National Technical University of Athens

Modified Gravity Cosmology II

Emmanuel N. Saridakis

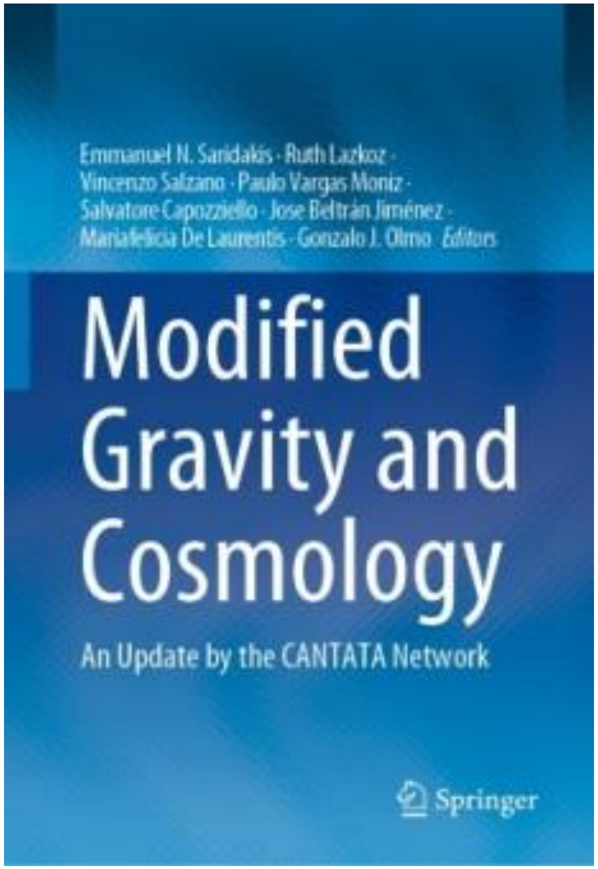
National Observatory of Athens

SIGRAV 2022

CANTATA

Cosmology and Astrophysics Network for Theoretical Advances and Training Actions

Bibliography



- Modified Gravity and Cosmology: An Update by the CANTATA Network
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- Modified Gravity and Cosmology,
  T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, Phys.Rept. 513 (2012) 1-189, 1106.2476 [astro-ph.CO]

- Extended Theories of Gravity
  S. Capozziello, M. De Laurentis, Phys.Rept. 509 (2011) 167-321, 1108.6266 [gr-qc]

- f(T) teleparallel gravity and cosmology
  YF. Cai, S. Capozziello, M. De Laurentis, E. N. Saridakis, Rept.Prog.Phys. 79 (2016) 10, 106901, 1511.07586 [gr-qc]

Summary:

ΛCDM concordance model is almost perfect!

$$H(t)^2 + \frac{k}{a(t)^2} = \frac{8\pi G}{3}\left[\rho_{dm}(t) + \rho_b(t) + \rho_r(t)\right] + \frac{\Lambda}{3}$$

$$\dot{H}(t) - \frac{k}{a(t)^2} = -4\pi G\left[\rho_{dm}(t) + p_{dm}(t) + \rho_b(t) + p_b(t) + \rho_r(t) + p_r(t)\right]$$

$$w_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1$$

Issues of ΛCDM Paradigm

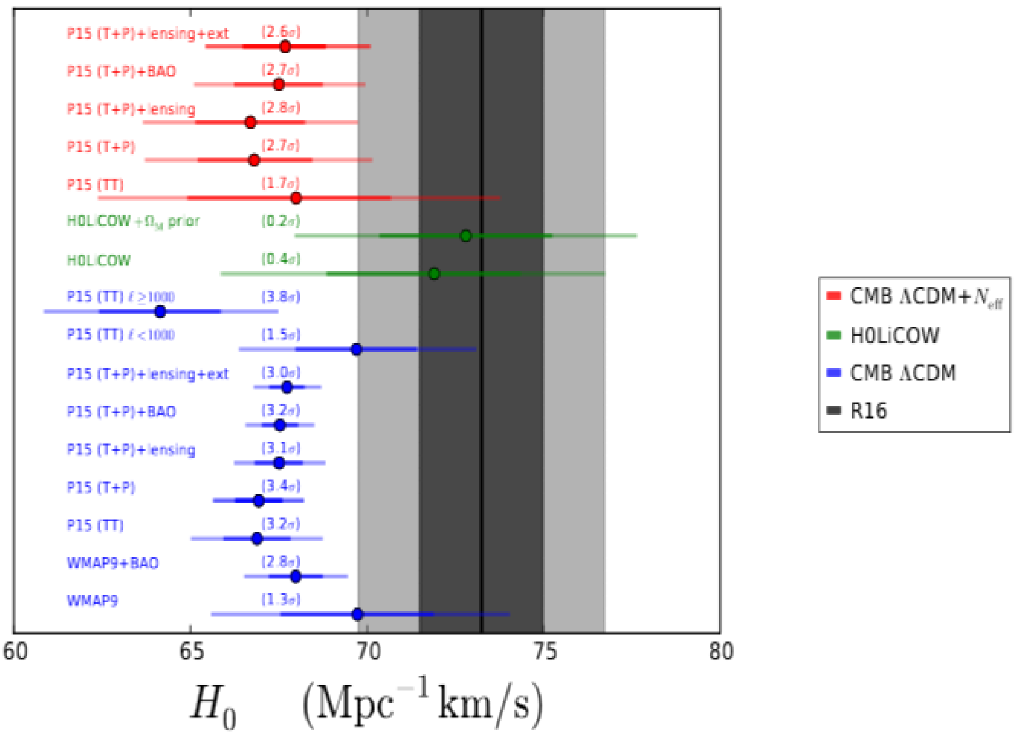
- ΛCDM is a successful cosmological model:

1) Describes the evolution of the universe at the background level
2) Describes the evolution of the universe at the perturbation level

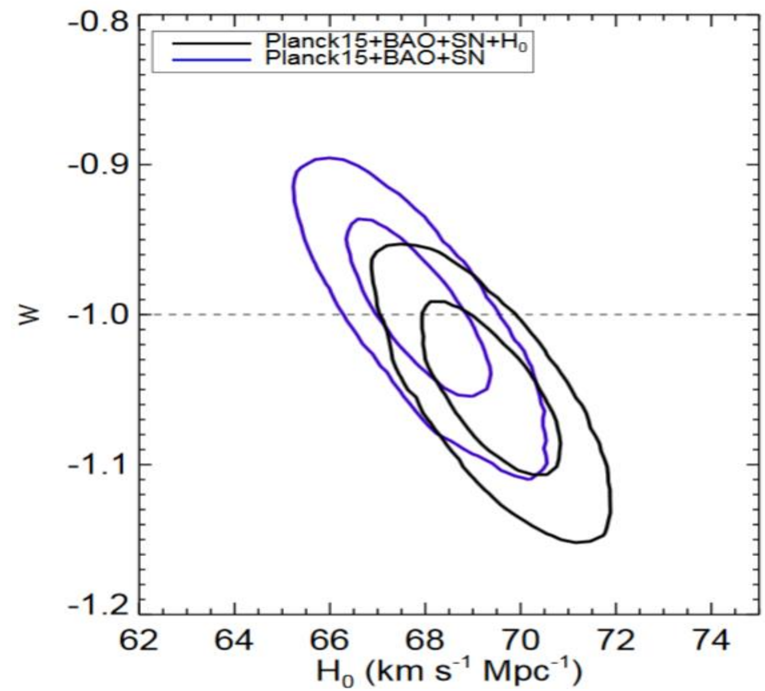
- However there are open issues:
  1) General Relativity is non-renormalizable. It cannot get quantized.
  2) The cosmological-constant problem. Calculation of Λ gives a number 120 orders of magnitude larger than observed.
     Worst error in the ~~history of physics~~, ~~history of science~~, history
  3) How to describe primordial universe (inflation)
  4) Tensions with some data sets, e.g. H0, fσ8, AL data
  5) Missing galaxy satellites, curspy-core problems.

Tension1 – H0

- Tension between the data (direct measurements) and Planck/ΛCDM (indirect measurements). The data indicate a lack of “gravitational power”.



