the hadronic running of the electroweak couplings from lattice QCD

and the connection with $(g-2)_\mu$

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based on work with Mainz lattice group
arXiv:2203.08676, arXiv:2206.06582

8 July 2022
ICHEP 2022, Bologna

the running of the electromagnetic coupling

the QED coupling $\alpha = g^2/(4\pi)$ runs with energy

- in the Thomson limit ($q^2 \to 0$), the fine-structure constant is known at 0.23 ppb
  $\alpha^{-1} = \alpha(0)^{-1} = 137.035\,999\,139(31)$ [PDG 2018, CODATA 2014]
- at the Z pole, $\hat{\alpha}^{(5)}(M_Z)^{-1} = 127.955(10)$ (in the $\overline{\text{MS}}$ scheme)

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)} \qquad \Delta\alpha_{\text{had}}(q^2) = 4\pi\alpha \,\mathrm{Re}\, \bar{\Pi}(q^2), \qquad \bar{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

main uncertainty in the Z-pole value: the hadronic contribution to the running,
proportional to the subtracted hadronic vacuum polarization (HVP) function $\bar{\Pi}(q^2)$

- extracted from the exp. R-ratio data via dispersive integral (data-driven method) [Erler 1999; Davier *et al.* 2017; PDG 2018]

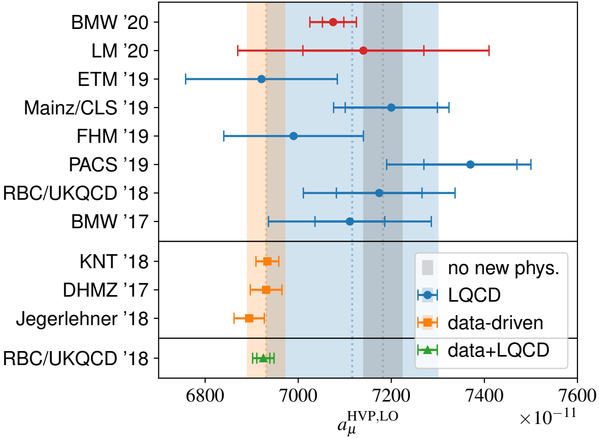
$$\mathrm{Re}\, \bar{\Pi}(q^2) = \frac{q^2}{12\pi} P \int_{m_\pi^2}^{\infty} \frac{R(s)}{s(s - q^2)}\, \mathrm{d}s\,, \qquad \mathrm{Im}\, \bar{\Pi}(q^2) = \frac{R(q^2)}{12\pi}, \qquad R(s) = \frac{\sigma_{e^+e^- \to \text{hadrons}}(s)}{4\pi\alpha^2/(3s)}$$

$$\Delta\alpha_{\text{had}}^{(3)}(4\,\mathrm{GeV}^2) = 58.71(50) \times 10^{-4}, \qquad \Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027\,64(7), \qquad \text{(on-shell scheme)}$$

- or computed on the lattice [Burger *et al.* 2015; Francis *et al.* Lattice 2015; Borsanyi *et al.* 2018; MC *et al.* Lattice 2019; MC *et al.* 2022-03]

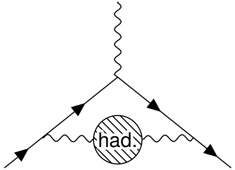
$$\bar{\Pi}(-Q^2) = -\frac{1}{3} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}Q\cdot x} \left\langle j_\mu^\gamma(x) j_\mu^\gamma(0) \right\rangle$$

motivation – the leading-order HVP contribution to a_μ



$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^{\infty} \frac{\hat{K}(s)}{s^2} R(s)\,\mathrm{d}s$$

$$\hat{K}(4m_\pi^2) \approx 0.63, \quad \lim_{s\to\infty} \hat{K}(s) = 1$$



- no new physics: the value of $a_\mu^{\mathrm{HVP,LO}}$ that matches the experimental result without BSM contributions
  $\Rightarrow$ 4.2σ tension with the data-driven estimate (white paper average)
- excluding BMW '20, the lattice results are compatible with both "data-driven" and "no new physics"
  but new subpercent-precision lattice results (BMW '20) points to a no new physics solution
- the experimental error will shrink: target $\delta a_\mu \approx 15 \times 10^{-11}$, 2 ‰ of $a_\mu^{\mathrm{HVP,LO}}$

motivation – hypotetical scenario

what if the lattice QCD confirms the larger value of $a_\mu^{\mathrm{HVP,LO}}$?

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^{\infty} \mathrm{d}s \, \frac{\hat{K}(s)}{s^2} R(s), \qquad \bar{\Pi}(-Q^2) = \frac{Q^2}{12\pi^2} \int_{m_\pi^2}^{\infty} \mathrm{d}s \, \frac{R(s)}{s(s+Q^2)}$$

with $\hat{K}(s) > 0 \Rightarrow$ $R(s)$ is larger for some s, $\bar{\Pi}(-Q^2)$ is also larger!

$\Delta\alpha_{\mathrm{had}}$ is affected differently depending on where the hadronic cross-section increases

[Passera, Marciano, Sirlin 2008; Crivellin, Hoferichter, Manzari, Montull 2020; Keshavarzi, Passera, Marciano, Sirlin 2020]

- at higher $\sqrt{s}$: the increase in $\Delta\alpha^{(5)}(M_Z)$ is in tension with global SM fits
  $\Rightarrow M_W$, $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lep}}$, M_H exclude shifts for $\sqrt{s} > 0.7\,\mathrm{GeV}$ at $95\,\%$ C.L.
- below $0.7\,\mathrm{GeV}$ (ρ-resonance region): no significant change in $\Delta\alpha_{\mathrm{had}}$, no tension in global SM fits,
  a $9\,\%$ increase of the integrated cross-section would solve the $(g-2)_\mu$ discrepancy
  $\Rightarrow$ the experimental hadronic cross-section data is wrong?
  tension between BaBar and KLOE in the $\sigma_{e^+e^-\to\pi^+\pi^-}$ for $s \in [0.6, 0.9]\,\mathrm{GeV}^2$, but it is not enough!

can lattice say something more about this?

yes, by studying the running with energy of the electromagnetic coupling!

