Black holes and Nilmanifolds: Quasinormal Modes as the fingerprints of Extra Dimensions?

Anna Chrysostomou [1,2,3], Alan S. Cornell [1], Aldo Deandrea [2,3], Étienne Ligout [2,4], Dimitrios Tsimpis [2,3]

[1]University of Johannesburg [2]Université de Lyon-1 [3]Institut de physique des 2 infinis de Lyon [4]École normale supérieure de Lyon

Abstract

Quasinormal modes (QNMs), the damped oscillations in spacetime that emanate from a perturbed body as it returns to an equilibrium state, have served for several decades as a theoretical means of studying n-dimensional black hole spacetimes. These black hole QNMs can in turn be exploited to explore beyond the Standard Model (BSM) scenarios and quantum gravity conjectures. With the establishment of the LIGO-Virgo-KAGRA (LVK) network of gravitational-wave (GW) detectors, there now exists the possibility of comparing computed QNMs against GW data from compact binary coalescences. Encouraged by this development, we investigate whether QNMs can be used in the search for signatures of extra dimensions. To address a gap in the BSM literature, we focus here on higher dimensions characterised by negative Ricci curvature. As a first step, we consider a product space comprised of a 4D Schwarzschild black hole spacetime and a 3D nilmanifold (twisted torus); we model the black hole perturbations as a scalar test field. We find that the extra-dimensional geometry can be stylised in the QNM effective potential as a squared mass-like term. We then compute the QNM spectrum using three different numerical methods and determine constraints for the extra dimensions for a toy BSM model.

Black hole perturbations: quasinormal modes

A stationary, spherically-symmetric, 4D black hole (BH) can be described mathematically by the Schwarzschild vacuum solution to the Einstein field equations (under geometric units $G = c = 1$),

$$ds^2_{BH} = g^{BH}_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(\sin^2 d\theta^2 + d\phi^2) \ , \qquad f(r) = 1 - \frac{2M}{r} \ .$$

Per the "no-hair" conjecture, such a BH is fully characterised by its mass M; classically, energy cannot escape from within the event horizon $r_H = 2M$. To model the evolution of the background fields $g^{\rm BG}_{\mu\nu}$ and $\Phi^{\rm BG}$ in the wake of a BH perturbation, we introduce the perturbed fields $\tilde{g}_{\mu\nu} = g^{\rm BG}_{\mu\nu} + h_{\mu\nu}$ and $\tilde{\Phi} = \Phi^{\rm BG} + \varphi$ into the Einstein field equations, linearise the system with respect to the perturbations $h_{\mu\nu}$ and φ, and determine the vacuum solution [1].

For a toy model, we may consider a scalar test field that contributes negligibly to the energy-density of the system. Then the equations of motion for $h_{\mu\nu}$ and φ decouple, and we can set $h_{\mu\nu}$ to zero. The problem reduces to that of a scalar field evolving on a fixed background, with energy purely ingoing at $r = r_H$ and outgoing at $r = \infty$. We define this behaviour with a variable-separable **quasinormal mode (QNM)** and a damped **quasinormal frequency (QNF)**,

$$\Phi^s_{n\ell m}(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{\ell,m}^{\infty} \frac{\psi_{sn\ell}(r)}{r} \, Y_{\ell m}(\theta,\phi) \, e^{-i\omega t} \ , \qquad \omega_{sn\ell} = \omega_R - i\omega_I \ ,$$

- ⋆ $\mathbb{R}e\{\omega\}$: physical oscillation frequency; $\mathbb{I}m\{\omega\}$: inverse damping rate (dissipative boundary conditions ⇒ "quasi"),
- ⋆ s: spin of the perturbing field,
- ⋆ m, ℓ: azimuthal, angular/multipolar number for spherical harmonic decomposition in θ, ϕ,
- ⋆ n: overtone number labelling QNMs by monotonically increasing multiples of $|\mathbb{I}m\{\omega\}|$.