[Bernal, Verde, Riess, JCAP1610]

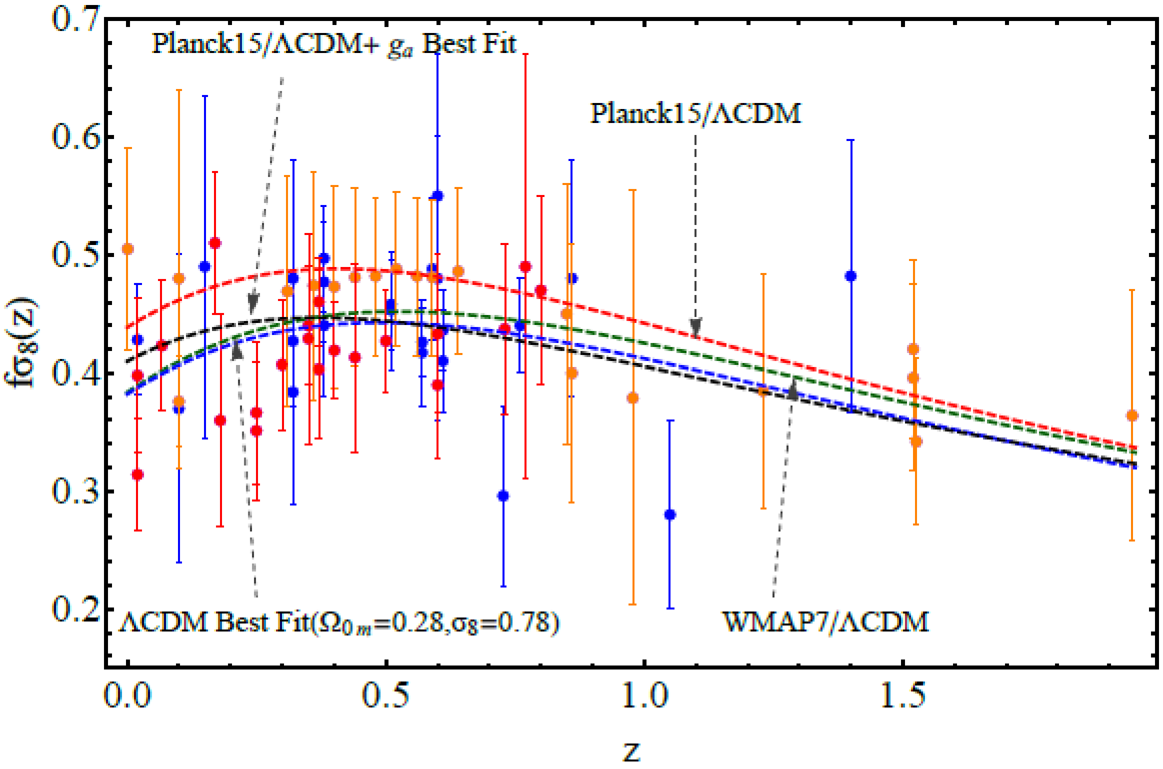


[Riess et al, Astrophys.J 826]

Tension2 – fσ8

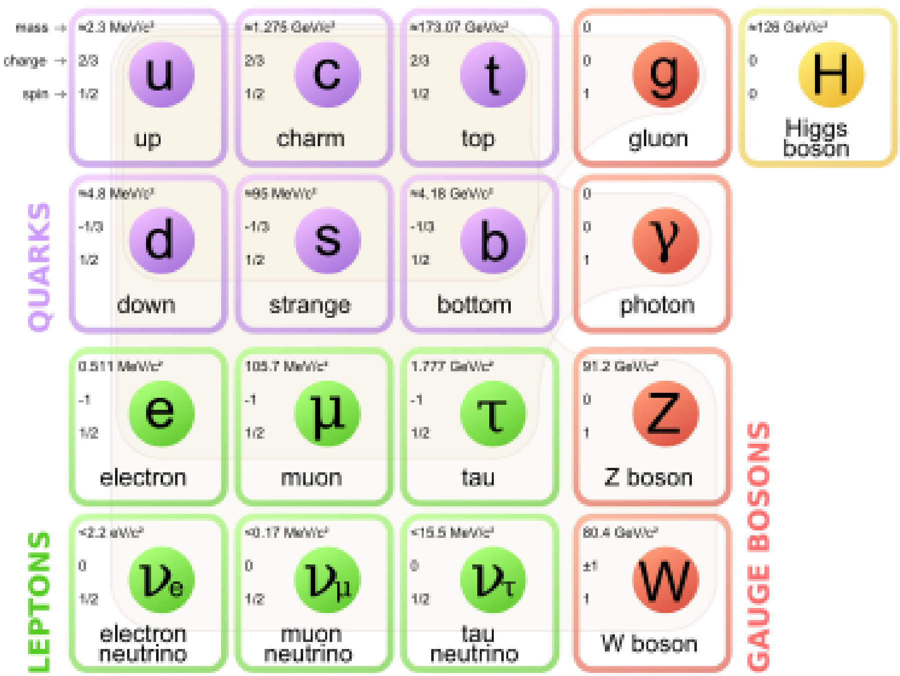
- Tension between the data and Planck/ΛCDM. The data indicate a lack of "gravitational power" in structures on intermediate-small cosmological scales.

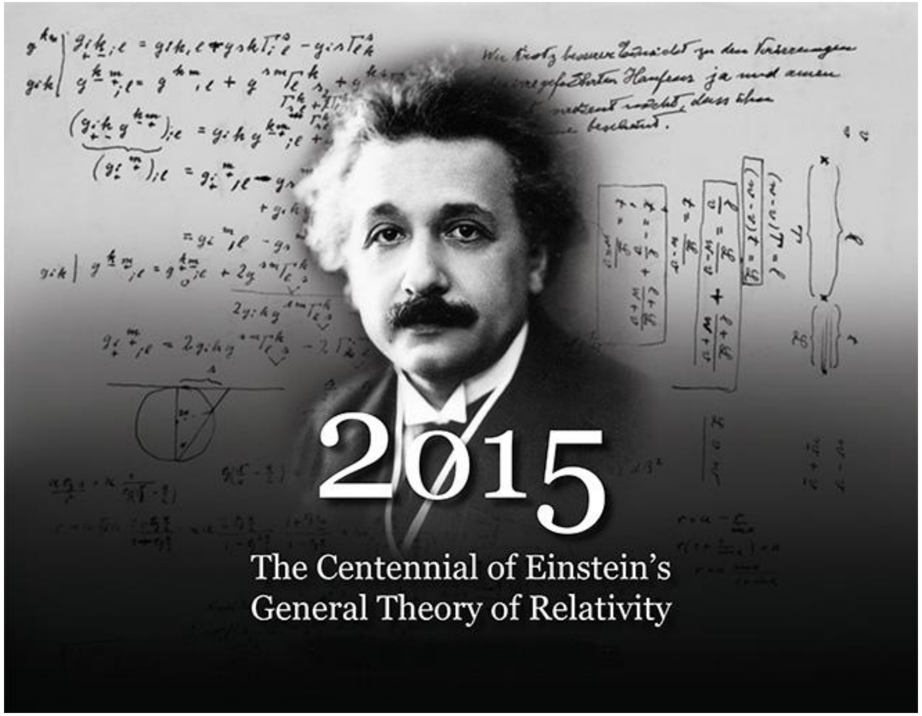
Parameter	Planck15/ΛCDM [12]	WMAP7/ΛCDM [45]
$\Omega_b h^2$	0.02225 ± 0.00016	0.02258 ± 0.00057
$\Omega_c h^2$	0.1198 ± 0.0015	0.1109 ± 0.0056
n_s	0.9645 ± 0.0049	0.963 ± 0.014
H_0	67.27 ± 0.66	71.0 ± 2.5
Ω_{0m}	0.3156 ± 0.0091	0.266 ± 0.025
w	-1	-1
σ_8	0.831 ± 0.013	0.801 ± 0.030



