NLO and NNLO hadronic vacuum polarization contributions to the muon g-2 in the space-like region

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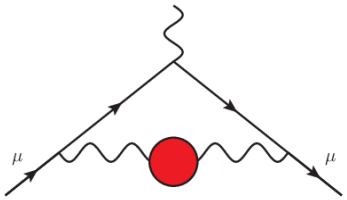
Summary

- Brief summary of time-like method for LO hadronic vacuum polarization contribution to muon g-2
- Space-like method for LO hadronic vacuum polarization contribution to muon g-2
- NLO hadronic vacuum polarization contributions
- NNLO hadronic vacuum polarization contributions

The content is based on E.Balzani, S.L. and M.Passera, arXiv:2112.05704

Reminder: Time-like method



leading order (LO) hadronic vacuum polarization contribution to muon g-2.

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi^2} \int\limits_{s_0=m_{\pi^0}^2}^{\infty} \frac{ds}{s}\, K^{(2)}(s/m^2){\rm Im}\Pi(s) \qquad = 6931(40)\times 10^{-11}\ ({\rm WP20})$$

$$\text{optical theorem} \to {\rm Im}\Pi(s) = \frac{\alpha}{3}R(s) \qquad R(s) = \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{4\pi\alpha^2/(3s)}$$

- $R(s)$ fluctuating at low energy due to resonance and particle production threshold effects
- $K^{(2)}(s/m^2)$: 1-loop QED g-2 contribution with a massive photon of mass $\sqrt{s}$

$$K^{(2)}(s/m^2) = \int\limits_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

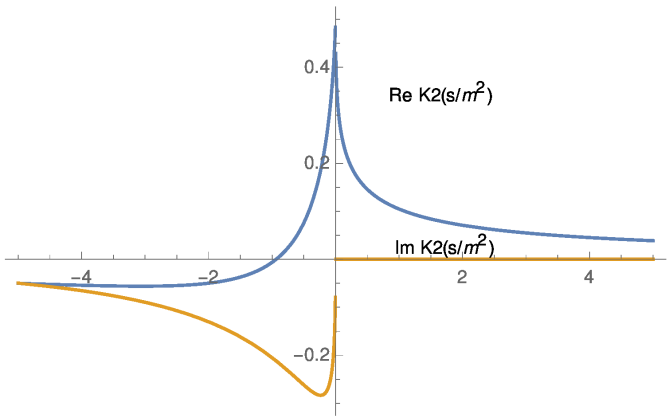
$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right) \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$$

$$K^{(2)}(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\mathrm{Im}K^{(2)}(z')}{z'-z}, \quad z>0 \qquad \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{s-q^2} = \frac{\Pi(q^2)}{q^2}\,, \quad q^2<0$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{LO}) = \frac{\alpha}{\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\,K^{(2)}(s/m^2)\mathrm{Im}\Pi(s) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\,\Pi(t)\mathrm{Im}K^{(2)}(t/m^2)$$

the imaginary part for $z<0$ is

$$\mathrm{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)\left[\frac{z^2}{2} - z + \frac{z-2z^2+\frac{z^3}{2}}{\sqrt{z(z-4)}}\right]$$



$$K^{(2)}(0) = 1/2\,, \qquad K^{(2)}(z) \to 1/(3z) \quad z\to\infty$$

$K^{(2)}(z)>0$ for $z>0$; there is no cut for $0<z<4$.

$\mathrm{Im}K^{(2)}(z)$ espressed in terms of $y(z)$ is simpler

$$\mathrm{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)F^{(2)}(1/y(z)) \ , \qquad F^{(2)}(u) = \frac{u+1}{u-1}u^2$$

changing again variable in the dispersive integral $t \to y \to x$ $(t < 0 \to y < -1 \to 0 < x < 1)$

$$t(x) = m^2 \frac{x^2}{x-1} \ , \qquad x = 1 + 1/y$$

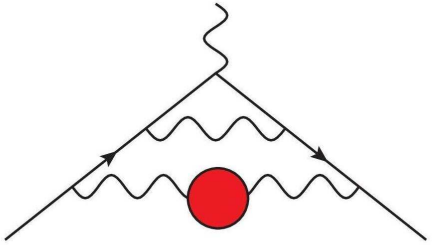
$$a_\mu^{\mathrm{HVP}}(\mathrm{LO}) = \frac{\alpha}{\pi} \int_0^1 dx \ \kappa^{(2)}(x) \Delta\alpha_{\mathrm{had}}(t(x))$$

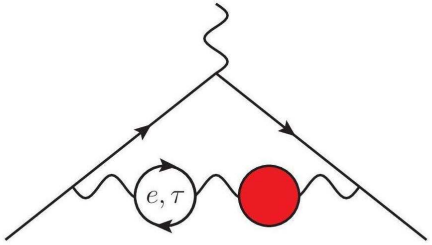
Lautrup Peterman de Rafael 1975

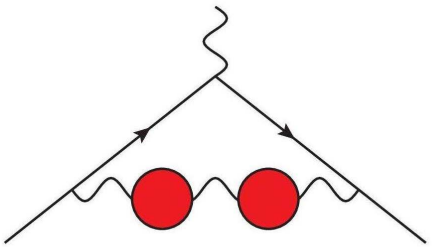
- $\kappa^{(2)}(x) = 1 - x$ simple space-like kernel
- $\Delta\alpha_{\mathrm{had}}(t) = -\Pi(t)$ hadronic contribution to the running of the effective fine-structure constant in the space-like region

The above expression was proposed for the first time (Carloni Calame Passera Trentadue Venanzoni 2015) to determine a_μ^{HVP} measuring the electromagnetic effective coupling in the space-like region through scattering data.

NLO hadronic vacuum polarization contributions



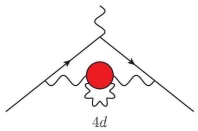




- Class a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = -209.0 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4b) = +106.8 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4c) = +3.5 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; total) = -98.7(9) \times 10^{-11}$$

(Krause 1996, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)



HVP insertion with internal corrections already incorporated in LO

We write the time-like expression

$$a_\mu^{\rm HVP}({\rm NLO};4a) = \frac{\alpha}{\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\,2K^{(4)}(s/m^2)\,{\rm Im}\Pi(s)$$