$\bar{\Pi}(-Q^2)$ for space-like $Q^2 > 0$ is also measured in t-channel scattering experiments, e.g. MUonE

the time-momentum representation (TMR) method

uses the zero-momentum-projected Euclidean-time correlator

$$G(t) = -\frac{1}{3}\int \mathrm{d}^3x \sum_{k=1}^{3} \langle j_k(x) j_k(0) \rangle,$$

and known kernel functions $K(t, Q^2)$ and $w(t)$ [Bernecker, Meyer 2011; Francis *et al.* 2013]

$$\bar{\Pi}(-Q^2) = \int_0^\infty \mathrm{d}t\, K(t,Q^2) G(t), \qquad K(t,Q^2) = t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right),$$

$$a_\mu^{\mathrm{HVP,LO}} = \int_0^\infty \mathrm{d}t\, w(t) G(t), \qquad w(t) = 4\pi^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2) K(t,Q^2)$$

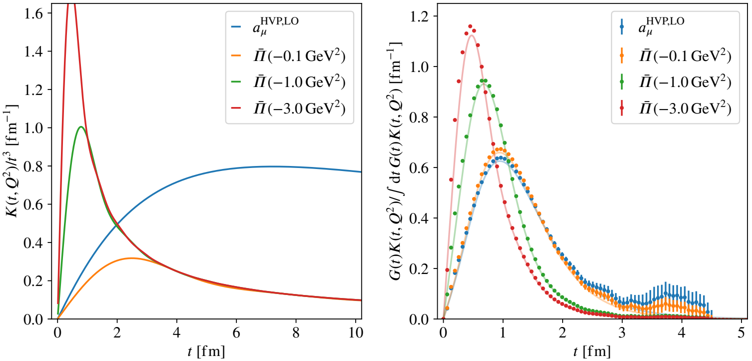
w.r.t. the traditional approach

- same statistical power
- better understanding of the systematics
  - finite-size effects correction • (improved) bounding method
- in principle, $\bar{\Pi}(-Q^2)$ can be computed for any Q^2

similar alternative approach: time moments [Chakraborty 2014]

the TMR method – the kernel

- the $a_\mu^{\mathrm{HVP,LO}}$ kernel is very long range
- the $\bar{\Pi}(-Q^2)$ kernel has a shorter range depending on Q^2



$\bar{\Pi}(-Q^2)$: $K(t,Q^2) = t^2 - \dfrac{4}{Q^2}\sin^2\left(\dfrac{Qt}{2}\right)$, $\qquad a_\mu^{\mathrm{HVP,LO}}$: $w(t) = 4\pi^2 \displaystyle\int_0^\infty \mathrm{d}Q^2\, f(Q^2) K(t,Q^2)$

statistical uncertainty and systematic effects

controlling the tail of the correlator ⇒ main source of statistical uncertainty

- the bounding method is used [Lehner LGT2016; Gérardin, MC *et al.* 2019]
- crucial to control the statistical error on $a_\mu^{\mathrm{HVP,LO}}$, less critical for $\bar{\Pi}(-Q^2)$
- alternative: address the signal-to-noise ratio problem with multi-level sampling [Dalla Brida, Giusti, Harris, Pepe 2020]

simulations of finite-size lattices ⇒ correction of finite-size effects

- computing FSE on the zero-momentum correlator $G(t)$ [Lüscher 1991; Lellouch, Lüscher 2000; Meyer 2011; Hansen, Patella 2019; 2020]
- with $m_\pi L \approx 4$ ($L \approx 6\,\mathrm{fm}$ at the physical point), about $1\,\%$ shift fully under control

extrapolation to the continuum limit, extra- or interpolation to physical meson masses

- thanks to ensembles around physical meson masses
- continuum-limit extrapolation with $\sim a^2$ and $\sim a^3$ effects at $Q^2 \gtrsim 2.5\,\mathrm{GeV}^2$
- use fine ensembles at finite-T? [MC, Harris, Meyer, Toniato, Török 2022]

QED and strong isospin breaking corrections

- $\delta a_\mu^{\mathrm{HVP}} = 72(34) \times 10^{-11}$, about $1\,\%$ of the total [Aoyama *et al.* 2020]
- only available on a few ensembles, work in progress ⇒ included as systematics [Risch, Wittig Lattice 2019; Lattice 2021]

scale setting systematics

- a $1\,\%$ uncertainty on the scale is a $\approx 2\,\%$ systematic error on $a_\mu^{\mathrm{HVP,LO}}$
  ⇒ a per-mille level scale determination is needed

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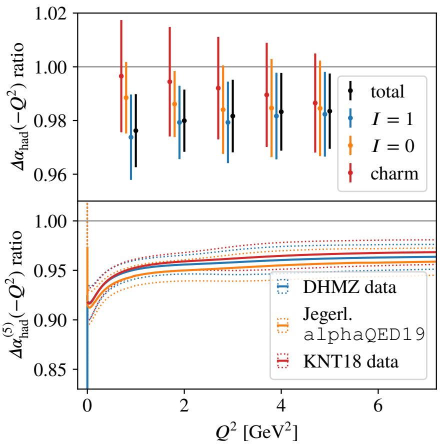
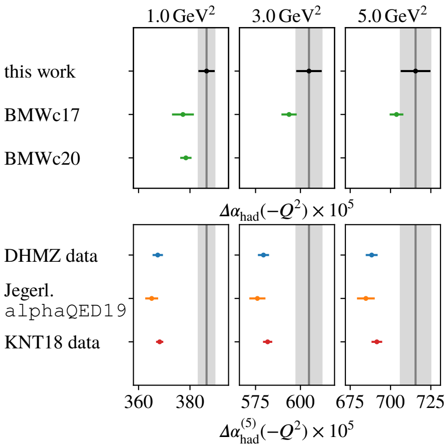
running with energy – results and comparison





conclusions

we computed on the lattice the HVP contribution to the running of α

$$\Delta\alpha_{\text{had}}^{(5)}(-Q^2) = 0.007\,16(9) \qquad \text{at } Q^2 = 5\,\text{GeV}^2$$

- comparable precision with the data-driven estimate • but up to 3.5σ tension with it
- precision limited by current scale setting on CLS ensembles
- full correction for isospin breaking effects still missing [work in progress: Risch, Wittig Lattice 2019; Lattice 2021]
- space-like $Q^2 \Leftrightarrow t$-channel scattering, input for MUonE?