Knowledge of Physics

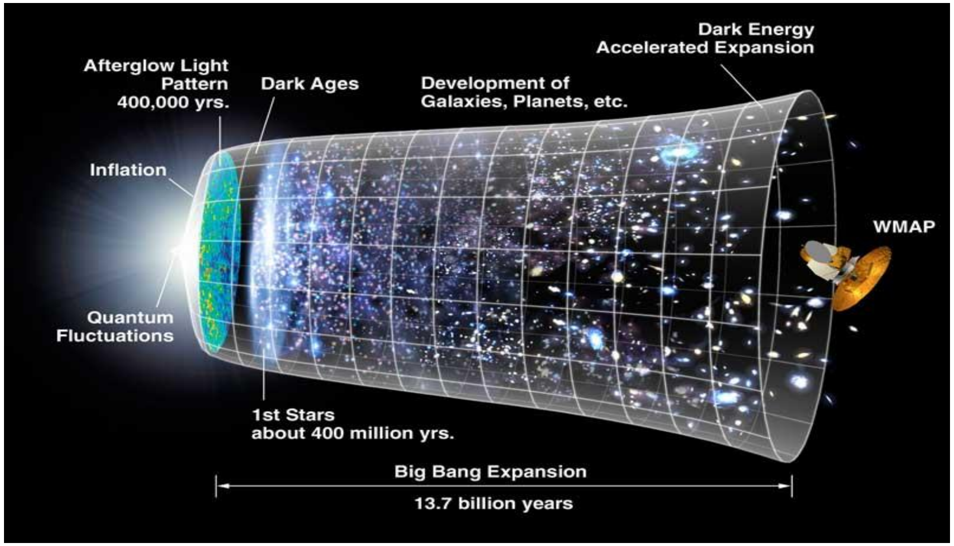
Knowledge of Physics: Standard Model + General Relativity

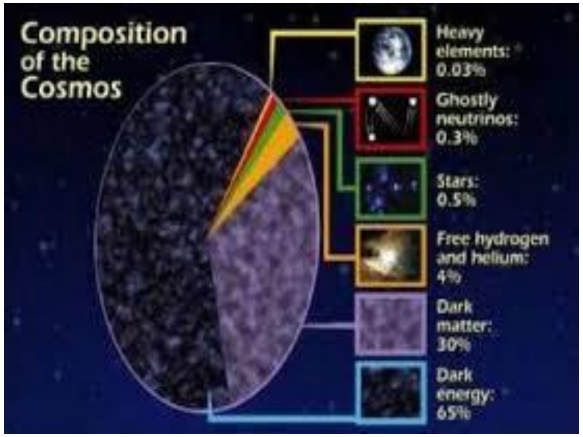




Modified/new knowledge of physics

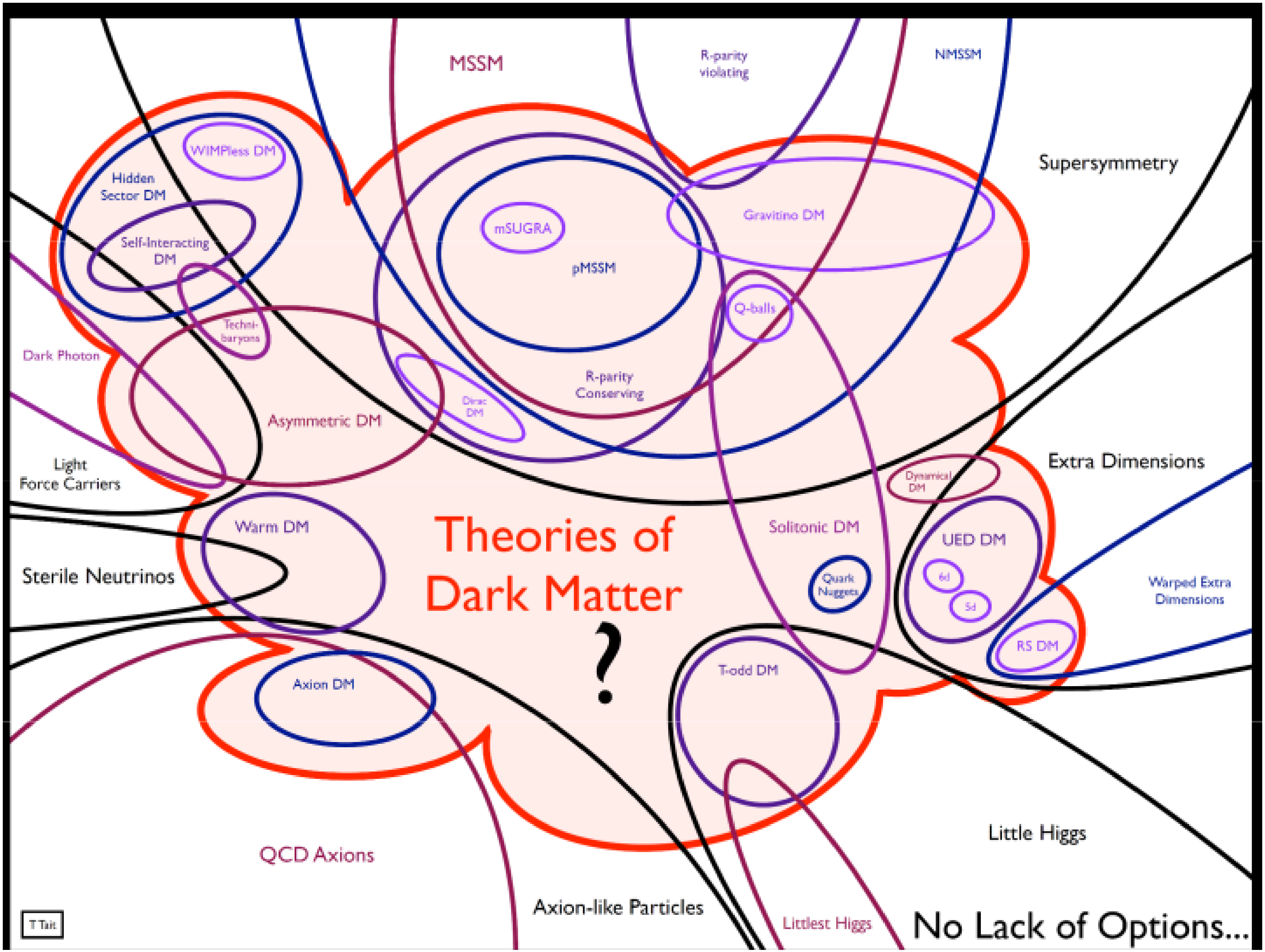
So can our knowledge of Physics describes all these?





Most probably, no!

We definitely need new physics for Inflation and Dark matter. Maybe for dark energy.


MSSM
R-parity violating
NMSSM
Supersymmetry
WIMPless DM
Hidden Sector DM
Self-Interacting DM
Techni-baryons
mSUGRA
pMSSM
Gravitino DM
Q-balls
Dark Photon
R-parity Conserving
Dirac DM
Asymmetric DM
Light Force Carriers
Extra Dimensions
Dynamical DM
Warm DM
Theories of Dark Matter
?
Solitonic DM
UED DM
6d
5d
Quark Nuggets
Sterile Neutrinos
Warped Extra Dimensions
RS DM
Axion DM
T-odd DM
Little Higgs
QCD Axions
Axion-like Particles
Littlest Higgs
No Lack of Options...
T Tait

Why Modified Gravity?

We need to modify something:

