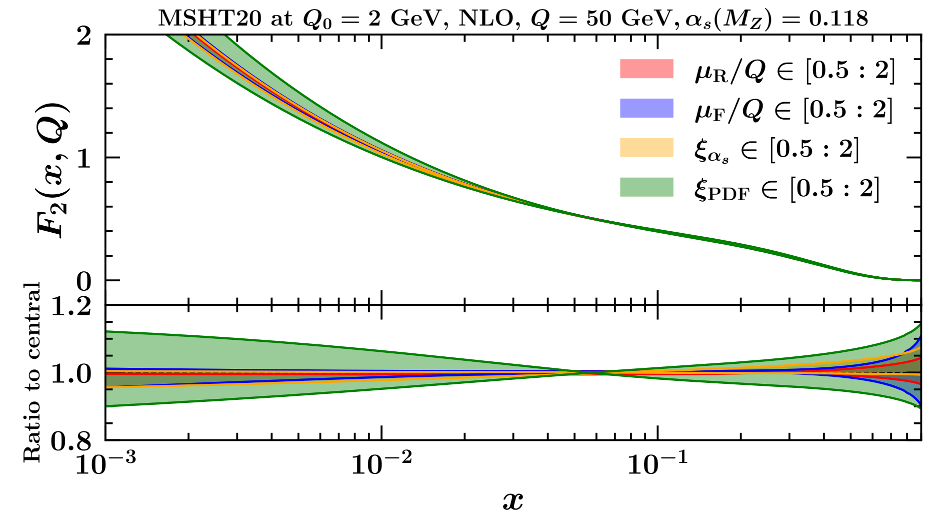
RGE uncertainties $F_2(x)$ @ NLO

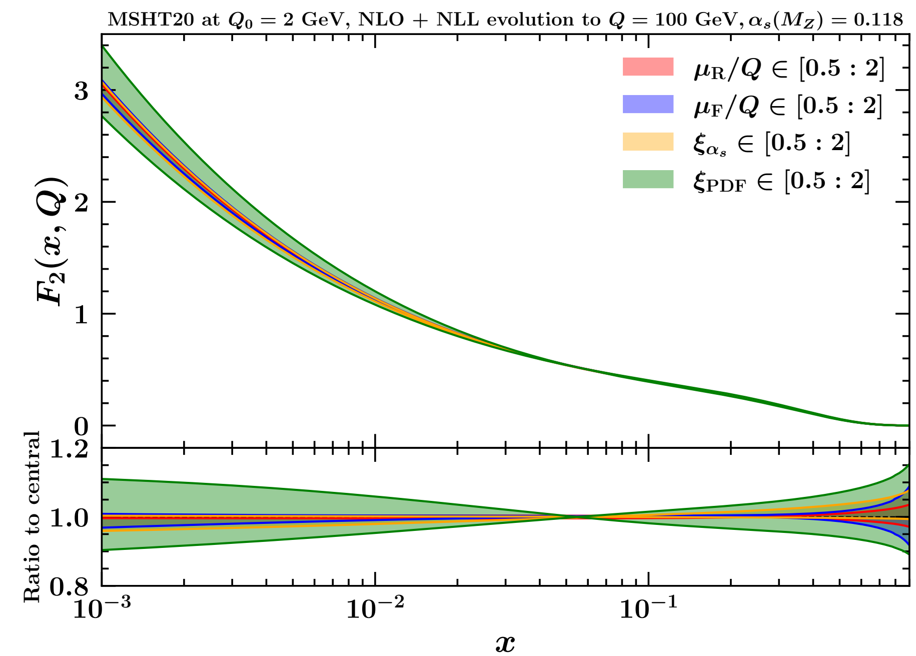
- ξ-uncertainties increase with Q
- become dominant also at large-x