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027\,73(15)$$

- employing the Euclidean split technique and pQCD for the running at large $Q^2 \Rightarrow$ no R-ratio data dependency
- the tension is strongly diminished • the result agrees with ones based on the R-ratio within the uncertainties

and the $Z\gamma$-mixing HVP contribution to the running of $\sin^2\theta_W$

$$\bar{\Pi}^{08}(-Q^2) = 0.007\,04(17) \qquad \text{for } Q^2 \gtrsim 7\,\text{GeV}^2$$

- using flavor separation on the lattice • most precise determination to date

running to Z pole

we use the Euclidean split technique (or Adler function approach) [Eidelman *et al.* 1999, Jegerlehner 2008]

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) = \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2) + \left[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)\right] + \left[\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2)\right]_{\mathrm{pQCD}}$$

1. $\Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)$ with Q_0^2 between 3 and $7\,\mathrm{GeV}^2$ is evaluated on the lattice

$$\Delta\alpha^{(5)}_{\mathrm{had}}(-5\,\mathrm{GeV}^2) = 0.007\,16(9)$$

2. $[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)]$ from either pQCD or R-ratio data (KNT18)

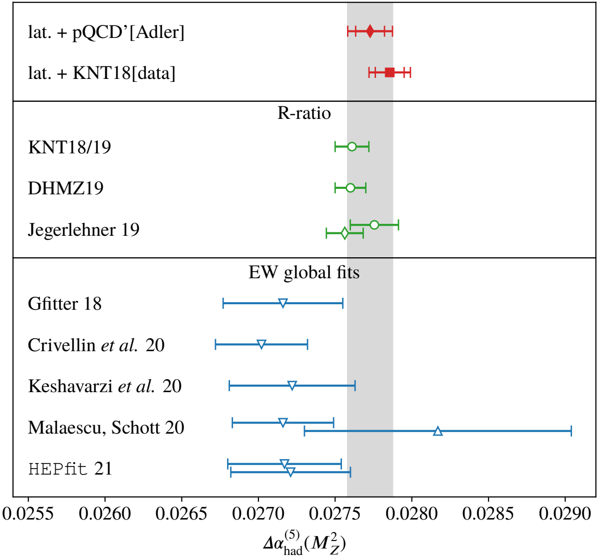
$$\left[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-5\,\mathrm{GeV}^2)\right] = 0.020\,53(11) \quad \text{or} \quad 0.020\,66(9)$$

3. $[\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2)]_{\mathrm{pQCD}} = 0.000\,045(2)$ has a negligible error [Jegerlehner 1986, 2020]

using pQCD ⇒ result independent from R-ratio input

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) = 0.027\,73(15)$$

running to Z pole – results and comparison



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$a_\mu^{\mathrm{HVP,LO}}$ TMR windows

the windows are obtained by multiplying the TMR integrand by three different window functions

$$a_\mu^{\mathrm{HVP,X}} = \int_0^\infty \mathrm{d}t\, w(t) G(t) f^{\mathrm{X}}(t, \dots),$$

$$f^{\mathrm{SD}}(t, t_0) = 1 - \Theta(t, t_0, \Delta),$$

$$f^{\mathrm{W}}(t, t_0, t_1) = \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta),$$

$$f^{\mathrm{LD}}(t, t_1) = \Theta(t, t_1, \Delta),$$

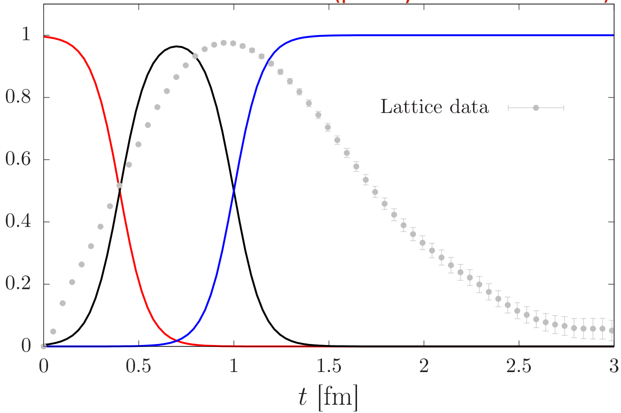
with the smooth step function

$$\Theta(t, t', \Delta) = \frac{1}{2}\left[1 + \tanh\left(\frac{t - t'}{\Delta}\right)\right]$$

and the standard choice [Lehner Lattice 2017]

$$t_0 = 0.4\,\mathrm{fm}, \quad t_1 = 1.0\,\mathrm{fm}, \quad \Delta = 0.15\,\mathrm{fm}$$

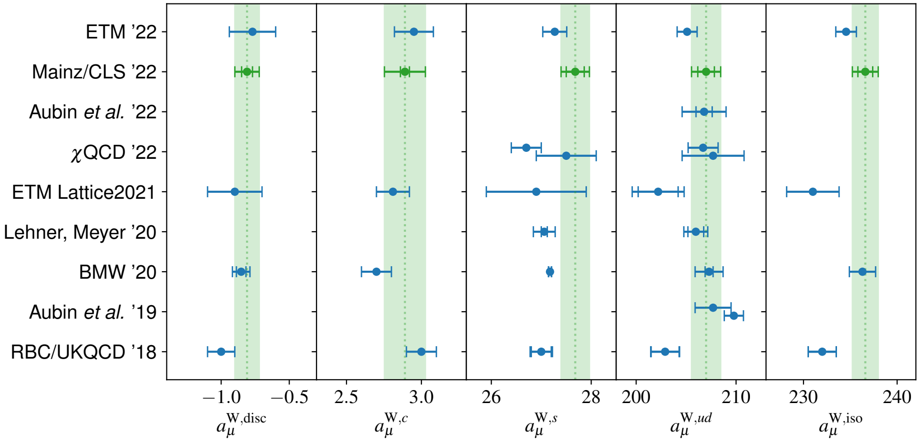
(plot by Antoine Gérardin)



“intermediate” window ⇒ higher precision and lower systematics on the lattice

$a_\mu^{\mathrm{HVP,W}}$ intermediate window – results

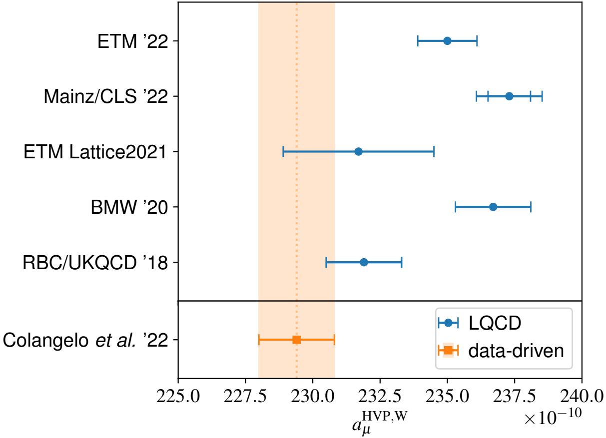
different components of $a_\mu^{\mathrm{HVP,W}}$ in isosymmetric QCD in 10^{-10} units [MC *et al.* arXiv:2206.06582; Kuberski at SchwingerFest 2022]