$2K^{(4)}$: anomaly from all 2-loop QED diagrams with 1 massless photon and 1 photon of mass $\sqrt{s}$ (factor 2 due to normalization chosen)

$$\begin{aligned}K^{(4)}(z) = {}& \left(\frac{z^2}{2}-\frac{7z}{6}+\frac{1}{2}\right)\Big[-3{\rm Li}_3(-y)-6{\rm Li}_3(y)+2\left({\rm Li}_2(-y)+2{\rm Li}_2(y)\right)\ln y+\frac{1}{2}\left(\ln^2 y+\pi^2\right)\ln(y+1)+\ln(1-y)\ln^2 y\Big]\\ &+\frac{\left(-\frac{z^3}{6}+\frac{z^2}{4}-\frac{7z}{6}-\frac{4}{z-4}+\frac{13}{3}\right)\left({\rm Li}_2(-y)+\frac{\ln^2 y}{4}+\frac{\pi^2}{12}\right)}{\sqrt{(z-4)z}}+\frac{\left(-\frac{7z^3}{12}+\frac{17z^2}{6}-2z\right)\left({\rm Li}_2(y)-\frac{1}{4}\ln^2 y+\ln(1-y)\ln y-\frac{\pi^2}{6}\right)}{\sqrt{(z-4)z}}\\ &+\left(-\frac{29z^2}{96}+\frac{53z}{48}+\frac{2}{z-4}-\frac{1}{3z}+\frac{19}{24}\right)\ln^2 y+\frac{\left(\frac{23z^3}{144}-\frac{115z^2}{72}+\frac{127z}{36}-\frac{4}{3}\right)\ln y}{\sqrt{(z-4)z}}+\frac{\left(-\frac{7z^3}{48}+\frac{17z^2}{24}-\frac{z}{2}\right)\ln y\ln z}{\sqrt{(z-4)z}}\\ &+\frac{1}{6}\pi^2\left(-\frac{z^2}{2}+\frac{5z}{24}-\frac{2}{z}+\frac{9}{4}\right)+\frac{5}{96}z^2\ln^2 z+\left(\frac{23z^2}{144}-\frac{7z}{36}+\frac{1}{z-4}+\frac{19}{12}\right)\ln z+\frac{115z}{72}-\frac{139}{144}\end{aligned}$$

Barbieri Remiddi 1975

$$K^{(4)}(0) = \frac{197}{144}+\frac{1}{12}\pi^2-\frac{1}{2}\pi^2\ln 2+\frac{3}{4}\zeta(3) = -0.328479$$ 2-loop g-2

As in the LO case we write the dispersive relation for $K^{(4)}(z)$ and $\Pi(q^2)$

$$K^{(4)}(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\mathrm{Im}K^{(4)}(z')}{z'-z}, \;\; z>0 \qquad \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{s-q^2} = \frac{\Pi(q^2)}{q^2}\;, \quad q^2<0$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \frac{\alpha}{\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\, 2K^{(4)}(s/m^2)\,\mathrm{Im}\Pi(s) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\Pi(t)\,\mathrm{Im}2K^{(4)}(t/m^2)$$

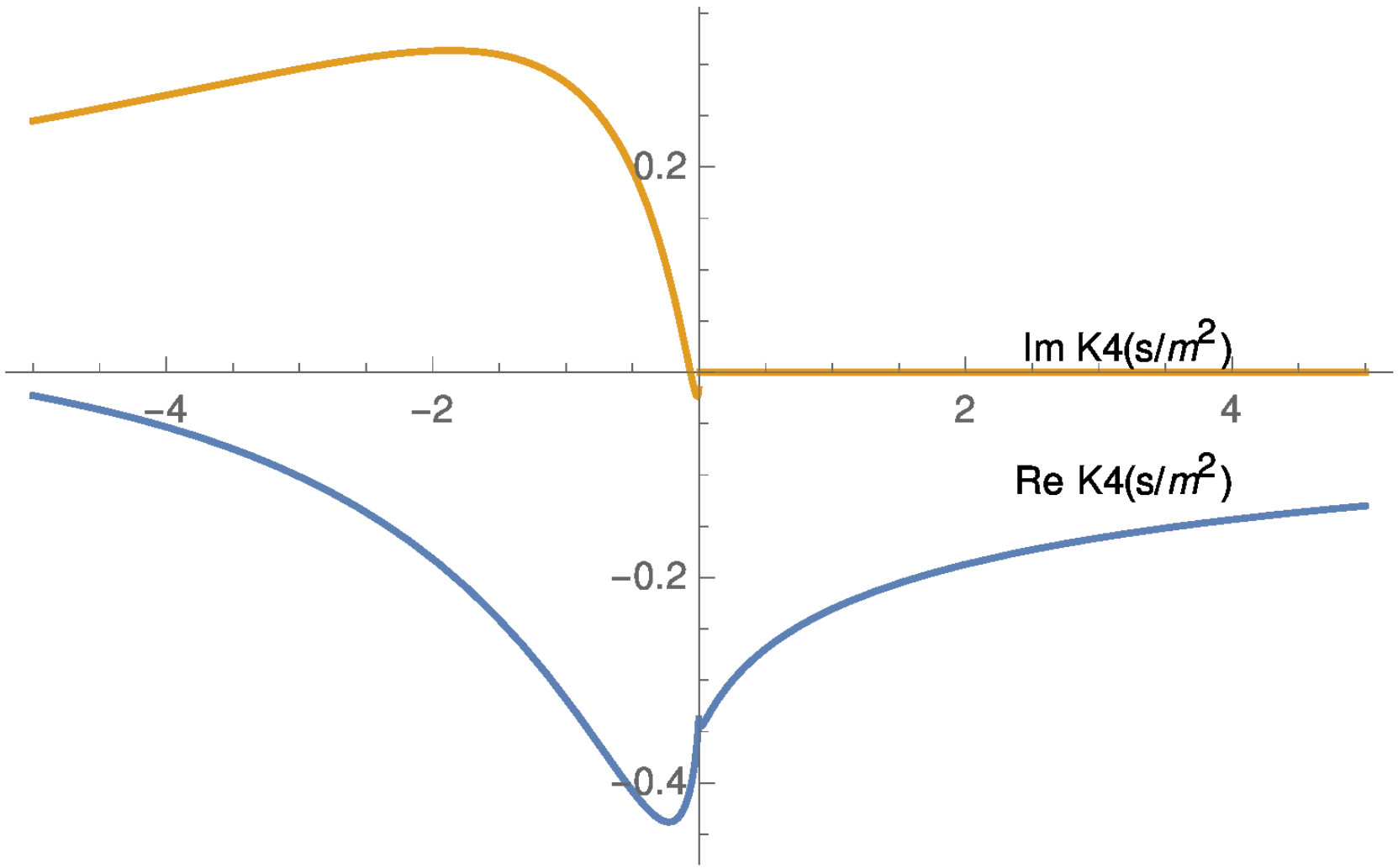
The imaginary part for $z<0$ is obtained from $K^{(4)}(z)$

$$\mathrm{Im}K^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(1/y(z)) \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}} < -1$$

$$\begin{aligned} F^{(4)}(u) =& \frac{-3u^4-5u^3-7u^2-5u-3}{6u^2}\left(2\mathrm{Li}_2(-u)+4\mathrm{Li}_2(u)+\ln(-u)\ln\left((1-u)^2(u+1)\right)\right) \\ &+\frac{(u+1)\left(-u^3+7u^2+8u+6\right)}{12u^2}\ln(u+1)+\frac{\left(-7u^4-8u^3+8u+7\right)}{12u^2}\ln(1-u) \\ &+\frac{23u^6-37u^5+124u^4-86u^3-57u^2+99u+78}{72(u-1)^2u(u+1)}+\frac{12u^8-11u^7-78u^6+21u^5+4u^4-15u^3+13u+6}{12(u-1)^3u(u+1)^2}\ln(-u) \end{aligned}$$

$\mathrm{Im}K^{(4)}(z)$ also found independently by A.V.Nesterenko arXiv:2112.05009.