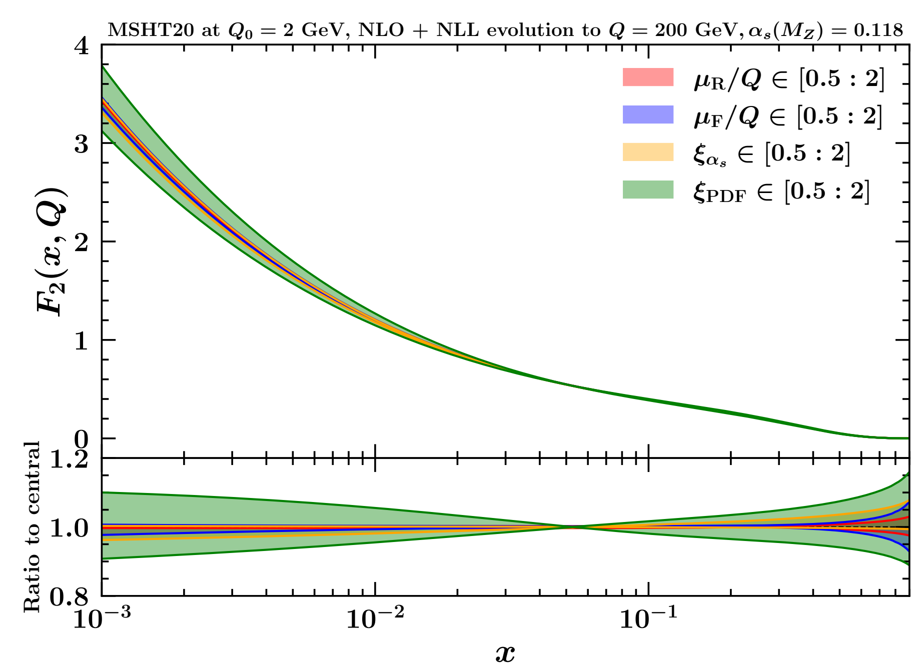




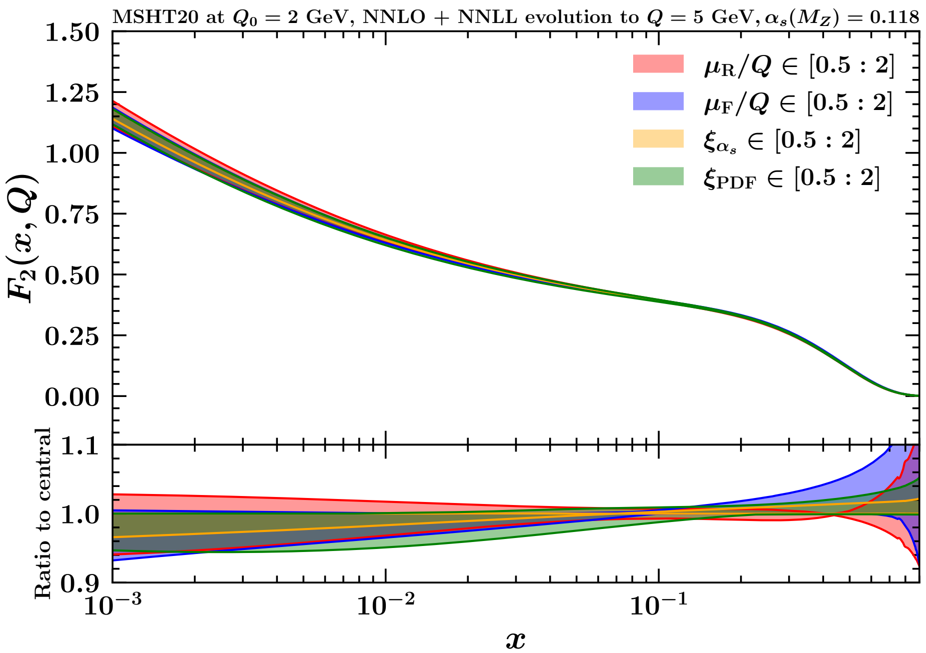


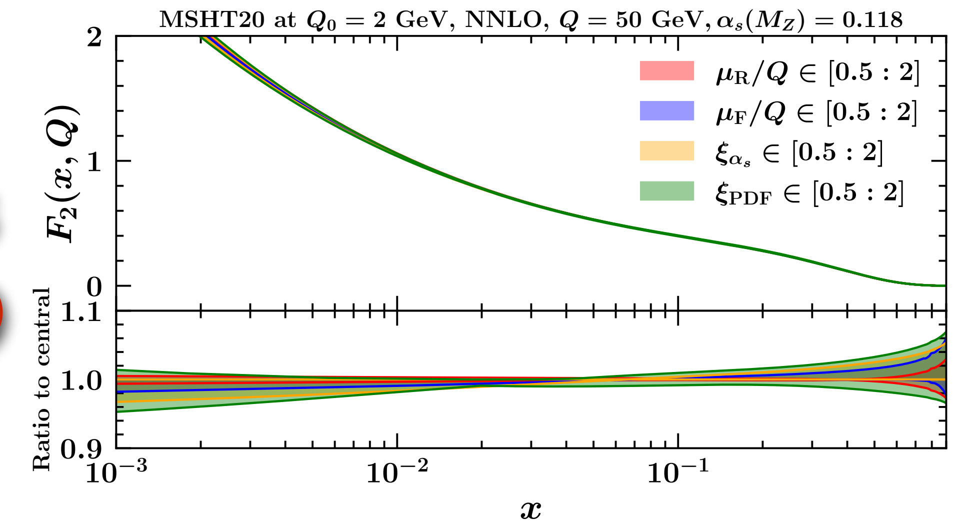


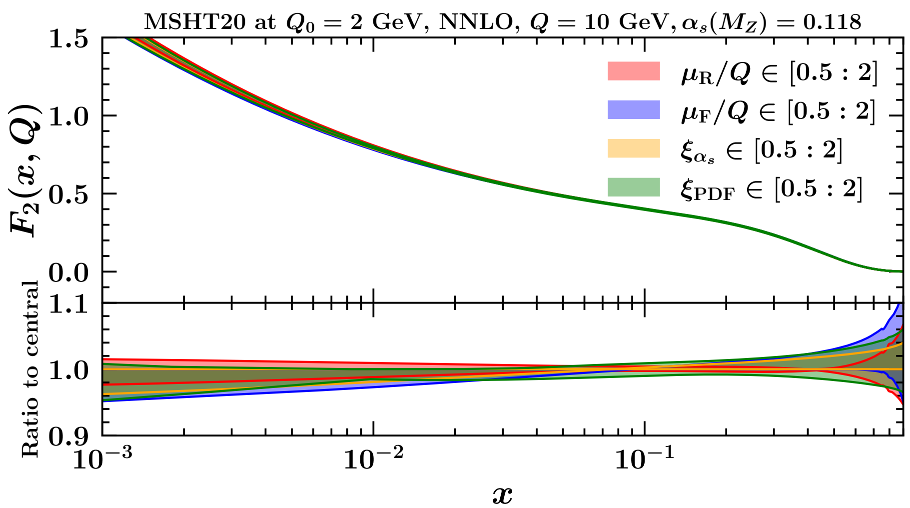




RGE uncertainties $F_2(x)$ @ NNLO







- same features with reduced bands at NNLO