Astrophysically, QNFs are characterised by their source and are independent of its initial perturbing stimulus. The QNF spectrum is in turn dominated by the least-damped / longest-lived *fundamental mode: $n = 0$, $\ell = m = 2$.*

Negatively-curved extra dimensions: nilmanifolds

While the parameter space of flat and positively-curved extra dimensions has been probed and constrained [2], models involving higher-dimensional manifolds with negative Ricci curvature have remained under-explored. Phenomenologically, studies on compact negative spaces suggest that these models could be used to address the hierarchy problem [3] and cosmological observations [4, 5]. Motivated by these, let us begin with the following definition:

Any Lie group G of dimension d can be understood as a d-dimensional differentiable manifold. To make G compact, we quotient by the lattice Γ. For "nilpotent groups", there is always a Γ.

Based on this premise, we can construct a unique 3D nilmanifold through the *Heisenberg algebra,*

$$[Z_1, Z_2] = -\mathtt{f} Z_3 \ , \quad [Z_1, Z_3] = [Z_2, Z_3] = 0 \ ,$$

and exploit the veilbein formalism to encode its geometric properties [6],

$$de = \mathtt{f} e^1 \wedge e^2 \ , \quad de^1 = 0 \ , \quad de^2 = 0 \ ,$$

$$e^1 = r^1 dy^1 \ , \quad e^2 = r^2 dy^2 \ , \quad e^3 = r^3(dy^3 + N r^1 dy^2) \ , \quad N = r^1 r^2 \mathtt{f}/r^3 \ .$$

In this way, the manifold is fully characterised and we can obtain the most general minimal left-invariant metric,

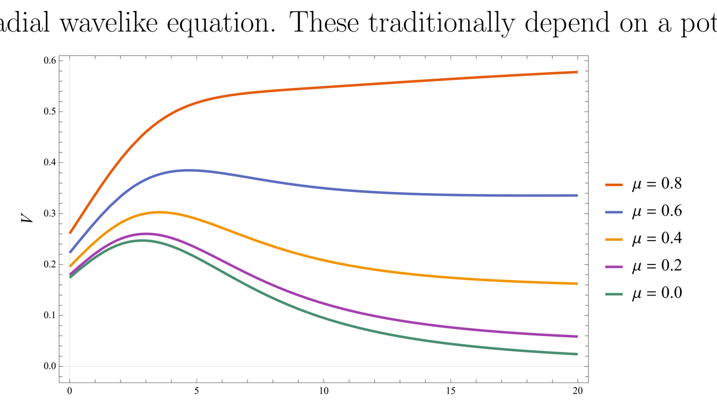
$$ds^2_{nil} = \delta_{ab} e^a e^b = (r^1 dy^1)^2 + (r^2 dy^2)^2 + (r^3 dy^3 + N r^1 r^3 dy^2)^2 \ .$$

A torus and a twisted torus, where the twist parameter is the geometric flux $\mathtt{f}$

How to isolate black hole quasinormal modes in a Schwarzschild-nilmanifold extra-dimensional setup

The effective potential

As an inherently dissipative system, the QNM boundary-value problem is non-Hermitian; the corresponding eigenfunctions are not normalisable and do not form a complete set. However, we can make use of established numerical techniques, informed by scattering theory, to compute QNFs from the radial wavelike equation. These traditionally depend on a potential barrier.



The effective scalar potential of the Schwarzschild-nilmanifold setup for increasing values of μ. The tortoise coordinate r_* is defined through $dr_* = dr/f(r)$.

For $\mu = 0$, $V \to \infty$ as $r_* \to \infty$ and the effective potential has a distinct peak. For $\mu \neq 0$, $V \to \mu^2$ as $r_* \to +\infty$; the curve is smoothed for increasing μ^2 and the QNM behaviour is lost. Beyond some critical μ, we therefore consider QNMs to be an inappropriate probe for extra dimensions. From the QNF spectrum computed below, we find this critical μ when $\mu > \mathbb{R}e\{\omega\}$. In this way, we extract an upper bound for μ based on QNM numerics.