$a_\mu^{\mathrm{HVP,W}}$ intermediate window – results

compared to the data-driven result

[MC *et al.* arXiv:2206.06582; Kuberski at SchwingerFest 2022]



conclusions (II)

we computed on the lattice the intermediate window $a_\mu^{\mathrm{HVP,W}}$ [MC *et al.* arXiv:2206.06582; Kuberski at SchwingerFest 2022]

$$a_\mu^{\mathrm{HVP,W}} = 237.30(0.79)_{\mathrm{stat}}(1.22)_{\mathrm{syst}} \times 10^{-10}$$

which is 3.8σ above the recent data-driven evaluation of $229.4(1.4) \times 10^{-10}$ [Colangelo *et al.* 2022]

w.r.t. the results for the HVP contribution to running of the couplings

- additional statistics, two additional finer ($a \approx 0.039\,\mathrm{fm}$) and coarser ($\approx 0.993\,\mathrm{fm}$) lattice spacings
- alternative renormalization/improvement: Z_V and c_V [Heitger, Joswig 2021], b_V and $\bar{b}_V$ with SF [Fritzsch 2018]
- alternative scale setting with f_π-rescaling [Feng *et al.* 2011] $\Rightarrow$ no intermediate scale needed
- large variation of fits averaged with Akaike information criterion (AIC), $w_n = N \exp\{-(\chi^2 + 2k - 2n)/2\}$

[Akaike 1974; Borsanyi *et al.* 2021; Jay, Neil 2021; Djukanovic *et al.* 2021]

outlook

- new lattice results points to a possible solution of the muon $g-2$ puzzle
- without introducing tension in the global SM fits
- the discrepancy with the data-driven method, and in turns with the R-ratio data, needs an explanation
- new results for both intermediate and short-distance windows by ETMc [talk by Frezzotti]
- the long ($t > 1.0\,\mathrm{fm}$) window requires more time

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thanks

for your attention!

questions?

backup slides

motivation – t-channel scattering

the leading hadronic contribution to $(g-2)_\mu$ from the running of α [Lautrup, Peterman, de Rafael 1972]

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\,(1-x)\Delta\alpha_{\mathrm{had}}(Q^2), \qquad Q^2 = \frac{x^2 m_\mu^2}{1-x},$$

with the integrand peaked at $x \approx 0.914$, $Q^2 \approx 0.108\,\mathrm{GeV}^2$. [Carloni Calame *et al.*2015]

upcoming MUonE experiment @ CERN: measure the energy dependence of α at space-like Q^2 [Abbiendi *et al.*2017]

- independent determination of $a_\mu^{\mathrm{HVP,LO}}$
- kinematic range $0 < x < 0.932$, corresponding to $Q^2 \lesssim 0.14\,\mathrm{GeV}^2$
- $0.932 < x < 1$ or $Q^2 \gtrsim 0.14\,\mathrm{GeV}^2$ accounts for $13\,\%$ of $a_\mu^{\mathrm{HVP,LO}}$
- test run in 2023!

$\Rightarrow$ lattice input for the intermediate region $Q^2 = 0.14 - 4\,\mathrm{GeV}^2$

the running of the electroweak mixing angle

the electroweak mixing (Weinberg) angle θ_{W} parametrizes the mixing between the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ sectors of the Standard Model. At tree level,

$$\sin^2\hat{\theta}_{\mathrm{W}} = \frac{g'^2}{g^2+g'^2}, \quad \text{or} \quad \sin^2\theta_{\mathrm{W}} = 1 - \frac{M_W^2}{M_Z^2},$$

where g and g' are the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ coupling respectively

- Z vector coupling $v_f = T_f - 2Q_f \sin^2\theta_f^{\mathrm{eff}}$
- weak charge of the proton $Q_W(p) \sim 1 - 4\sin^2\theta_{\mathrm{W}}(0)$

like the couplings, the mixing angle is renormalization scheme and energy dependent

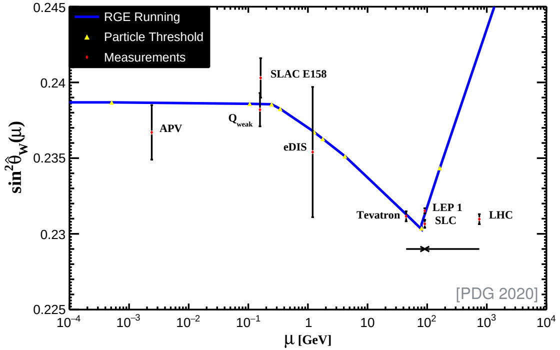
$$\sin^2\theta_{\mathrm{W}}(Q) = \sin^2\theta_{\mathrm{W}}(0)\left[1 + \Delta\sin^2\theta_{\mathrm{W}}(Q^2)\right],$$

and the leading hadronic contribution to the running [Jegerlehner 1986; 2011]

$$\Delta_{\mathrm{had}}\sin^2\theta_{\mathrm{W}}(Q^2) = -\frac{4\pi\alpha}{\sin^2\theta_{\mathrm{W}}}\bar{\Pi}^{Z\gamma}(Q^2), \qquad \bar{\Pi}^{Z\gamma}(Q^2) = \Pi^{Z\gamma}(Q^2) - \Pi^{Z\gamma}(0),$$

is proportional to the subtracted $Z\gamma$-mixing HVP function

the running of the electroweak mixing angle



experimental values measured at colliders enter global SM fits [PDG 2022]

$$\sin^2\hat{\theta}_W(M_Z) = 0.231\,22(4)$$

upcoming experiments at low Q^2

- P2 @ MESA, Mainz [Becker *et al.* 2018]
  $0.13\,\%$ target precision at $Q^2 = 0.005\,\mathrm{GeV}^2$
- MOLLER @ JLab [Benesch *et al.* 2014]

the running to the Thomson limit is affected by non-perturbative QCD physics that

- can be extracted from hadronic cross-section data [Erler, Ferro-Hernández 2017]

$$\sin^2\hat{\theta}_W(0) = 0.238\,68(5)(2), \qquad (\overline{\text{MS}}\text{ scheme})$$

  with additional input for flavor separation
- or can be computed on the lattice [Burger *et al.* 2015; Francis *et al.* 2015; Gülpers *et al.* 2015]
  ⇒ lattice easily provides flavour separation

ensembles

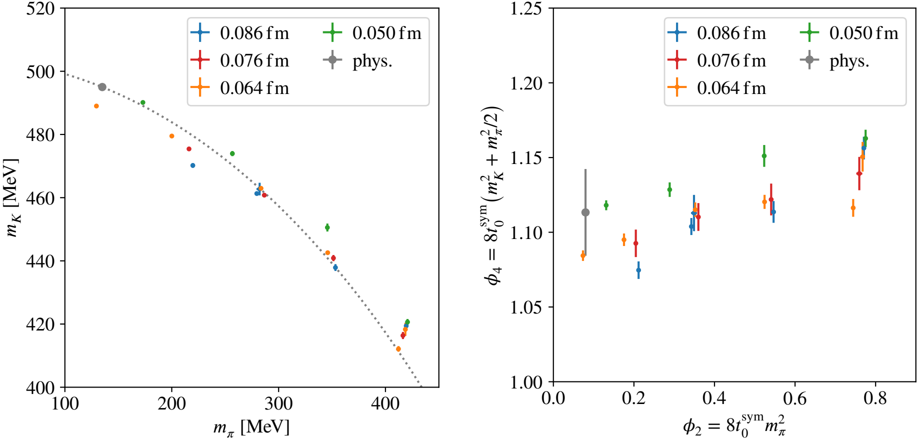
from the CLS initiative

[Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]

tree-level Lüscher-Weisz gauge action, non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions,
open BCs in time, except B450, N451, D450, and E250 that have periodic BCs in time,