NLO class $4a$

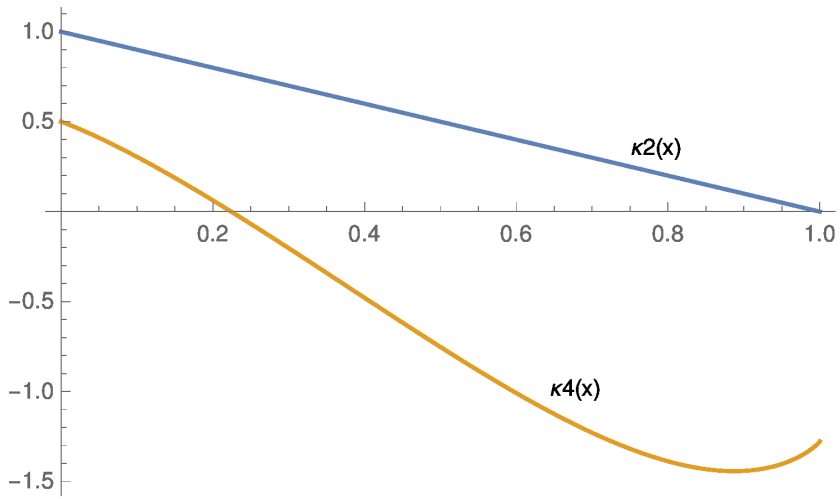


$K^{(4)}(0) = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328479$ 2-loop g-2 $\qquad K^{(4)}(z) \to \frac{1}{z}\left(-\frac{23\ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right)$

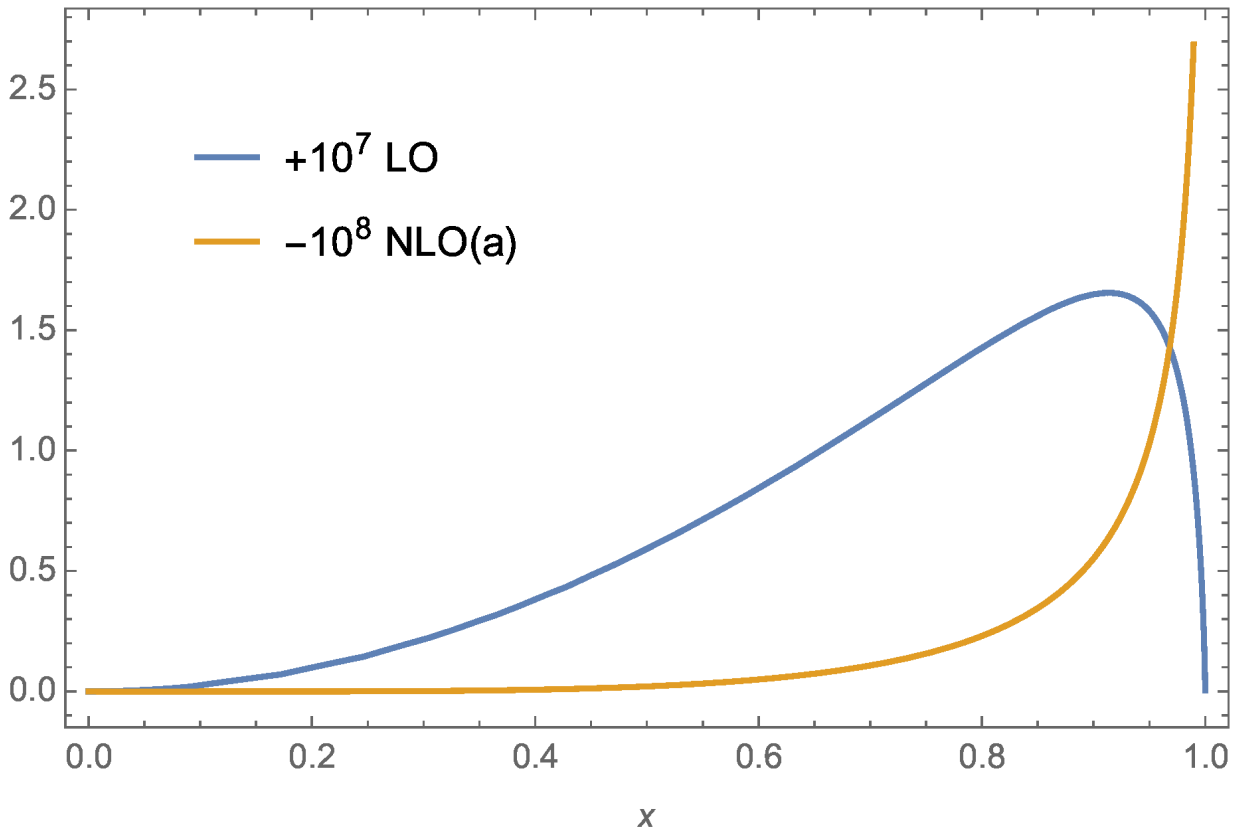
$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = \left(\frac{\alpha}{\pi}\right)^2 \int\limits_0^1 dx\, \kappa^{(4)}(x) \Delta\alpha_{\rm had}(t(x))$$

space-like kernel $\kappa^{(4)}(x)$:

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$



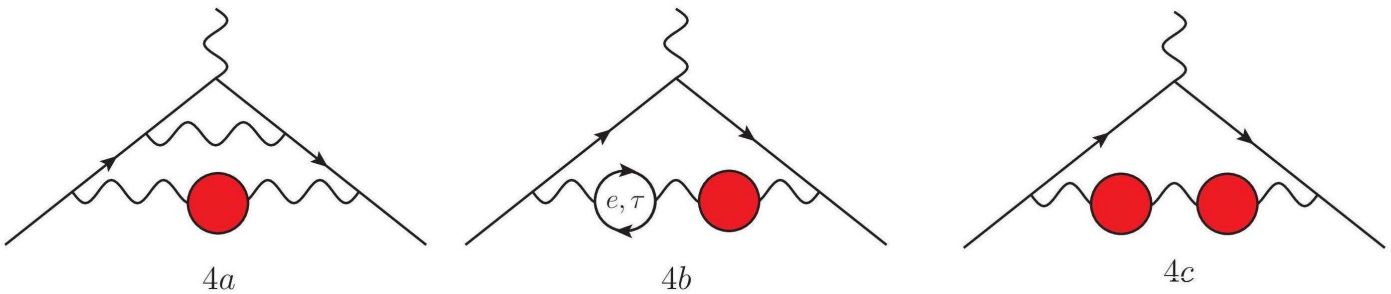
- $\kappa^{(4)}(1) = -\frac{23}{18}, \quad \kappa^{(4)}(0) = \frac{1}{2}$;
- $\kappa^{(4)}(x)$ provides stronger weight a large $q^2 < 0$ $(x \to 1)$ than $\kappa^{(2)}(x)$



the integrands $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))$ (blue) $\quad (\alpha/\pi)^2\kappa^{(4)}(x)\Delta\alpha_{\rm had}(t(x))$ (orange)