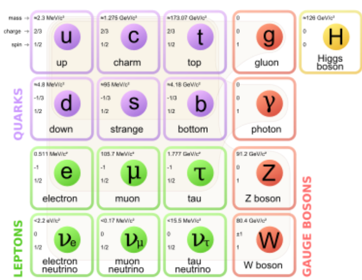
The universe content

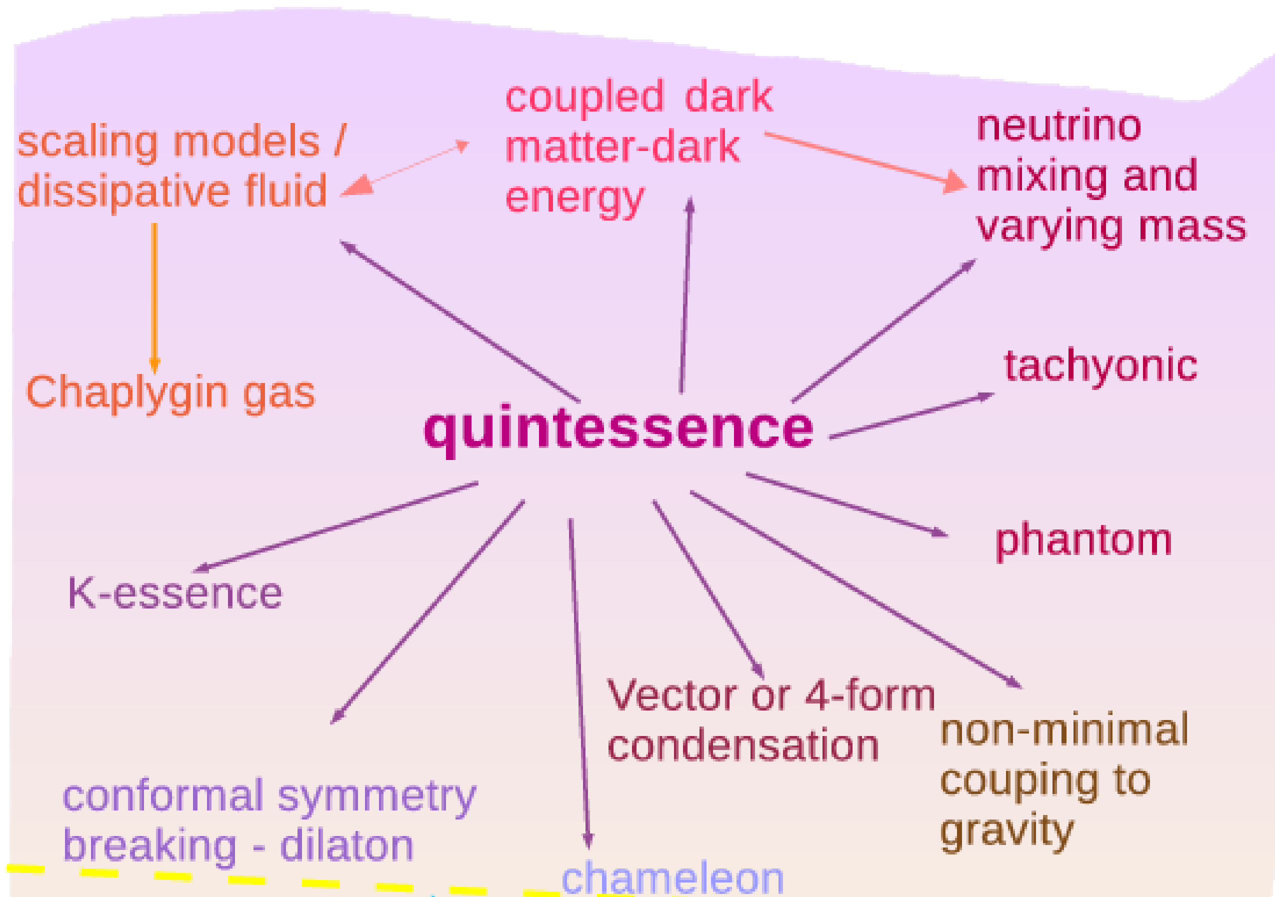
or

The theory of Gravity

Dark Energy-Inflation

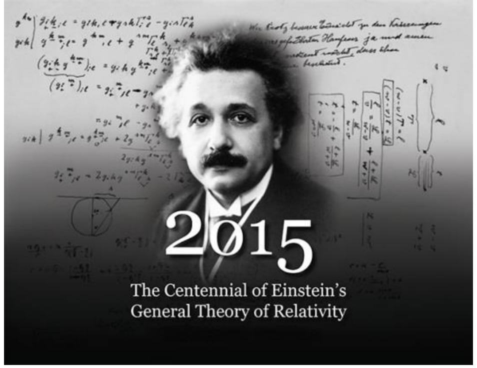
- Add a scalar field φ in the Universe content





General Relativity

- Einstein 1915: **General Relativity**:



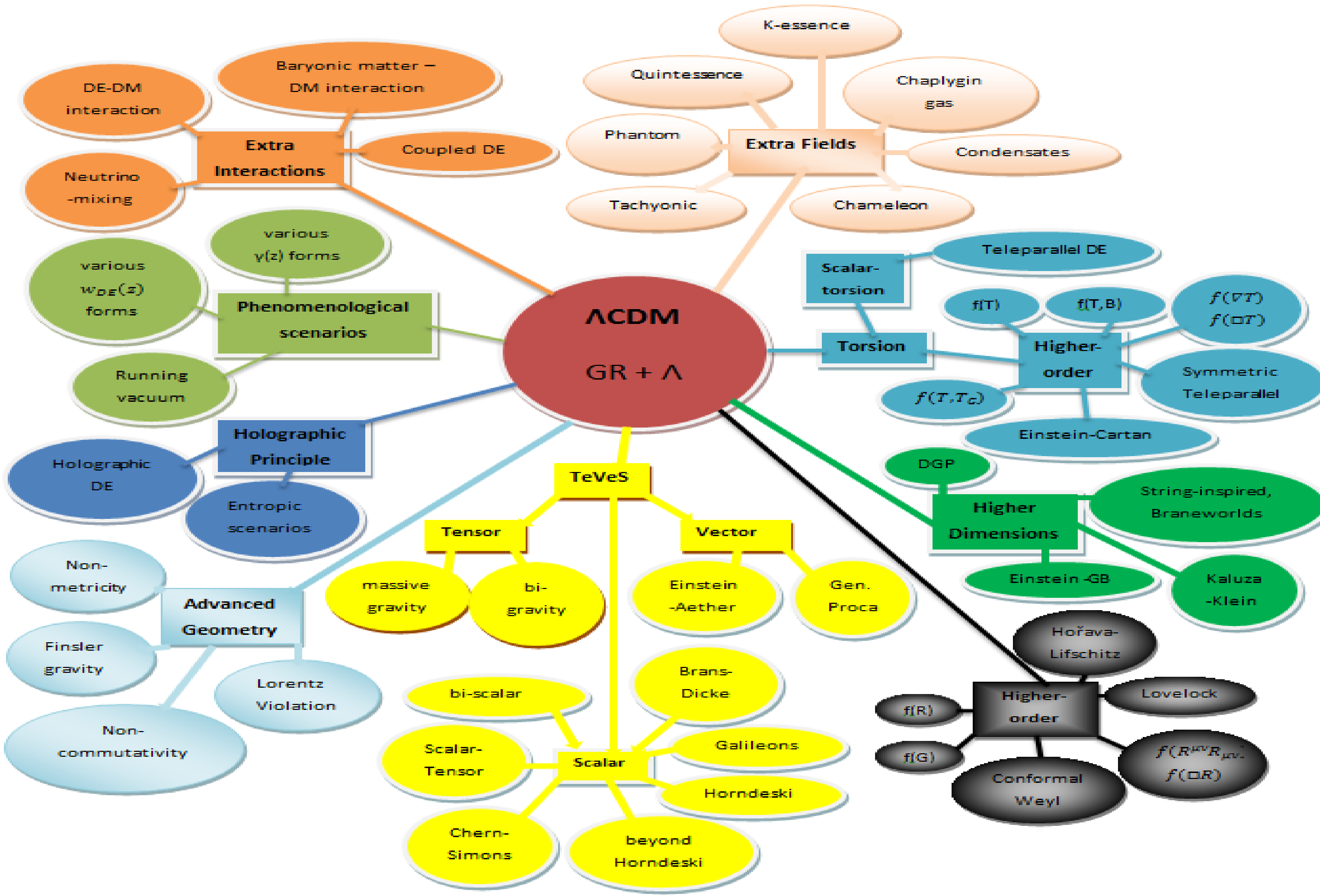
energy-momentum source of spacetime Curvature

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda\right]+\int d^4x\, L_m\left(g_{\mu\nu},\psi\right)$$

$$\Rightarrow \quad R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda = 8\pi G\, T_{\mu\nu}$$

with $\quad T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta L_m}{\delta g_{\mu\nu}}$

Modified Gravity



Cosmology-background

- Homogeneity and isotropy: $ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$