	T/a	L/a	t_0^{sym}/a^2	a [fm]	L [fm]	m_π, m_K [MeV]		$m_\pi L$	#cnfg (con., dis.)	
H101	96	32	2.860	0.086	2.8	415		5.8	2 000	-
H102	96	32			2.8	355	440	5.0	1 900	1 900
H105	96	32			2.8	280	460	3.9	1 000	1 000
N101	128	48			4.1	280	460	5.8	1 155	1 155
C101	96	48			4.1	220	470	4.6	2 000	2 000
B450	64	32	3.659	0.076	2.4	415		5.1	1 600	-
S400	128	32			2.4	350	440	4.3	1 720	1 720
N451	128	48			3.7	285	460	5.3	1 000	1 000
D450	128	64			4.9	215	475	5.3	500	500
H200	96	32	5.164	0.064	2.1	420		4.4	1 980	-
N202	128	48			3.1	410		6.4	875	-
N203	128	48			3.1	345	440	5.4	1 500	1 500
N200	128	48			3.1	285	465	4.4	1 695	1 695
D200	128	64			4.1	200	480	4.2	2 000	1 000
E250	192	96			6.2	130	490	4.1	485	485
N300	128	48	8.595	0.050	2.4	420		5.1	1 680	-
N302	128	48			2.4	345	460	4.2	2 190	2 190
J303	192	64			3.2	260	475	4.2	1 040	1 040
E300	192	96			4.8	175	490	4.3	600	600

ensemble landscape



lattice correlators

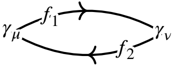
on $N_\mathrm{f} = 2+1$ ensembles from the CLS initiative [Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]
with $\mathrm{SU}(3)_F$ notation, in the isospin-symmetric limit (light quark ℓ: either u or d):

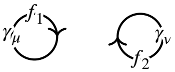
$I=1$ contribution: $$G^{33}_{\mu\nu}(x) = \frac{1}{2} C^{\ell,\ell}_{\mu\nu}(x),$$

$I=0$ contribution: $$G^{88}_{\mu\nu}(x) = \frac{1}{6}\Big[C^{\ell,\ell}_{\mu\nu}(x) + 2C^{s,s}_{\mu\nu}(x) + 2D^{\ell-s,\ell-s}_{\mu\nu}(x)\Big],$$

Z-γ mixing: $$G^{08}_{\mu\nu}(x) = \frac{1}{2\sqrt{3}}\Big[C^{\ell,\ell}_{\mu\nu}(x) - C^{s,s}_{\mu\nu}(x) + D^{2\ell+s,\ell-s}_{\mu\nu}(x)\Big],$$

where the connected and disconnected Wick's contractions are

$C^{f_1,f_2}_{\mu\nu} = -\Big\langle$ 

$\Big\rangle$, $D^{f_1,f_2}_{\mu\nu} = \Big\langle$ 

$\Big\rangle$

and the relevant correlators are given by (note: $G^{\ell}_\mathrm{con} = 2G^{33}$ and $G^{s}_\mathrm{con} = 3G^{88}_\mathrm{con} - G^{33}$)

$$G^{\gamma\gamma} = G^{33} + \frac{1}{3}G^{88} + \frac{4}{9}C^{c,c},$$

$$G^{Z\gamma} = \left(\frac{1}{2} - \sin^2\theta_\mathrm{W}\right)G^{\gamma\gamma} - \frac{1}{6\sqrt{3}}G^{08} + \frac{4}{9}\left(\frac{3}{8} - \sin^2\theta_\mathrm{W}\right)C^{c,c}.$$

renormalization and $\mathcal{O}(a)$ improvement

for the local current [Bhattacharya *et al.* 2006, […], Gérardin, Harris, Meyer 2018]

$$V^3_{\mu,R} = Z_V\big(1 + 3\bar{b}_V a m_q^{\mathrm{av}} + b_V a m_{q,\ell}\big) V_\mu^{3,I} = Z_3 V_\mu^{3,I},$$

$$\begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}_R = Z_V \begin{pmatrix} 1 + 3\bar{b}_V a m_q^{\mathrm{av}} + b_V \frac{a(m_{q,\ell}+2m_{q,s})}{3} & \left(\frac{b_V}{3} + f_V\right)\frac{2a(m_{q,\ell}-m_{q,s})}{\sqrt{3}} \\ r_V d_V \frac{a(m_{q,\ell}-m_{q,s})}{\sqrt{3}} & r_V 1 + (3\bar{d}_V + d_V) a m_q^{\mathrm{av}} \end{pmatrix} \begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}^I = \begin{pmatrix} Z_8 & Z_{80} \\ Z_{08} & Z_0 \end{pmatrix} \begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}^I$$

where

$$V_\mu^{a,I} = V_\mu^a + a c_V \partial_0 T_{0\mu}^a, \qquad V_\mu^{0,I} = V_\mu^0 + a\bar{c}_V \partial_0 T_{0\mu}^0.$$

while for the conserved current

$$V^a_{\mu,R} = V_\mu^a + a c_V^{\mathrm{cs}} \partial_0 T_{0\mu}^a, \qquad V^0_{\mu,R} = V_\mu^0 + a\bar{c}_V^{\mathrm{cs}} \partial_0 T_{0\mu}^0.$$

$\Rightarrow$ we use only the conserved vector current for the flavor-singlet component, and we set

$$f_V = 0, \qquad \bar{c}_V = c_V \qquad \bar{c}_V^{\mathrm{cs}} = c_V^{\mathrm{cs}}.$$

bounding method

$$G(t) = \sum_{n=0}^{\infty} \frac{Z_n^2}{2E_n} \mathrm{e}^{-E_n t}$$

for a correlator with positive spectral decomposition, and $t > t_\mathrm{c}$

$$0 \leq G(t_\mathrm{c})\mathrm{e}^{-E_\mathrm{eff}(t_\mathrm{c})(t-t_\mathrm{c})} \leq G(t) \leq G(t_\mathrm{c})\mathrm{e}^{-E_0(t-t_\mathrm{c})},$$

where $E_\mathrm{eff}(t) = -(1/a)\log G(t+a)/G(t)$ is the effective mass
and E_0 is the ground state in the given channel, depending on the volume L^3 and on m_π