We consider our higher-dimensional manifold as a direct product of a 4D flat spacetime and the defined 3D nilmanifold, in which a 4D Schwarzschild BH is embedded. The resultant metric and oscillating scalar field can be written as

$$ds^2_{7D} = ds^2_{BH} + ds^2_{nil} \ , \qquad \Psi^s_{n\ell m}(\mathbf{z}) = \sum_{n=0}^{\infty}\sum_{\ell,m}^{\infty} \frac{\psi_{sn\ell}(r)}{r} \, Y^s_{m\ell}(\theta,\phi) \, Z(y^1, y^2, y^3) \, e^{-i\omega t} \ .$$

To extract the QNMs, we first apply the Laplacian. For this unusual spacetime, we can account for the higher dimensions by exploiting the separability of the metric. Recall that *the Laplacian of a product space is the sum of its parts*

$$\nabla^2\Psi(\mathbf{z}) = \left(\nabla^2_{BH} + \nabla^2_{nil}\right)\Phi^s_{n\ell m}(\mathbf{x}) Z(\mathbf{y}) \ .$$

However, if we choose to encode the extra-dimensional behaviour through an effective mass term representing a Kaluza-Klein tower of states, then we may describe the 7D scalar field evolution through a *4D "massive" Klein-Gordon equation:*

$$\nabla^2_{nil} Z(\mathbf{y}) = -\mu^2 Z(y^1, y^2, y^3) \quad \Rightarrow \quad \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi\right) - \mu^2\Psi = 0 \ .$$

Finally, we may extract the characteristic wavelike equation containing the QNF and the effective scalar potential,

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\psi = 0 \ , \qquad V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right) \ .$$

Schwarzschild-nilmanifold quasinormal frequencies

Dolan and Ottewill's expansion method [7] expresses the QNF as a series of inverse multipolar numbers. By substituting this and a novel ansatz into the wavelike equation provided, they determined a highly accurate iterative technique for QNF computation. We apply their method here to the "massive" case; we obtain a series expansion in terms of μ and ℓ, and QNFs in excellent agreement with those computed using the modified WKB [8] and Pöschl-Teller [9] methods.

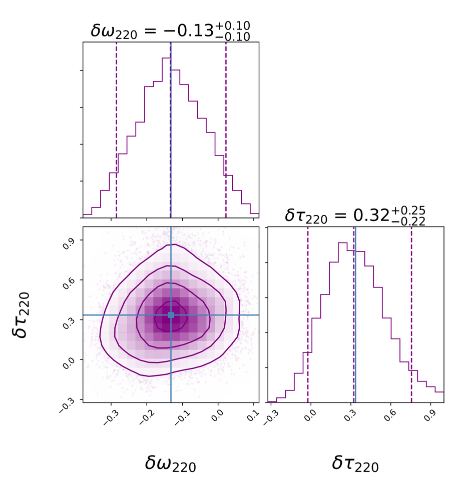
μ	$\mathbb{R}e\{\omega(\ell,\mu)\}$	f (Hz)	$\delta\omega_{\mu=0}$	$\mathbb{I}m\{\omega(\ell,\mu)\}$	τ (ms)	$\delta\tau_{\mu=0}$
0.0	0.4836	115.03	0.0000	−0.0968	6.9157	0.0000
0.1	0.4868	115.78	0.00654	−0.0957	6.9940	0.0113
0.2	0.4963	118.05	0.0262	−0.0924	7.2429	0.0473
0.3	0.5124	121.87	0.0594	−0.0868	7.7104	0.1149
0.4	0.5352	127.30	0.1066	−0.0787	8.5074	0.2302
0.5	0.5653	134.44	0.1687	−0.0676	9.8937	0.4306
0.6	0.6032	143.47	0.2472	−0.0532	12.5900	0.8206
0.7	0.6500	154.60	0.3440	−0.0343	19.4890	1.8181

For increasing μ, we compute the real and imaginary parts of the QNF using the Dolan-Ottewill method, as well as the observed frequency f and damping time τ corresponding to the final BH of event GW150914. The terms $\delta\omega_{\mu=0}$ and $\delta\tau_{\mu=0}$ represent parametric deviations from the $\mu = 0$ value of the QNF and its damping.