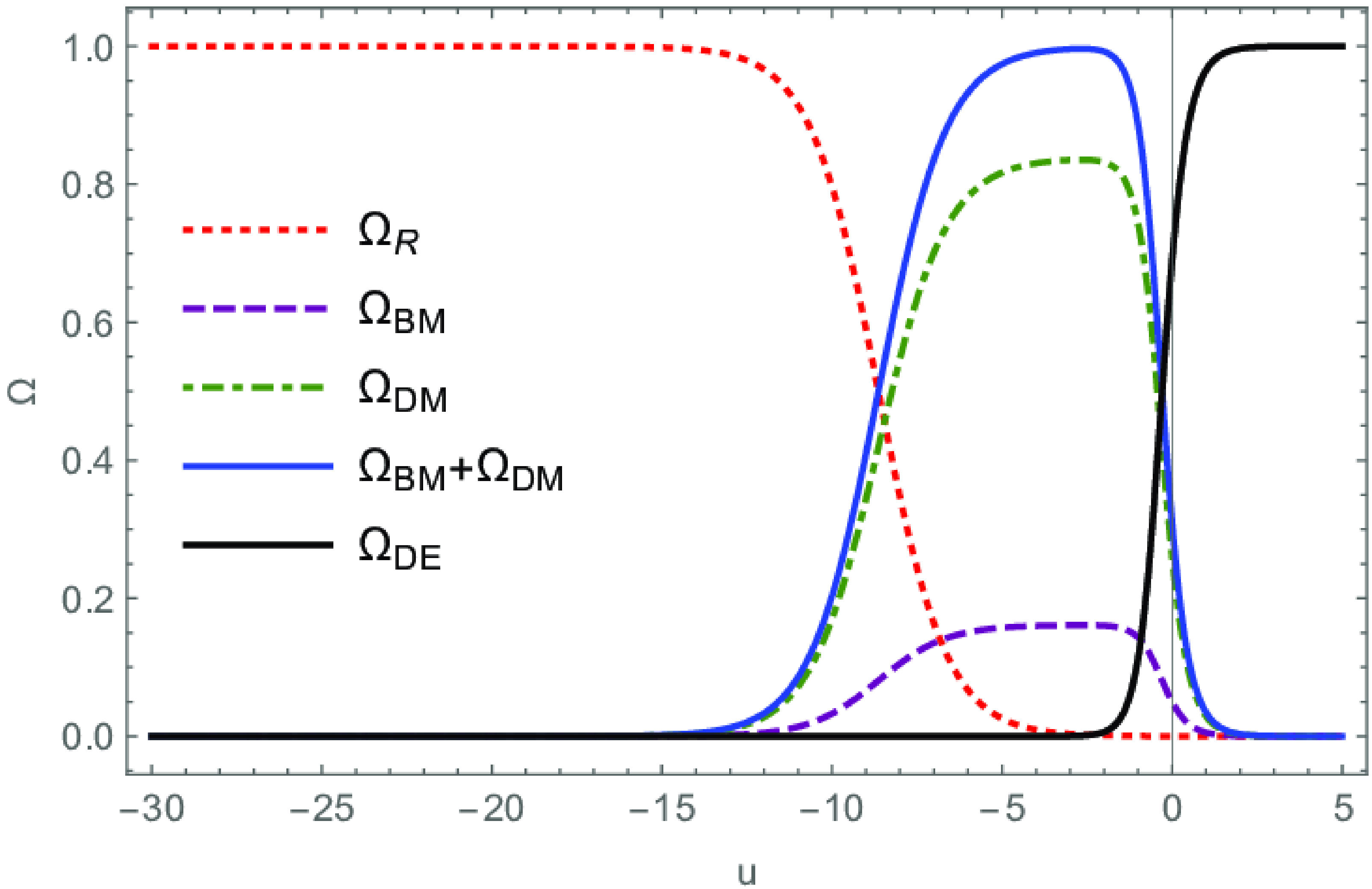
- Background evolution (Friedmann equations) in flat space

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_{DE}\right)$$
$$\dot{H} = -4\pi G\left(\rho_m + p_m + \rho_{DE} + p_{DE}\right),$$

(the effective DE sector can be either Λ or any possible modification)

- One must obtain a H(z) and Ωm(z) and wDE(z) in agreement with observations (SNIa, BAO, CMB shift parameter, H(z) etc)

Cosmology-background



Cosmology-perturbations

- Perturbation evolution: $\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff}\,\rho\,\delta \approx 0$ where $\delta \equiv \delta\rho/\rho$

  where $G_{\rm eff}(z,k)$ is the effective Newton's constant, given by

  $$\nabla^2\phi \approx 4\pi G_{\rm eff}\rho\,\delta,$$

  under the scalar metric perturbation $ds^2 = -(1+2\phi)dt^2 + a^2(1-2\psi)d\vec{x}^{\,2}$

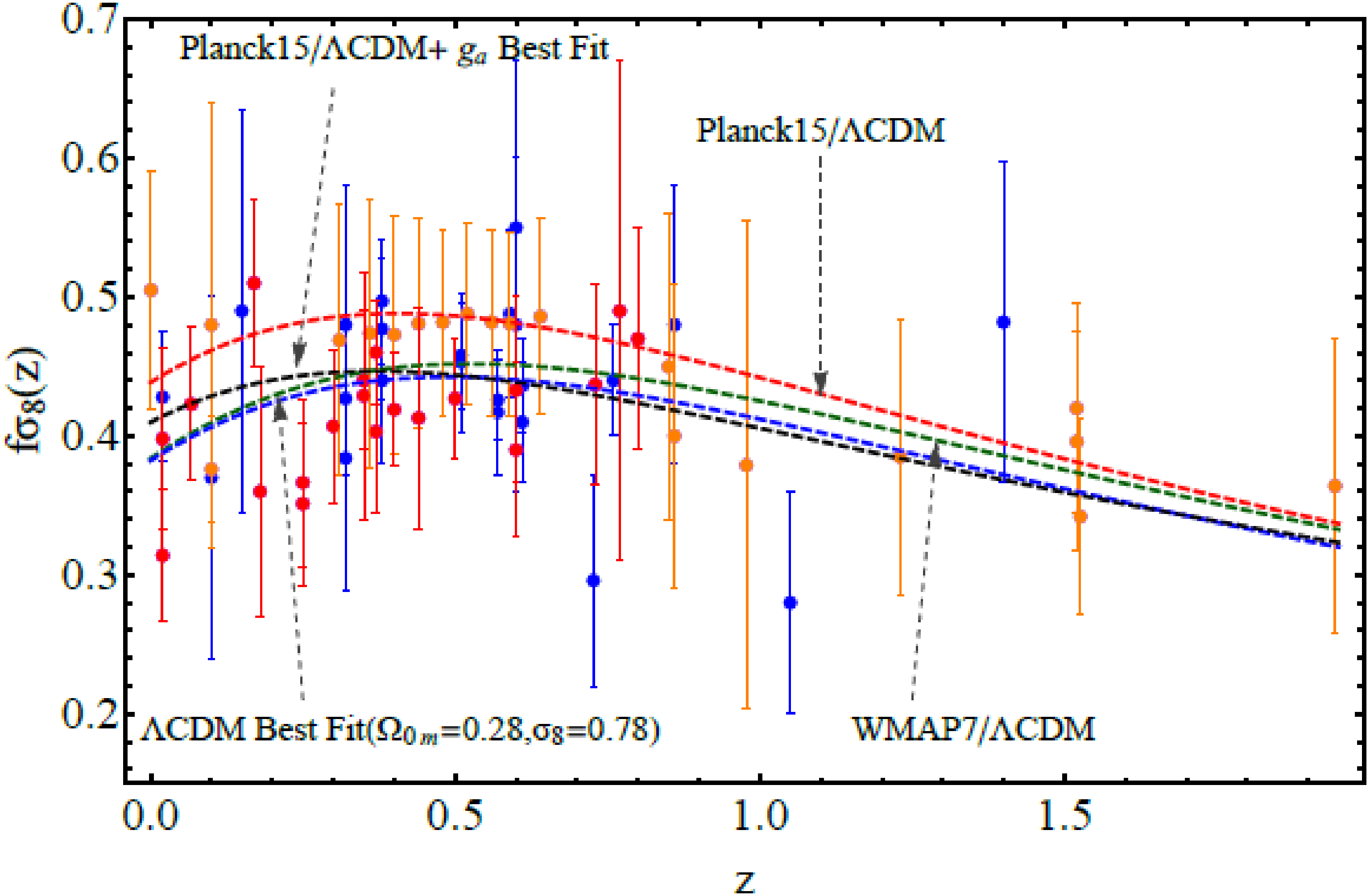
- Hence: $\delta'' + \left(\frac{(H^2)'}{2\,H^2} - \frac{1}{1+z}\right)\delta' \approx \frac{3}{2}(1+z)\frac{H_0^2}{H^2}\frac{G_{\rm eff}(z,k)}{G_N}\,\Omega_{0m}\delta$

  with $f(a) = \frac{dln\delta}{dlna}$ the growth rate, with $f(a) = \Omega_{\rm m}(a)^{\gamma(a)}$ and $\Omega_{\rm m}(a) \equiv \frac{\Omega_{0m}\,a^{-3}}{H(a)^2/H_0^2}$

- One can define the observable: $f\sigma_8(a) \equiv f(a)\cdot\sigma(a) = \frac{\sigma_8}{\delta(1)}\,a\,\delta'(a)$

  with $\sigma(a) = \sigma_8\frac{\delta(a)}{\delta 1}$ the z-dependent rms fluctuations of the linear density field within spheres of radius $R = 8h^{-1}\mathrm{Mpc}$, and σ8 its value today.