- LO integrand has a peak at $x \approx 0.914$
- NLO has an (integrable) logarithmic singularity at $x \to 1$

NLO class $4b$ and $4c$



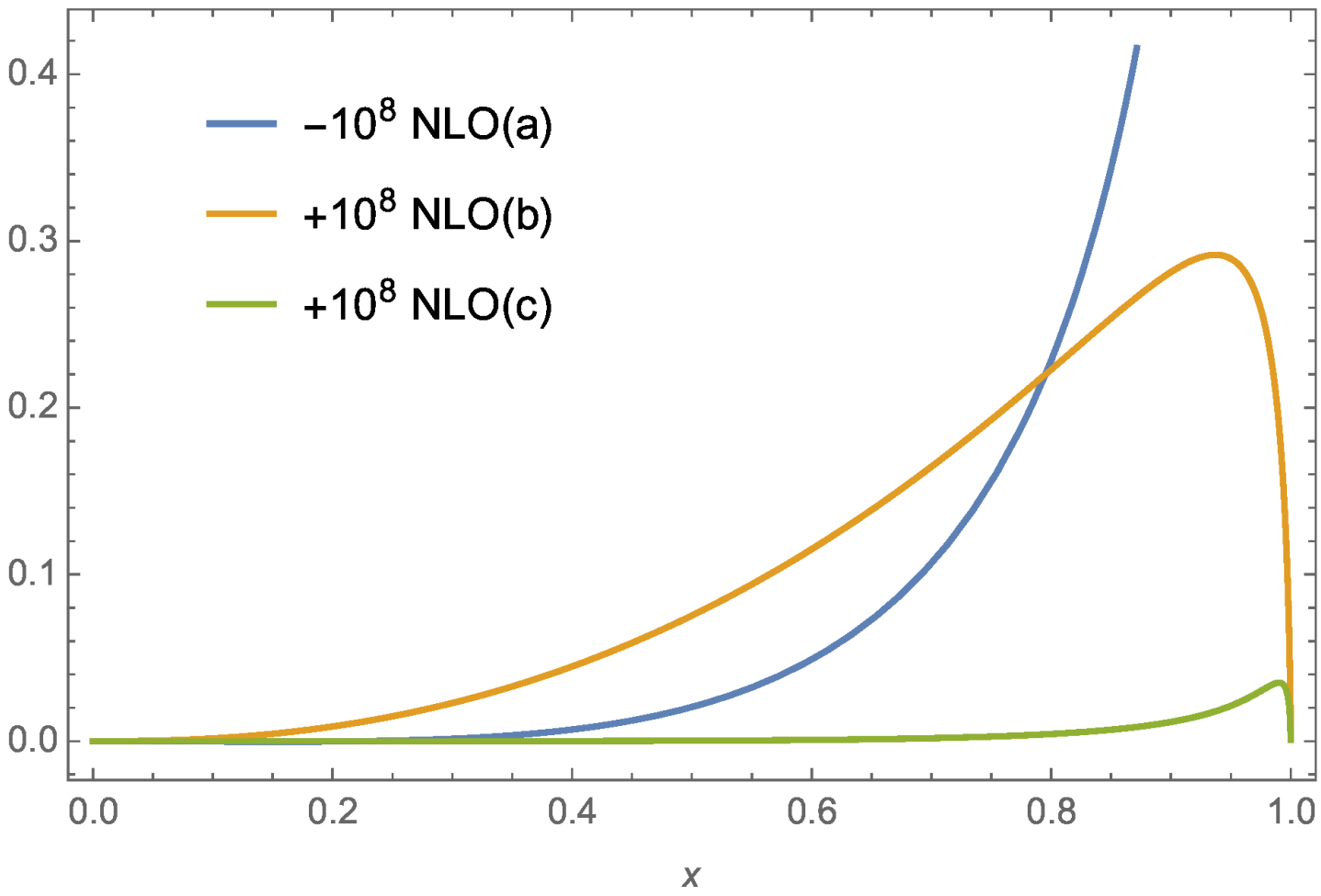
starting from time-like expressions one finds

$$a_\mu^{\rm HVP}({\rm NLO}; 4b) = \frac{\alpha}{\pi} \int_0^1 dx\ \kappa^{(2)}(x) \Delta\alpha_{\rm had}(t(x))\ 2 \left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x)) \right)$$

$$a_\mu^{\rm HVP}({\rm NLO}; 4c) = \frac{\alpha}{\pi} \int_0^1 dx\ \kappa^{(2)}(x) \left(\Delta\alpha_{\rm had}(t(x)) \right)^2$$

$\Pi_l^{(2)}(t) = -\Delta\alpha_{\rm l}(t)$ renormalized one-loop QED vacuum polarization function

$$\Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi}\right) \left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l \left(\frac{1}{2} - \frac{\beta_l^2}{6} \right) \ln \frac{\beta_l - 1}{\beta_l + 1} \right] \ , \quad \beta_l = \sqrt{1 - 4m_l^2/t}$$



the 3 NLO integrands

NLO class $4a$: *approximated* space-like kernels $\bar{\kappa}^{(4)}(x)$

Asymptotic expansion for large s of $K^{(4)}(s/m^2)$ in powers of $r = m^2/s$ (Lautrup 1997)

$$K^{(4)}(r) = r\left(\frac{23\ln r}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right) + r^2\left(\frac{19\ln^2 r}{144} + \frac{367\ln r}{216} - \frac{37\pi^2}{48} + \frac{8785}{1152}\right)$$
$$+ r^3\left(\frac{141\ln^2 r}{80} + \frac{10079\ln r}{3600} - \frac{883\pi^2}{240} + \frac{13072841}{432000}\right) + \ldots$$

from this expansion we derive *approximated* space-like kernel $\bar{\kappa}^{(4)}(x)$

We use the *modified* ansatz of [Groote Körner Pivovarov 2002] [Chakraborty Davies Kobonen Lepage VandeWater 2018]

$$K^{(4)}(s/m^2) = r\int_0^1 \mathrm{d}\xi\left[\frac{L(\xi)}{\xi + r} + \frac{P(\xi)}{1 + r\xi}\right] \qquad L(\xi) = G(\xi) + H(\xi)\ln\xi$$

$G(\xi) = \sum_{i=0}^{n-1} g_i\xi^i$, $H(\xi) = \sum_{i=0}^{n-1} h_i\xi^i$, $P(\xi) = \sum_{i=0}^{n-1} p_i\xi^i$. Integrating and expanding in r, the coefficients g_i, h_i and p_i fit the coefficients of $r^{i+1}\ln r$, $r^{i+1}\ln^2 r$, r^{i+1}, respectively.