Constraints from gravitational waves: PyRing

The post-merger phase of a binary BH coalescence is dominated by the QNFs. With the Python package PyRing [10, 11], we can combine observed GW data with simulation and numerically-generated waveform templates to perform parameter estimation, QNM analyses, and parametrised tests of general relativity (GR). We consider the latter here: an agnostic test of how far the GW data deviates from the QNFs predicted by GR.

$$\omega = \omega^{\rm GR}(1 + \delta\omega) \ , \quad \tau = \tau^{\rm GR}(1 + \delta\tau) \ .$$



As a proof-of-concept, we perform a rudimentary parameter estimation of the GR deviations using PyRing for event GW150914 (GW data sampled at 4096 Hz). We narrow priors to reduce computation cost. With Corner, we plot the 2D posteriors and 1D histograms on $(\delta\omega, \delta\tau)$, where $(0,0)$ is the GR-predicted value. The 90% credible region is demarcated by dashed lines and contours; the blue line indicates the mean.

In their most recent catalogue of GR testing [12], the LVK collaboration reported the hierarchical combination of their strongest bounds on GR deviations to date:

$$\delta\omega = 0.02^{+0.07}_{-0.07} \ , \quad \delta\tau = 0.13^{+0.21}_{-0.22}$$

If we set $\omega^{\rm GR} = \omega^{\mu=0}$ and use the computed QNF series expansion for the dominant QNM $\omega(\ell = 2, \mu)$, we can use the real part of the QNF to constrain μ:

$$0.1747 < \mu < 0.3681$$

Outlook

From our QNM numerics, the requirement that $\mathbb{R}e\{\omega\} > \mu$ gives an upper bound on μ; from GW analyses, we suggest that this bound may be further constrained. Our next immediate step is to subject the mass spectrum of the nilmanifold model studied in Ref. [6] to this constraint in order to extract tangible bounds on the radius of the nilmanifold extra dimensions herein constructed.

References

[1] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[2] P. A. Zyla *et al.* (Particle Data Group), PTEP **2020**, 083C01 (2020).
[3] D. Orlando and S. C. Park, JHEP **08**, 006 (2010), arXiv:1006.1901 [hep-th] .
[4] G. D. Starkman, D. Stojkovic, and M. Trodden, Phys. Rev. Lett. **87**, 231303 (2001), arXiv:hep-th/0106143 .
[5] C.-M. Chen, P.-M. Ho, I. P. Neupane, N. Ohta, and J. E. Wang, JHEP **10**, 058 (2003), arXiv:hep-th/0306291 .
[6] D. Andriot, G. Cacciapaglia, A. Deandrea, N. Deutschmann, and D. Tsimpis, JHEP **06**, 169 (2016), arXiv:1603.02289 [hep-th] .
[7] S. R. Dolan and A. C. Ottewill, Class. Quant. Grav. **26**, 225003 (2009), arXiv:0908.0329 [gr-qc] .
[8] R. A. Konoplya, Phys. Rev. D **68**, 024018 (2003), arXiv:gr-qc/0303052 .
[9] H.-J. Blome and B. Mashhoon, Physics Letters A **100**, 231 (1984).
[10] G. Carullo, W. Del Pozzo, and J. Veitch, Phys. Rev. D **99**, 123029 (2019), arXiv:1902.07527 [gr-qc] .
[11] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel, and S. A. Teukolsky, Phys. Rev. Lett. **123**, 111102 (2019), arXiv:1905.00869 [gr-qc] .
[12] R. Abbott *et al.* (LIGO Scientific, VIRGO, KAGRA), (2021), arXiv:2112.06861 [gr-qc] .

 a.chrysostomou@ip2i.in2p3.fr