Cosmology-perturbations



Scalar-Tensor Theories

- Add a scalar field:

$$L = \frac{1}{16\pi}\sqrt{-g}\left[f(\phi)R - s(\phi)\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + L_m\left(h(\phi)g_{\mu\nu}, \psi\right)$$

Conformal Transf. to Jordan frame: $h(\phi)g_{\mu\nu} \to g_{\mu\nu}$

Scalar-Tensor Theories

- Add a scalar field:

$$L = \frac{1}{16\pi}\sqrt{-g}\left[f(\phi)R - s(\phi)\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + L_m\left(h(\phi)g_{\mu\nu},\psi\right)$$

Conformal Transf. to Jordan frame: $h(\phi)g_{\mu\nu} \rightarrow g_{\mu\nu}$

- Redefinition of ϕ :

$$L = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2V(\phi)\right] + L_m\left(g_{\mu\nu},\psi\right)$$

- Brans-Dicke for $\omega \rightarrow const., \ V \rightarrow 0$
- GR for $\omega \rightarrow \infty, \ \omega'/\omega^2 \rightarrow 0, \ V \rightarrow const.$

Scalar-Tensor Theories

- Field equations:

$$\phi G_{\mu\nu} + \left[\Diamond\phi + \frac{\omega}{2\phi}(\nabla\phi)^2 + V \right] g_{\mu\nu} - \nabla_\mu \nabla_\nu \phi - \frac{\omega}{\phi} \nabla_\mu \phi \nabla_\nu \phi = 8\pi\, T_{\mu\nu}$$

$$(2\omega+3)\Box\phi + \omega'(\nabla\phi)^2 + 4V - 2\phi V' = 8\pi\, T$$

- For Brans-Dicke:
  - PPN parameters: $\beta_{PPN} = 1, \;\; \gamma_{PPN} = \frac{1+\omega}{2+\omega} \;\; \Rightarrow \;\; \omega \geq 40000$
  - Newton's constant: $G = \left(\frac{4+2\omega}{3+2\omega} \right) \frac{1}{\phi}$ with $\frac{\dot{G}}{G} \leq 1.7\ 10^{-12}\ yr^{-1}$

Brans-Dicke Cosmology

- Friedmann-Robertson-Walker metric: $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$
- Friedmann equations:

$$H^2 = \frac{8\pi}{3\phi}\rho_m - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} + \frac{V}{3\phi}$$

$$2\dot{H} + 3H^2 = -\frac{1}{\phi}\left(8\pi p_m + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi} + 2H\dot{\phi} + \ddot{\phi}\right) + \frac{V}{\phi}$$

- Scalar-field equation:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{8\pi}{2\omega+3}\left(\rho_m - 3p_m\right) = 0 + \frac{2}{2\omega+3}\left(2V - \phi\frac{dV}{d\phi}\right)$$

- Matter equation: $\dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0$

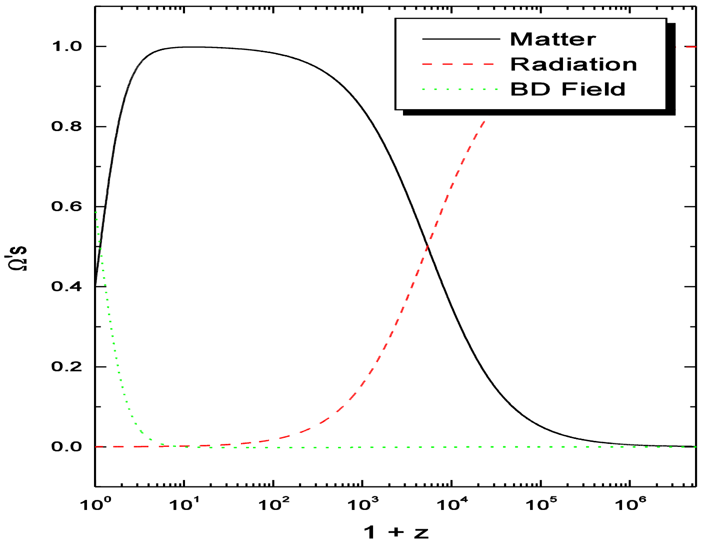
Dark Energy in Brans-Dicke Cosmology

- Effective Dark Energy sector:

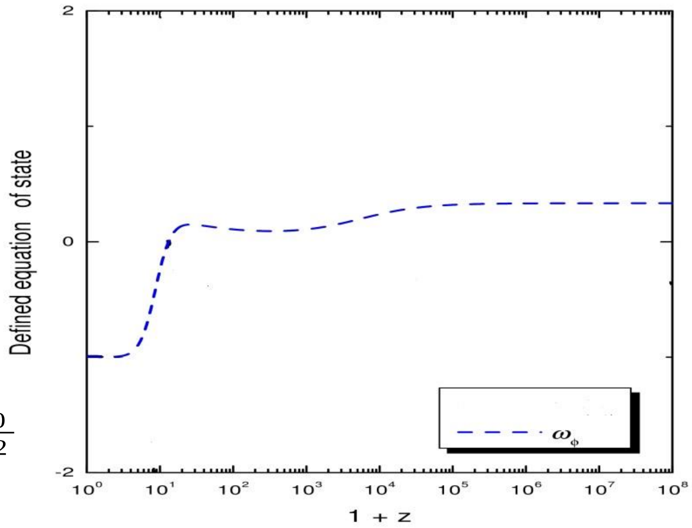
$$\rho_{DE} = \frac{3}{8\pi}\left(-H\dot{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi}\right) + \frac{V}{8\pi}$$

$$p_{DE} = \frac{1}{8\pi}\left(\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi} + 2H\dot{\phi} + \ddot{\phi}\right) - \frac{V}{8\pi}$$

$$\Rightarrow w_{DE} = \frac{p_{DE}}{\rho_{DE}}$$



$$V(\phi) = \frac{V_0}{\phi^2}$$



Scalar-Tensor Theories

- Most general 4D scalar-tensor theories having second-order field equations:

$$L_H = \sum_{i=2}^{5} L_i$$

$$L_2[K] = K(\phi, X)$$

$$L_3[G_3] = -G_3(\phi, X)\Diamond\phi$$

$$X = -\partial^\mu\phi\partial_\mu\phi/2$$

$$L_4[G_4] = G_4(\phi, X)R + G_{4,X}\left[(\Diamond\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right]$$

$$L_5[G_5] = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}\left[(\Diamond\phi)^3 - 3(\Diamond\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]$$

[G. Horndeski, Int. J. Theor. Phys. 10]

Horndeski Theories

- Most general 4D scalar-tensor theories having second-order field equations:

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$$L_5[G_5] = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}\left[(\Diamond\phi)^3 - 3(\Diamond\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]$$

[G. Horndeski, Int. J. Theor. Phys. 10]

- Coincides with Generalized Galileon theories

$$\phi \to \phi + c, \quad \partial_\mu\phi \to \partial_\mu\phi + b_\mu$$

[Nicolis,Rattazzi,Trincherini, PRD 79]

Horndeski Cosmology (background)

- Field Equations: $L.H.S = R.H.S$
- In flat FRW:
- $$2XK_{,X} - K + 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi} - 6H^2G_4 + 24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} - 6H\dot{\phi}G_{4,\phi} + 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi X}) = -\rho_m$$
- $$K - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}) + 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4,X} - 4H\dot{X}G_{4,X} - 8\dot{H}XG_{4,X} - 8HX\dot{X}G_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X} - 2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} - 4H^2X^2\ddot{\phi}G_{5,XX} + 4HX(\dot{X} - HX)G_{5,\phi X} + 2\left[2(\dot{H}X + H\dot{X}) + 3H^2X\right]G_{5,\phi} + 4HX\dot{\phi}G_{5,\phi\phi} = -p_m$$
- $$\frac{1}{a^3}\frac{d}{dt}(a^3J) = P_\phi$$

with $J = \dot{\phi}K_{,X} + 6HXG_{3,X} - 2\dot{\phi}G_{3,\phi} + 6H^2\dot{\phi}(G_{4,X} + 2XG_{4,XX}) - 12HXG_{4,\phi X} + 2H^3X(3G_{5,X} + 2XG_{5,XX}) - 6H^2\dot{\phi}(G_{5,\phi} + XG_{5,\phi X})$