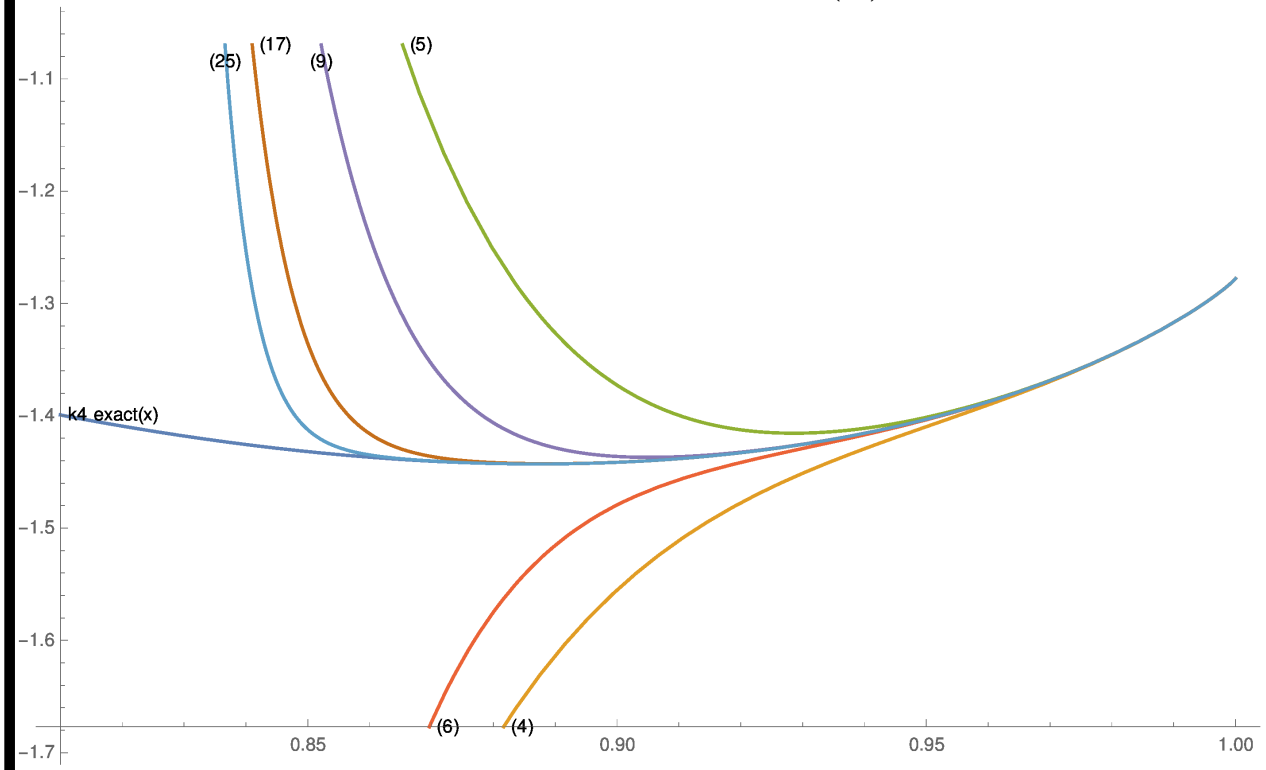
$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; 4a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 \mathrm{d}x\, \bar{\kappa}^{(4)}(x)\, \Delta\alpha_{\mathrm{h}}(t(x)),$$

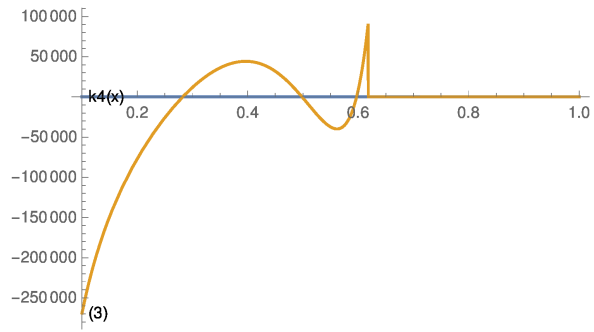
$$\bar{\kappa}^{(4)}(x) = \begin{cases} \frac{2-x}{x(1-x)}\, P\!\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618\ldots \\ \frac{2-x}{x^3}\, L\!\left(\frac{1-x}{x^2}\right) 1, & x_\mu < x < 1 \end{cases}$$

- *Original* ansatz had $\ln^2 r$ terms not fitted (*i.e.* $H = 0$) $\rightarrow$ Error of 6% on $a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; total)$,
- Error $\boxed{\text{eliminated}}$ by our *exact* NLO kernel $\kappa^{(4)}(x)$!

NLO class $4a$: *approximated* space-like kernels $\bar{\kappa}_n^{(4)}(x)$

Plot of the approximated kernels $\bar{\kappa}_n^{(4)}(x)$ for $n = 4, 5, 6, 9, 17, 25$ compared with exact $\kappa^{(4)}(x)$