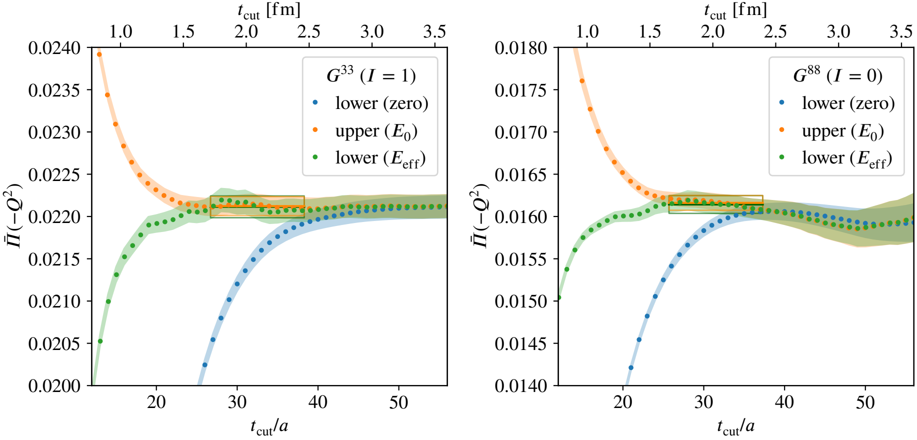
- for $I = 1$, $E_0 = m_\rho$ or $E_{2\pi}$,
- for $I = 0$, $E_0 = m_\omega \approx m_\rho$ or $E_{3\pi}$

improved bounding method: [Lehner LGT2016; Gérardin, MC *et al.* 2019]
if $E_0, \dots E_N$ and $Z_0, \dots Z_{N-1}$ are available, one can bound the subtracted correlator

$$\tilde{G}(t) = G(t) - \sum_{n=0}^{N-1} \frac{Z_n^2}{2E_n} \mathrm{e}^{-E_n t},$$

that approaches zero faster ⇒ dedicated spectroscopy effort

bounding method – example



$$0 \leq G(t_{\text{cut}})\mathrm{e}^{-E_{\text{eff}}(t_{\text{cut}})(t-t_{\text{cut}})} \leq G(t) \leq G(t_{\text{cut}})\mathrm{e}^{-E_0(t-t_{\text{cut}})}, \qquad \text{for } t \geq t_{\text{cut}},$$

with $aE_{\text{eff}}(t) = \log(G(t)/G(t+a))$ and E_0 ground state in the channel

correction of finite-size effects – 2019 paper

added to the $I = 1$ correlator $G^{33}(t)$, with two different strategies and $t_i = (m_\pi L/4)^2/m_\pi$ [Gérardin, MC *et al.* 2019]

$t < t_i$: correction from NLO χPT (scalar QED) [Francis *et al.* 2013; Della Morte *et al.* 2017]

$$G^{33}(t,L) - G^{33}(t,\infty) = \frac{1}{3}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\right)\frac{\vec{k}^2 + m_\pi^2}{\vec{k}^2}\mathrm{e}^{-2t\sqrt{\vec{k}^2+m_\pi^2}}$$

⇒ at longer distances, this accounts only for a fraction of the FSE correction.

$t > t_i$: Meyer-Lellouch-Lüscher (MLL) method using the time-like $F_\pi(\omega)$:
correction from *e.g.* GS parametrization of F_π as function of m_ρ, $g_{\rho\pi\pi}$ [Gounaris, Sakurai 1968]

$$G^{33}(t,\infty) = \int_0^\infty \mathrm{d}\omega\, \omega^2 \rho(\omega^2)\mathrm{e}^{-\omega t} \qquad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{\frac{3}{2}}\left|F_\pi(\omega)\right|^2$$

and the corresponding finite-volume correlator [Lüscher 1991; Lellouch, Lüscher 2000; Meyer 2011]

$$G^{33}(t,L) = \sum_n \left|A_n\right|^2 \mathrm{e}^{-\omega_n t} \qquad \text{with Lüscher's } \omega_n \text{ and LL's } A_n$$

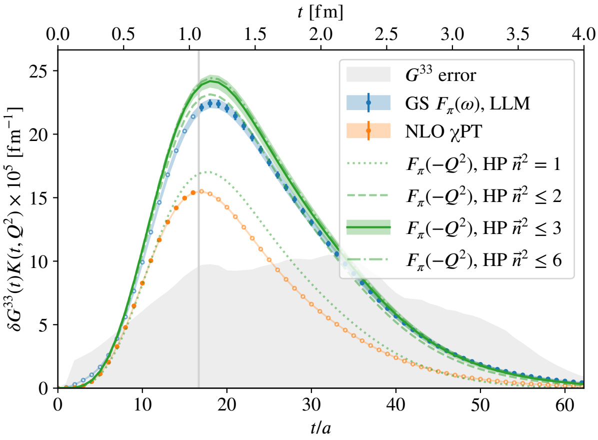
⇒ a model is used only for the small correction $G^{33}(t,L) - G^{33}(t,\infty)$

correction of finite-size effects – Hansen-Patella (HP) method

series expansion in $\exp\{-|\vec{n}|m_\pi L\}$, $\vec{n}^2 = 1, 2, 3, 6, \ldots$ [Hansen, Patella 2019; 2020]

- implementation by K. Miura
- neglects $\exp\{-1.93 m_\pi L\}$ contributions
- fast convergence at short and medium distances
- input: forward Compton amplitude, $F_\pi(Q^2)$ monopole ansatz with $M^2 = 0.517(23)\,\mathrm{GeV}^2 + 0.647(30) m_\pi^2$

[Brommel *et al.* 2007]



correction of finite-size effects

added to the $I = 1$ correlator $G^{33}(t)$, with two different strategies and $t_i = (m_\pi L/4)^2/m_\pi$ [Gérardin, MC *et al.* 2019]

$t < t_i$: HP method with $\vec{n}^2 \leq 3$, with the size of the $\vec{n}^2 = 3$ level included as a systematic error

$t > t_i$: average of MLL-GS and HP methods, with the half-difference included as an extra systematic error

$\Rightarrow$ a model is used only for the small correction $G^{33}(t, L) - G^{33}(t, \infty)$

explicit check of FSE with two pair of ensembles at different volume and otherwise identical simulation parameters