$P_\phi = K_{,\phi} - 2X(G_{3,\phi\phi} + \ddot{\phi}G_{3,\phi X}) + 6(2H^2 + \dot{H})G_{4,\phi} + 6H(\dot{X} + 2HX)G_{4,\phi X} - 6H^2XG_{5,\phi\phi} + 2H^3X\dot{\phi}G_{5,\phi X}$

[De Felice,Tsujikawa JCAP 1202]

Horndeski Cosmology (perturbations)

- Scalar perturbations: $ds^2 = -(1+2\psi)dt^2 + a^2(1-2\phi)\delta_{ij}dx^i dx^j$ $\Rightarrow L.H.S = R.H.S$
- No-ghost condition: $Q_S \equiv \dfrac{w_1\left(4w_1w_3 + 9w_2^2\right)}{3w_2^2} > 0$
- No Laplacian instabilities condition: $c_S^2 \equiv \dfrac{3\left(2w_1^2w_2H - 4w_2^2w_4 + 4w_1w_2\dot{w}_1 - 2w_1^2\dot{w}_2\right) - 6w_1^2\left(\rho_m + p_m\right)}{w_1\left(4w_1w_3 + 9w_2^2\right)} > 0$

with $\mathrm{w}_1 \equiv 2\left(G_4 - 2XG_{4,X}\right) - 2X\left(G_{5,X}\dot{\phi}H - G_{5,\phi}\right)$

$$\mathrm{w}_2 \equiv -2G_{3,X}X\dot{\phi} + 4G_4H - 16X^2G_{4,XX}H + 4\left(\dot{\phi}G_{4,\phi X} - 4HG_{4,X}\right)X + 2G_{4,\phi}\dot{\phi}$$
$$+ 8X^2G_{5,\phi X}H + 2HX\left(6G_{5,\phi} - 5HG_{5,X}\dot{\phi}\right) - 4G_{5,XX}\dot{\phi}X^2H^2$$

$$\mathrm{w}_3 \equiv 3X\left(K_{,X} + 2XK_{,XX}\right) + 6X\left(3X\dot{\phi}HG_{3,XX} - G_{3,\phi X}X - G_{3,\phi} + 6\dot{\phi}HG_{3,X}\right)$$
$$+ 18H\left(4HX^3G_{4,XXX} - HG_4 - 5X\dot{\phi}G_{4,\phi X} - G_{4,\phi}\dot{\phi} + 7HG_{4,X}X + 16HX^2G_{4,XX} - 2X^2\dot{\phi}G_{4,X\phi X}\right)$$
$$+ 6H^2X\left(2H\dot{\phi}G_{5,XXX}X^2 - 6X^2G_{5,\phi XX} + 13XH\dot{\phi}G_{5,XX} - 27G_{5,\phi X}X + 15H\dot{\phi}G_{5,X} - 18G_{5,\phi}\right)$$

$$\mathrm{w}_4 \equiv 2G_4 - 2XG_{5,\phi} - 2XG_{5,X}\ddot{\phi}$$

Inflation: scalar field

$$\mathrm{L} = \frac{1}{2}\partial_\mu \varphi \, \partial^\mu \varphi - \mathrm{V}(\varphi)$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \,,$$

$$H^2 = \frac{8\pi}{3m_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

Inflation: scalar field

$$\mathrm{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \mathrm{V}(\varphi)$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,$$

$$H^2 = \frac{8\pi}{3m_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

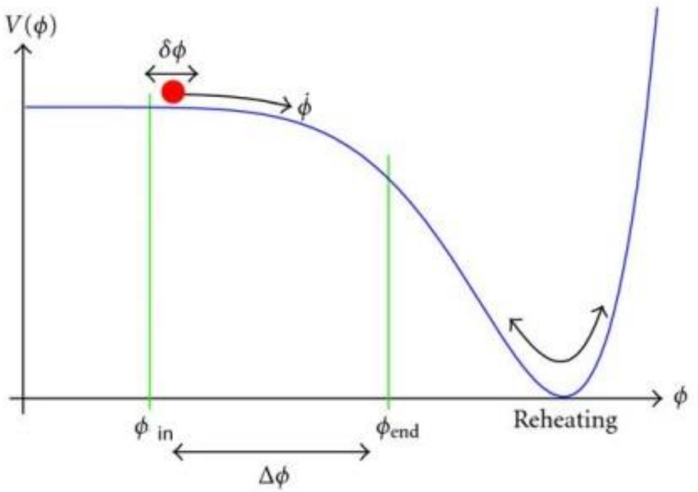
$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

- Slow-roll conditions: $\dot{\phi}^2/2 \ll V(\phi)$ and $|\ddot{\phi}| \ll 3H|\dot{\phi}|$

$$H^2 \simeq \frac{8\pi V(\phi)}{3m_{\rm pl}^2}\,,$$

$$3H\dot{\phi} \simeq -V_\phi(\phi)$$

$$N \equiv \ln\frac{a_f}{a} = \int_t^{t_f} H\mathrm{d}t \simeq \frac{8\pi}{m_{\rm pl}^2}\int_{\phi_f}^{\phi}\frac{V}{V_\phi}\mathrm{d}\phi$$



Primordial Spectra

The results for the power spectra of the scalar and tensor fluctuations created by inflation are

$$\Delta_{\rm s}^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \left. \frac{1}{8\pi^2} \frac{H^2}{M_{\rm pl}^2} \frac{1}{\varepsilon} \right|_{k=aH} , \tag{222}$$

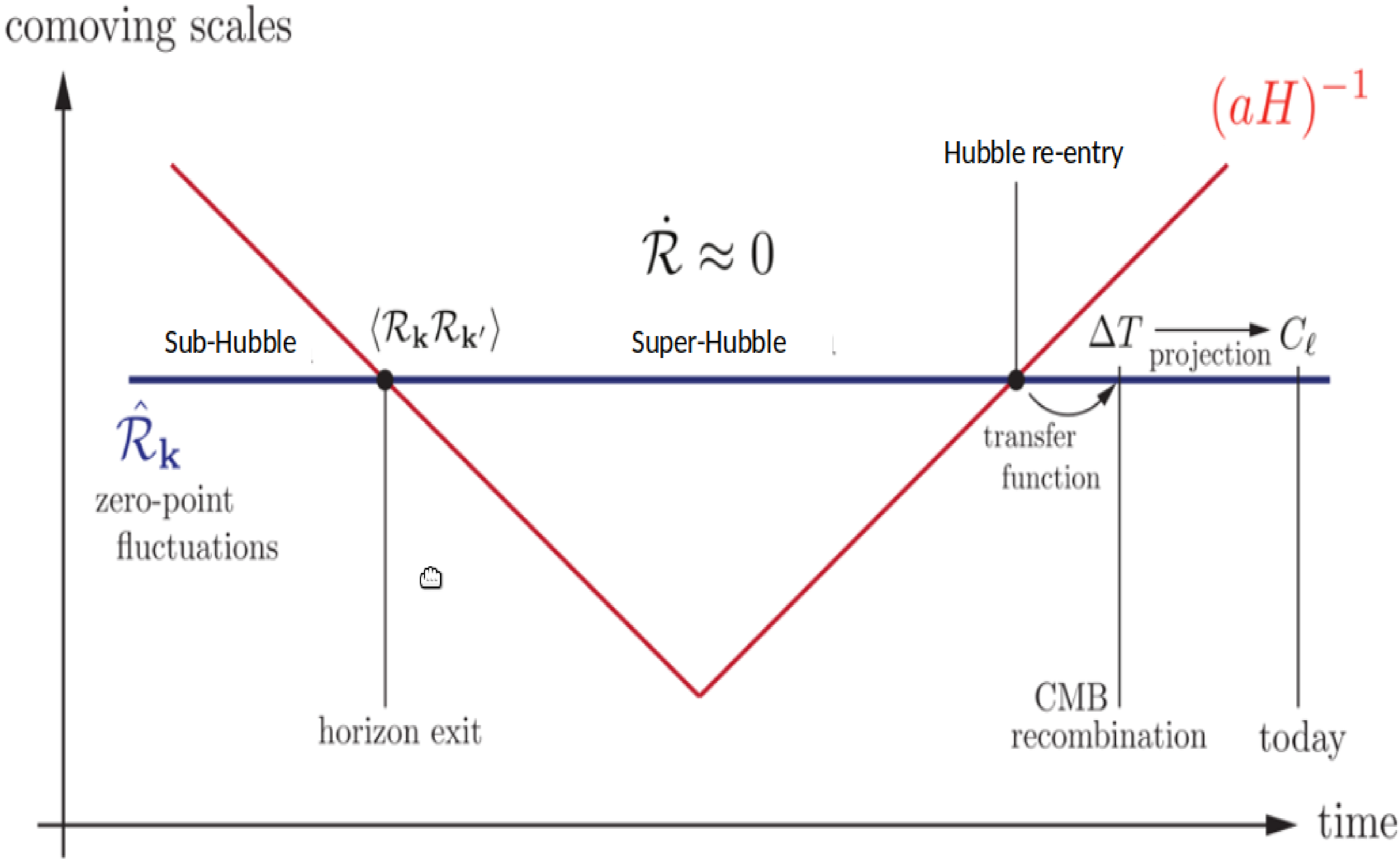
$$\Delta_{\rm t}^2(k) \equiv 2\Delta_h^2(k) = \left. \frac{2}{\pi^2} \frac{H^2}{M_{\rm pl}^2} \right|_{k=aH} , \tag{223}$$