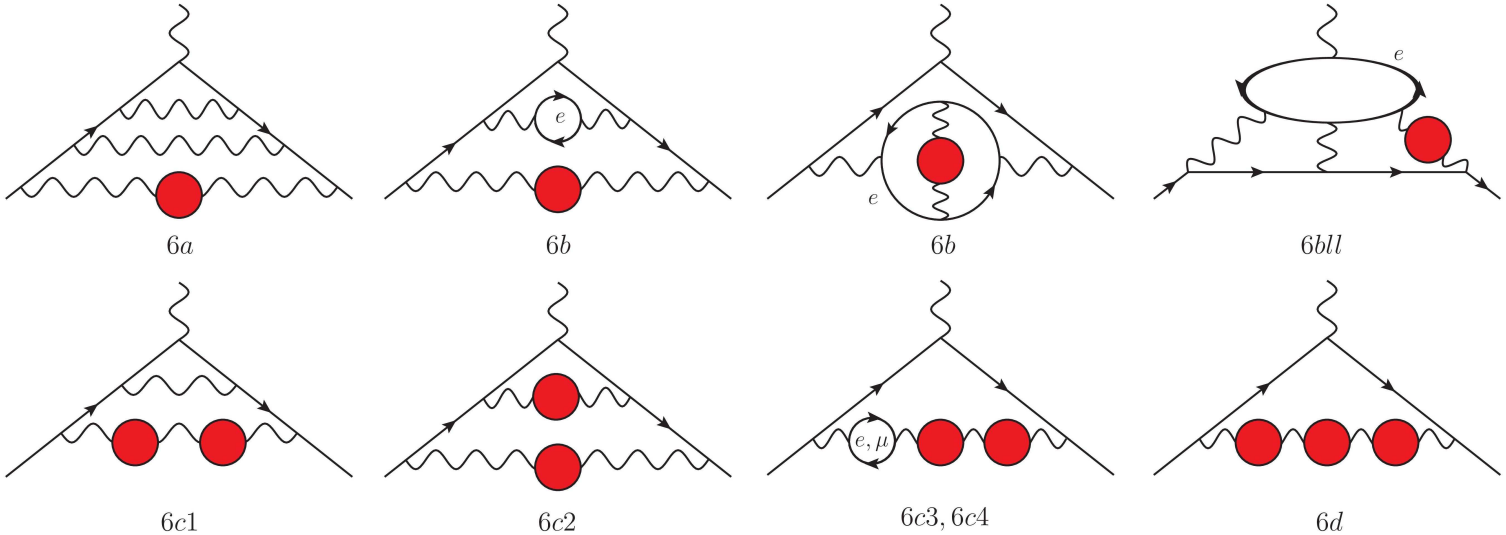




$n = 4$

- $\bar{\kappa}_n^{(4)}(x)$: good approximation for x near 1 ($t \to -\infty$);
- Discontinuity for $x = (\sqrt{5} - 1)/2 \approx 0.618$ ($t = -m^2$)
- Wild oscillations for small x, worse for large n.
- For $n = 25$ up to $\sim \pm 10^{30}$! But the integral reproduces the exact result with error 10^{-20} $\to$ deep numerical cancellations!.
- Large n not necessary! $\boxed{n = 4}$ reproduces $a_\mu^{\rm HVP}(\text{NLO}; 4a)$ with error $\lesssim 0.1\%$
- Useful method of approximation at NNLO

NNLO hadronic vacuum polarization contributions



$$a_\mu^{\rm HVP}({\rm NNLO}; 6a) = +8.0 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NNLO}; 6b) = -4.1 \times 10^{-11}$$

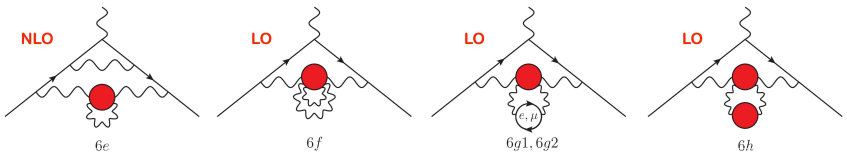
$$a_\mu^{\rm HVP}({\rm NNLO}; 6bll) = +9.1 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NNLO}; 6c) = -0.6 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NNLO}; 6d) = +0.005 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NNLO}; total) = +12.4(1) \times 10^{-11}$$

Kurz Liu Marquard Steinhauser 2014



HVP with internal corrections already incorporated in NLO and LO

$K^{(6a)}(s/m^2)$: Only the first 4 terms of the expansion in power series of $r = m^2/s$ are known
They contain terms with $r^n \ln r$, $r^n \ln^2 r$ and $r^n \ln^3 r$. As in NLO, we use an integral ansatz:

$$K^{(6a)}(s/m^2) = r \int_0^1 \mathrm{d}\xi \left[\frac{L^{(6a)}(\xi)}{\xi + r} + \frac{P^{(6a)}(\xi)}{1 + r\xi} \right] \qquad L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi) \ln \xi + J^{(6a)}(\xi) \ln^2 \xi$$

$G^{(6a)}$, $H^{(6a)}$,$J^{(6a)}$, $P^{(6a)}$ polynomials

$$G^{(6a)}(\xi) = \sum_{i=0}^{3} g_i^{(6a)} \xi^i, \quad H^{(6a)}(\xi) = \sum_{i=0}^{3} h_i^{(6a)} \xi^i, \quad J^{(6a)}(\xi) = \sum_{i=0}^{3} j_i^{(6a)} \xi^i, \quad P^{(6a)}(\xi) = \sum_{i=0}^{3} p_i^{(6a)} \xi^i$$

We integrate in ξ, expand in r, and we find $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$ and $p_i^{(6a)}$, $i = 0, 1, 2, 3$, in order to fit the known coefficients of the asymptotic expansion in r of $K^{(6a)}(s/m^2)$. Then approximated kernel $\bar{\kappa}^{(6a)}(x)$ is

$$a_\mu^{HVP}(\mathrm{NNLO}; 6a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 \mathrm{d}x \, \bar{\kappa}^{(6a)}(x) \, \Delta\alpha_\mathrm{h}(t(x)),$$

$$\bar{\kappa}^{(6a)}(x) = \begin{cases} \frac{2-x}{x(1-x)} \, P^{(6a)}\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618\ldots \\ \frac{2-x}{x^3} \, L^{(6a)}\left(\frac{1-x}{x^2}\right) 1, & x_\mu < x < 1 \end{cases}$$

The uncertainty due to the series approximation of $K^{(6a)}$ is estimated to be less than $O(10^{-12})$
The contributions of classes $(6b)$ and $(6bll)$ can be calculated similarly to class $(6a)$.

(6a)	
$j_0 = 0;$	$h_0 = -\frac{359}{36};$
$j_1 = -\frac{3793}{864};$	$h_1 = \frac{122293}{5184};$
$j_2 = \frac{35087}{21600};$	$h_2 = -\frac{43879427}{648000};$
$j_3 = \frac{1592093}{43200};$	$h_3 = \frac{14388407}{48000};$
$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$	
$g_1 = \frac{441277}{10368} + \pi^2\left(-\frac{355}{648} + \ln 4\right) + \frac{25\,\zeta(3)}{2};$	
$g_2 = -\frac{5051645167}{38880000} + \pi^2\left(\frac{221411}{32400} - 18\ln 2\right) - \frac{3919\,\zeta(3)}{60};$	
$g_3 = \frac{14588342017}{38880000} + \pi^2\left(-\frac{2479681}{64800} + 112\ln 2\right) + \frac{3113\,\zeta(3)}{10};$	
$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456\,a_4 + \frac{2085448\,\zeta(3)}{15} +$ $+\pi^2\left(-\frac{11944163099}{194400} + \frac{272}{3}(180 - 31\ln 2)\ln 2 + \frac{115072\,\zeta(3)}{3}\right) - \frac{575360\,\zeta(5)}{3};$	
$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360\,a_4 + 2255200\,\zeta(5) +$ $+\pi^2\left(\frac{23549054249}{32400} - 201122\ln 2 + \frac{98420\ln^2 2}{3} - 451040\,\zeta(3)\right) - \frac{57189259\,\zeta(3)}{36};$	
$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790\ln^4 2 + 1938960\,a_4 + \frac{77371609\,\zeta(3)}{20} +$ $+\pi^2\left(-\frac{729995599}{405} + 6(85313 - 13465\ln 2)\ln 2 + 1114360\,\zeta(3)\right) - 5571800\,\zeta(5);$	
$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{18} - 53340\ln^4 2 - 1280160\,a_4 + \frac{11057200\,\zeta(5)}{3} +$ $+\pi^2\left(\frac{5809659289}{4860} + 420\ln 2(-823 + 127\ln 2) - \frac{2211440\,\zeta(3)}{3}\right) - \frac{22833188\,\zeta(3)}{9};$	