- H105 and N101 at $m_\pi \approx 280\,\mathrm{MeV}$
- H200 and N202 at the $\mathrm{SU}(3)$-symmetric point

we observe a good agreement between the MLL-GS and HP methods, especially for $t \gtrsim 2\,\mathrm{fm}$,
with the two methods relying on very different input $\Rightarrow$ robustness of the evaluation of finite size effects

extrapolation to the physical point

a combined fit of $\bar{\Pi}^{33}$, $\bar{\Pi}^{88}$ and $\bar{\Pi}^{08}$, with two discretization each (one discr. for $\bar{\Pi}^{08}$) is used

$$\bar{\Pi}^{33,X}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \bar{\Pi}^{\rm sym} + \delta_2^X a^2/t_0^{\rm sym} + \gamma_1^{33}(\phi_2 - \phi_2^{\rm sym}) + \gamma_{\log}^{33} \log \phi_2/\phi_2^{\rm sym} + \eta_1(\phi_4 - \phi_4^{\rm sym}),$$
$$\bar{\Pi}^{88,X}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \bar{\Pi}^{\rm sym} + \delta_2^X a^2/t_0^{\rm sym} + \gamma_1^{88}(\phi_2 - \phi_2^{\rm sym}) + \gamma_2^{88}(\phi_2 - \phi_2^{\rm sym})^2 + \eta_1(\phi_4 - \phi_4^{\rm sym}),$$
$$\bar{\Pi}^{08,\rm CL}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \lambda_1(\phi_4 - 3/2\phi_2),$$

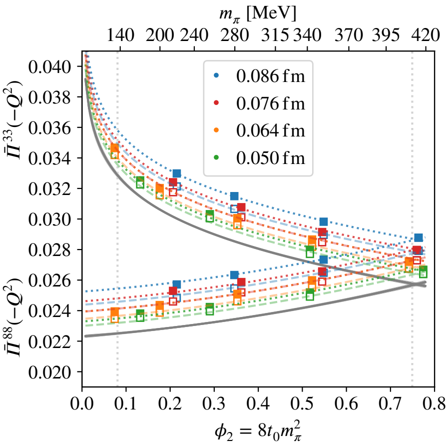
where $X = \rm CL$ or $\rm LL$, $\phi_2 = 8t_0 m_\pi^2$, $\phi_4 = 8t_0(m_K^2 + m_\pi^2/2)$.

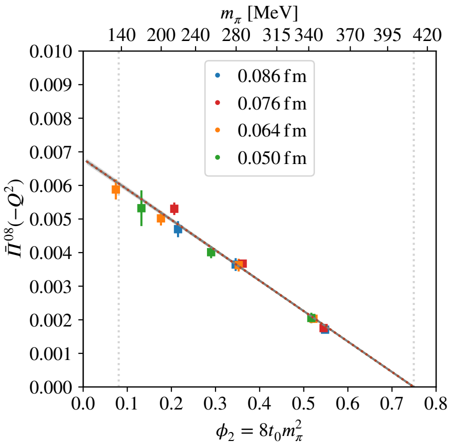
- we add also a $\delta_3^X a^3/(t_0^{\rm sym})^{3/2}$ $\Rightarrow$ better fit at large Q^2 $\Rightarrow$ smooth transition around $Q^2 = 2.5\,\mathrm{GeV}^2$
- $\sim a^2 \log a$ term? assuming free theory coeff., up to $0.4\,\%$ downward shift, within stat. error [MC, Harris, Meyer 2021]
- extrapolation of the charm contribution done separately

future improvement:
alternative renormalization/improvement, alternative scale setting, AIC (see HVP window paper) [MC *et al.* 2022-06]

extrapolation results

at $Q^2 = 1.0\,\mathrm{GeV}^2$





scale setting systematics

on the lattice, dimensionful quantities are expressed in terms of a reference energy scale Λ

the uncertainty $\Delta\Lambda$ on the scale Λ is a source of systematic error [Della Morte *et al.* 2017]

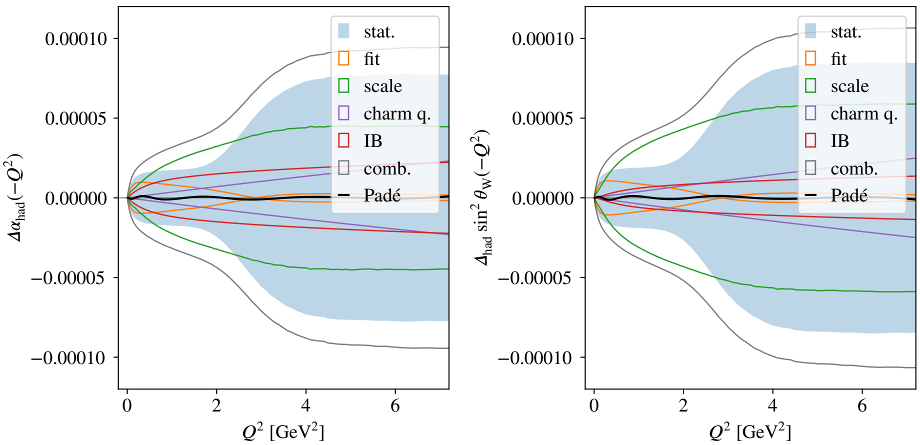
$$\frac{\Delta a_\mu^{\mathrm{HVP,LO}}}{a_\mu^{\mathrm{HVP,LO}}} = \left| \frac{\Lambda}{a_\mu^{\mathrm{HVP,LO}}} \frac{\mathrm{d}a_\mu^{\mathrm{HVP,LO}}}{\mathrm{d}\Lambda} \right| \frac{\Delta\Lambda}{\Lambda} = \left| \underbrace{\frac{m_\mu/\Lambda}{a_\mu^{\mathrm{HVP,LO}}} \frac{\partial a_\mu^{\mathrm{HVP,LO}}}{\partial m_\mu/\Lambda}}_{\approx 1.8} + \ldots \right| \frac{\Delta\Lambda}{\Lambda}$$

$$\frac{\Delta\bar{\Pi}}{\bar{\Pi}} = \left| \frac{\Lambda}{\bar{\Pi}} \frac{\mathrm{d}\bar{\Pi}}{\mathrm{d}\Lambda} \right| \frac{\Delta\Lambda}{\Lambda} = \left| \underbrace{\frac{Q/\Lambda}{\bar{\Pi}} \frac{\partial\bar{\Pi}}{\partial Q/\Lambda}}_{\approx 0.9 \text{ at } Q^2 = 1\,\mathrm{GeV}^2} + \ldots \right| \frac{\Delta\Lambda}{\Lambda}$$

⇒ sub-percent - permille level scale setting is needed

e.g. in [Mainz/CLS '19] $\Lambda^{-1} = \sqrt{8t_0} = 0.415(4)(2)\,\mathrm{fm}$ [Bruno, Korzec, Schaefer 2015]

running with energy – summary of systematics



the anomalous magnetic moment of the muon, today

current status from the Muon $g-2$ Theory initiative white paper [Aoyama *et al.* Physics Reports 887 (2020)]
and the recent Fermilab Muon $g-2$ experiment result [Abi *et al.*, Phys. Rev. Letter **126**, 141801 (2021)]

	$a_\mu \times 1 \times 10^{11}$		
QED	$116\,584\,718.931(104)$	up to 10th order	
EW	$153.6(1.0)$	two loops	
HVP, LO	$6\,931(40)$	e^+e^-	
	$7\,116(184)$	lattice, $udsc$	
HVP, NLO	$-98.3(7)$	e^+e^-	
HVP, NNLO	$12.4(1)$	e^+e^-	
HLbL	$90(17)$	pheno. + lattice	
total SM	$116\,591\,810(43)$	Muon $g-2$ Theory Initiative	[Aoyama *et al.* 2020]
experiment	$116\,592\,089(63)$	BNL E821	[Bennett *et al.* 2006]
	$116\,592\,040(54)$	FNAL E989 (Muon $g-2$ exp.)	[Abi *et al.* 2021]
	$116\,592\,061(41)$	experimental average	

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = 251(59) \times 10^{-11}\text{, a } 4.2\sigma \text{ tension}$$

- initial FNAL result based on $6\,\%$ of the target final dataset $\Rightarrow \delta a_\mu \approx 15 \times 10^{-11}$ with full statistics
- J-PARC E34 (Muon $g-2$/EDM experiment) with novel approach, data taking from 2027 [Abe *et al.* 2019]

the leading-order HVP contribution, on the lattice

[based on Gérardin, MC *et al.* 2019; see also the review Meyer, Wittig 2019, and white paper Aoyama *et al.* 2020]

the subtracted HVP function $\bar{\Pi}(q^2) = \mathrm{Re}\left[\Pi(q^2) - \Pi(0)\right]$ is given by the once-subtracted dispersion relation

$$\Pi(q^2) - \Pi(0) = -\frac{q^2}{\pi}\mathcal{P}\int_{m_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\,\Pi(s)}{s(s-q^2)}, \qquad \mathrm{Im}\,\Pi(s) = -\frac{R(s)}{12\pi}$$

$R(s)$ and $\Pi(q^2)$ at time-like $q^2 > 0$ are not directly accessible to lattice simulations in Euclidean spacetime, but the subtracted function $\bar{\Pi}(-Q^2)$ at space-like $q^2 = -Q^2 < 0$ is!

$$\bar{\Pi}(-Q^2) = \Pi(-Q^2) - \Pi(0), \qquad (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(-Q^2) = \int \mathrm{d}^4x\, e^{\mathrm{i}Qx}\left\langle j_\mu(x) j_\nu(0)\right\rangle$$

from it, one can extract [Lautrup, Peterman, de Rafael 1972; Blum 2003]

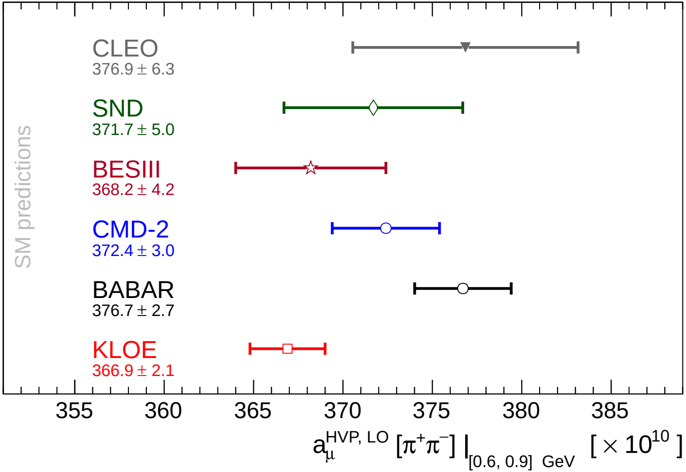
$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\bar{\Pi}(-Q^2), \qquad f(Q^2)\text{: known QED kernel}$$

first-generation lattice computation: discrete $4d$ Fourier transform of $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0)\rangle$

limitation: on a lattice in a finite box, $2\pi/L < Q \lesssim 2\pi/a \Rightarrow$ not enough resolution at small and high Q

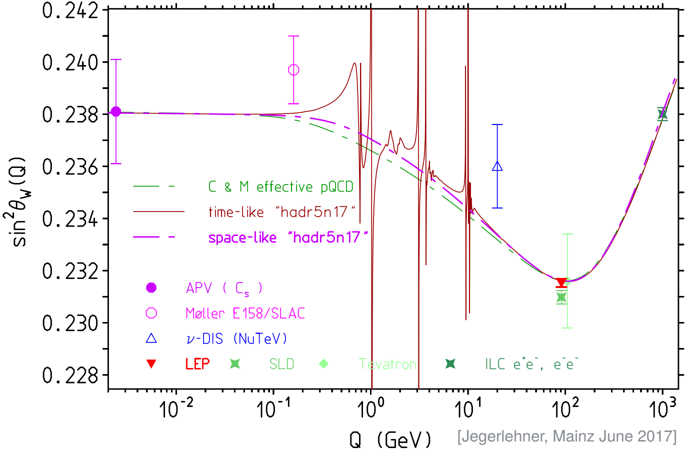
- hybrid method [Golterman, Maltman, Peris 2014]
- recent computations use the time-momentum representation (TMR) method [Bernecker, Meyer 2011; Francis *et al.* 2013]

tensions in $R(s)$: the $\pi^+\pi^-$ channel



- BaBar and KLOE contributions for $0.6\,\mathrm{GeV} < \sqrt{s} < 0.9\,\mathrm{GeV}$ disagree by 2.9σ
- not enough to explain the tension with BMW '20 lattice result

Euclidean vs. Minkowski running



[Jegerlehner, Mainz June 2017]

$\Delta\alpha_{\text{had}}(M_Z^2) - \Delta\alpha_{\text{had}}(-M_Z^2) \approx 40 \times 10^{-6}$ can be computed in perturbation teory [Jegerlehner 2008]

also, change of scheme: $\Delta\hat{\alpha}(M_Z) - \Delta\alpha(M_Z^2) = 0.007\,122(2)(5)$ [PDG 2022]