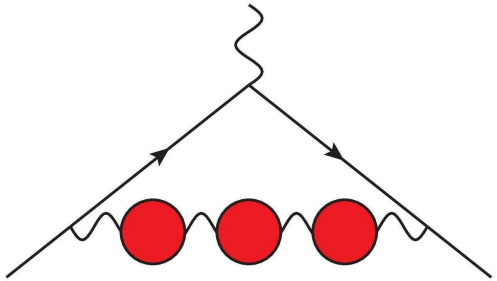
Table 1: The coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$, $p_i^{(6a)}$ $(i = 0,1,2,3)$. The superscript $(6a)$ has been dropped for simplicity. In the above coefficients, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

(6b)	
$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18}\ln\rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10}\ln\rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30}\ln\rho;$
$g_0 = \frac{1}{108}\left(259 - 72\rho^2 + 276\ln\rho\right);$	
$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\pi^2\rho}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2\right)\ln\rho + \frac{4}{3}\ln^2\rho + 2\,\zeta(3);$	
$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\pi^2\rho}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2\right)\ln\rho - 8\ln^2\rho - 12\,\zeta(3);$	
$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\pi^2\rho + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2\right)\ln\rho + \frac{112}{3}\ln^2\rho + 56\,\zeta(3);$	
$p_0 = -\frac{95519053063}{486000} - 7275\pi^2\rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9}\right)\ln\rho + \left(\frac{1135508}{9} + 96\rho^2\right)\zeta(3) +$ $+4720\ln^2\rho + \frac{1067115409\rho^2}{5400} + \pi^2\left(\frac{24382331}{810} - \frac{285184}{9}\ln 2\right) - 32\pi^2\rho^2(687 + \ln 4);$	
$p_1 = \frac{279489728279}{121500} + \frac{179283\pi^2\rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - \frac{1419328\pi^2}{9}\right)\ln\rho - \frac{10}{3}\left(446023 + 216\rho^2\right)\zeta(3) +$ $-\frac{174712}{3}\ln^2\rho - \frac{174350167\rho^2}{75} + \pi^2\left(-\frac{143574463}{405} + \frac{3352256\ln 2}{9}\right) + \frac{16}{3}\pi^2\rho^2(48481 + 90\ln 2);$	
$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2\rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3}\right)\ln\rho + \left(\frac{11034553}{3} + 1440\rho^2\right)\zeta(3) +$ $+148348\ln^2\rho + \frac{258653648\rho^2}{45} + \frac{4}{135}\pi^2(29597029 - 31048560\ln 2) - \frac{320}{3}\pi^2\rho^2(5989 + \ln 512);$	
$p_3 = \frac{72762177677}{19440} + 154035\pi^2\rho - \frac{7}{108}\left(-31650719 + 3973440\pi^2 + 8220240\rho^2\right)\ln\rho - \frac{280}{9}\left(78283 + 27\rho^2\right)\zeta(3) +$ $-100240\ln^2\rho - \frac{513692207\rho^2}{135} + \frac{35}{162}\pi^2(-2687659 + 2816064\ln 2) + \frac{140}{3}\pi^2\rho^2(9055 + \ln 4096);$	

Table 2: The coefficients $g_i^{(6b)}$, $h_i^{(6b)}$, $j_i^{(6b)}$, $p_i^{(6b)}$ $(i = 0,1,2,3)$. The superscript $(6b)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

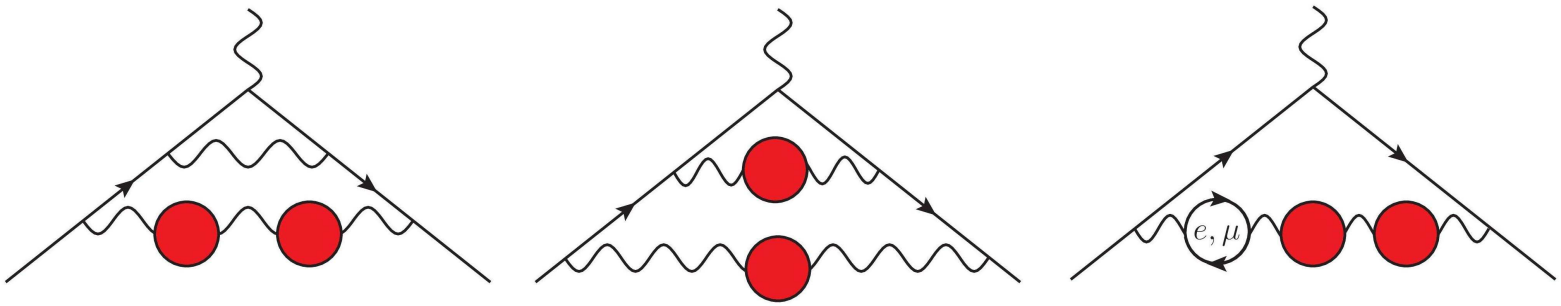
(6bll)	
$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2\ln\rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18}\ln\rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9}\ln\rho;$
$g_0 = \frac{43}{8} - 4\pi^2\rho + 15\rho^2 + \pi^2\rho^2 - 18\rho^2\ln\rho + 6\rho^2\ln^2\rho;$	
$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2\rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2\rho^2}{9} + \frac{607\rho^2}{18}\ln\rho - \frac{20\rho^2}{3}\ln^2\rho + \frac{2}{3}\zeta(3) + 2\rho^2\zeta(3);$	
$g_2 = -\frac{385}{162} - \frac{41\pi^2}{72} - \frac{28\pi^2\rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2\rho^2}{324} - \frac{1961\rho^2}{72}\ln\rho + 14\rho^2\ln^2\rho - \frac{5}{2}\zeta(3) - \frac{16\rho^2}{3}\zeta(3);$	
$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2\rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2\rho^2}{324} + \frac{6943\rho^2}{360}\ln\rho - 36\rho^2\ln^2\rho + \frac{42}{5}\zeta(3) + 15\rho^2\zeta(3);$	
$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2\left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972}\right) - \frac{4\pi^4}{135}\left(7817 + 3212\rho^2\right) +$ $+\left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9}\right)\rho^2\ln\rho + \left(-3388 - \frac{80\pi^2}{3}\right)\rho^2\ln^2\rho +$ $+\left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2\rho^2 - 160\rho^2\ln\rho\right)\zeta(3) - \frac{1280}{3}\rho^2\zeta(5);$	
$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324}\pi^2\left(-533001 + 9110736\rho + 3110417\rho^2\right) + \frac{2}{135}\pi^4\left(180247 + 73530\rho^2\right) +$ $+\left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3}\right)\rho^2\ln\rho + \frac{2}{3}\left(63269 + 300\pi^2\right)\rho^2\ln^2\rho +$ $+\frac{1}{45}\left(-13410977 + 100\left(-292301 + 432\pi^2\right)\rho^2 + 54000\rho^2\ln\rho\right)\zeta(3) + 3200\rho^2\zeta(5);$	
$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2\left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81}\right) - \frac{4}{9}\pi^4\left(14685 + 6032\rho^2\right) +$ $-\frac{1}{54}\left(190613 - 2847360\pi^2 + 11520\pi^4\right)\rho^2\ln\rho - 80\left(1347 + 5\pi^2\right)\rho^2\ln^2\rho +$ $-\frac{10}{9}\left(-658509 + \left(-1431463 + 1728\pi^2\right)\rho^2 + 2160\rho^2\ln\rho\right)\zeta(3) - 6400\rho^2\zeta(5);$	
$p_3 = \frac{49726331179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2\left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243}\right) + \frac{14}{27}\pi^4\left(8269 + 3419\rho^2\right) +$ $+\frac{7}{81}\left(-81551 - 401520\pi^2 + 1440\pi^4\right)\rho^2\ln\rho + \frac{140}{3}\left(1563 + 5\pi^2\right)\rho^2\ln^2\rho +$ $+\frac{35}{27}\left(-371889 + 16\left(-50437 + 54\pi^2\right)\rho^2 + 1080\rho^2\ln\rho\right)\zeta(3) + \frac{11200}{3}\rho^2\zeta(5);$	

Table 3: The coefficients $g_i^{(6bll)}$, $h_i^{(6bll)}$, $j_i^{(6bll)}$, $p_i^{(6bll)}$ $(i = 0,1,2,3)$. The superscript $(6bll)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.



6*d*

$$a_\mu^{HVP}(\mathrm{NNLO}; 6d) \;=\; \frac{\alpha}{\pi} \int\limits_0^1 \mathrm{d}x\, \kappa^{(2)}(x)\; [\Delta\alpha_\mathrm{h}(t(x))]^3 \,.$$



6c1 6c2 6c3, 6c4

$$a_\mu^{HVP}(\text{NNLO};6c) = a_\mu^{HVP}(\text{NNLO};6c1) + a_\mu^{HVP}(\text{NNLO};6c2) + a_\mu^{HVP}(\text{NNLO};6c3) + a_\mu^{HVP}(\text{NNLO};6c4)$$

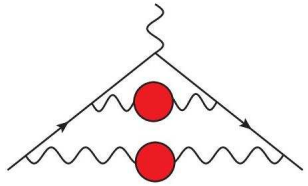
$$a_\mu^{HVP}(\text{NNLO};6c1) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \mathrm{d}x \left[\kappa^{(4)}(x) - \frac{2\pi}{\alpha}\kappa^{(2)}(x)\,\Delta\alpha_\mu^{(2)}(t(x))\right] [\Delta\alpha_{\rm h}(t(x))]^2$$

6c4 separated multiplicity=3

$$a_\mu^{HVP}(\text{NNLO};6c3) = \frac{3\alpha}{\pi}\int_0^1 \mathrm{d}x\,\kappa^{(2)}(x)\,[\Delta\alpha_{\rm h}(t(x))]^2\Delta\alpha_e^{(2)}(t(x)).$$

$$a_\mu^{HVP}(\text{NNLO};6c4) = \frac{3\alpha}{\pi}\int_0^1 \mathrm{d}x\,\kappa^{(2)}(x)\,[\Delta\alpha_{\rm h}(t(x))]^2\Delta\alpha_\mu^{(2)}(t(x)).$$

$a_\mu^{HVP}(6c1) = -5\times10^{-12}$, $a_\mu^{HVP}(6c2) = -1.8\times10^{-12}$, $a_\mu^{HVP}(6c3) = 0.9\times10^{-12}$, $a_\mu^{HVP}(6c4) = 0.1\times10^{-12}$



6c2

This class requires *double* integrals

$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \frac{\alpha^2}{\pi^4}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}\int_{s_0}^{\infty}\frac{\mathrm{d}s'}{s'}\,K^{(6c2)}(s/m^2,s'/m^2)\mathrm{Im}\Pi_\mathrm{h}(s)\,\mathrm{Im}\Pi_\mathrm{h}(s').$$

$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \left(\frac{\alpha}{\pi}\right)^2\int_{x_\mu}^{1}\mathrm{d}x\int_{x_\mu}^{1}\mathrm{d}x'\,\bar\kappa^{(6c2)}(x,x')\Delta\alpha_\mathrm{h}(t(x))\Delta\alpha_\mathrm{h}(t(x')),$$

$\bar\kappa^{(6c2)}(x,x')$ space-like bidimensional kernel, $\qquad x_\mu < \{x,x'\} < 1$

$$\bar\kappa^{(6c2)}(x,x') = \frac{2-x}{x^3}\frac{2-x'}{x'^3}\,G^{(6c2)}\left(\frac{1-x}{x^2},\frac{1-x'}{x'^2}\right)$$

From the <u>leading</u> terms of the known asymptotic expansion of $K^{(6c2)}(s/m^2,s'/m^2)$:
$s/s' \ll 1$ or $s/s' \approx 1$ or $s/s' \gg> 1$ and $s,s' \gg m^2$ we get the approximated space-like kernel

$$G^{(6c2)}(\xi,\xi') = \frac{1855-188\pi^2}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')}{\max(\xi,\xi')^2} + \frac{988\pi^2-9765}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')^2}{\max(\xi,\xi')^3} + \frac{6\left(435-44\pi^2\right)}{32\pi^2-315}\frac{\min(\xi,\xi')^3}{\max(\xi,\xi')^4}$$

Contribution of 6c2 class is -1.8×10^{-12}

The uncertainty of this leading order approximation is estimated to be $\sim 10^{-13}$

Conclusions

- We have provided simple analytic expressions to calculate HVP contrbutions to muon g-2 in the space-like region up to NNLO.
- Expressions are exact for LO and NLO; approximated for NNLO.
- These results can be employed in lattice QCD computations of a_μ^{HVP}
- These results can be employed in determinations of a_μ^{HVP} from scattering data, like those from MUonE experiment.

Thank You!