where

$$\varepsilon = -\frac{d \ln H}{dN} . \tag{224}$$

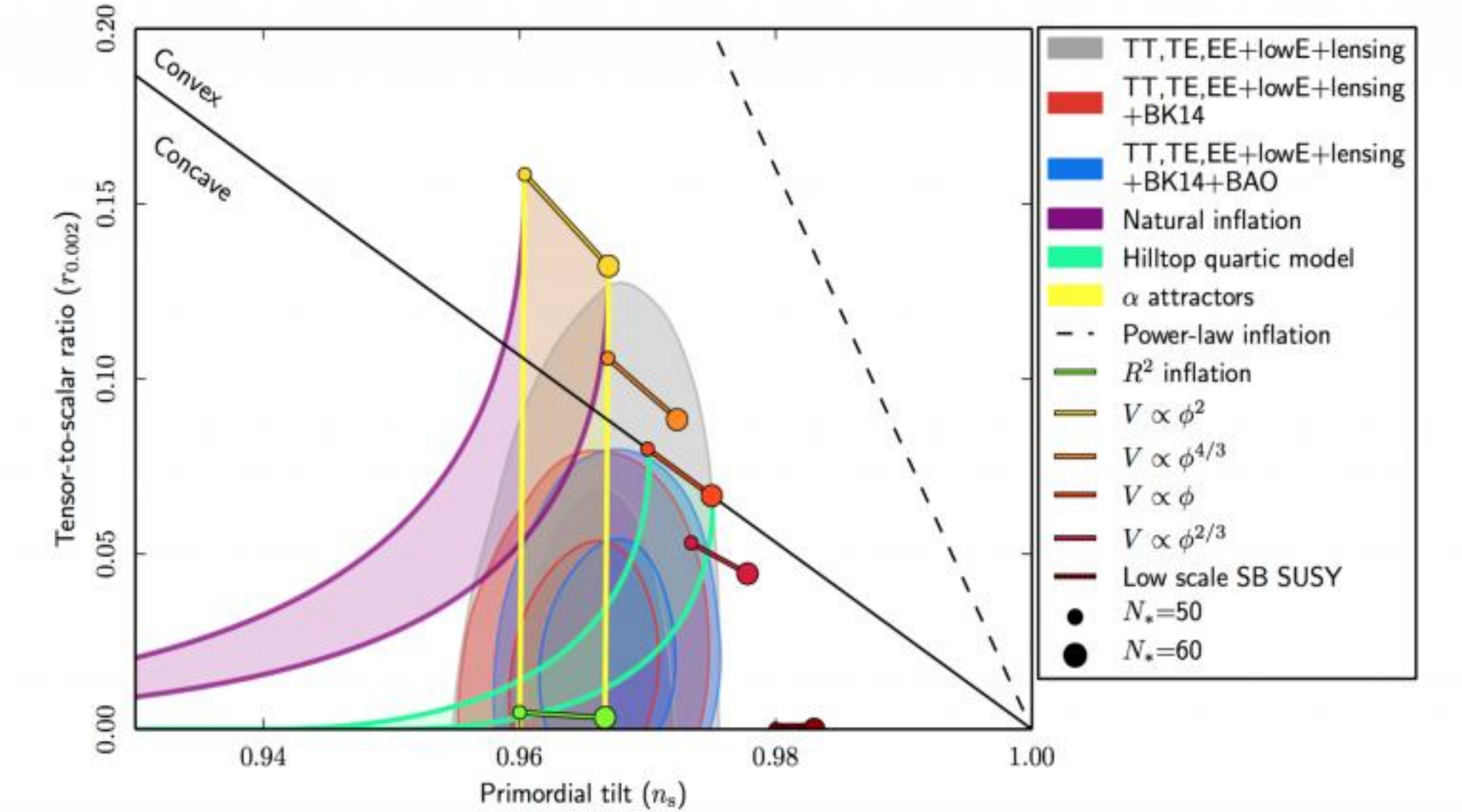
The horizon crossing condition $k = aH$ makes (222) and (223) functions of the comoving wavenumber k. The tensor-to-scalar ratio is

$$r \equiv \frac{\Delta_{\rm t}^2}{\Delta_{\rm s}^2} = 16\, \varepsilon_\star . \tag{225}$$



comoving scales
$(aH)^{-1}$
Hubble re-entry
$\dot{\mathcal{R}} \approx 0$
Sub-Hubble
$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle$
Super-Hubble
ΔT
projection
C_ℓ
$\hat{\mathcal{R}}_{\mathbf{k}}$
zero-point fluctuations
transfer function
horizon exit
CMB recombination
today
time

Simple Inflation models: problem



Inflation in Nonminimal Derivative Coupling

- $$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R-\frac{1}{2}(g_{\mu\nu}-\zeta G_{\mu\nu})\partial^\mu\phi\partial^\nu\phi-V(\phi)\right]+S_m+S_r$$

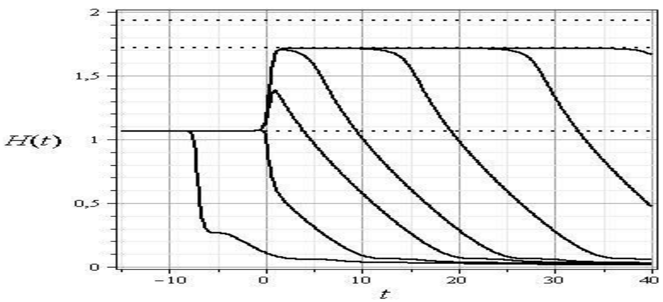
- In flat FRW:

$$H^2=\frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2}\left(1+9\zeta H^2\right)+V(\phi)+\rho_m+\rho_r\right]$$

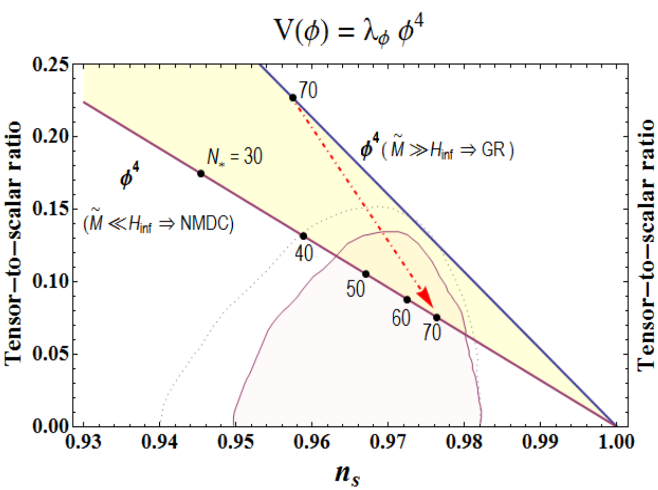
$$2\dot{H}+3H^2=-8\pi G\left[\frac{\dot{\phi}^2}{2}\left[1-\zeta\left(2\dot{H}+3H^2+\frac{4H\ddot{\phi}}{\dot{\phi}}\right)\right]-V(\phi)+p_m+p_r\right]$$

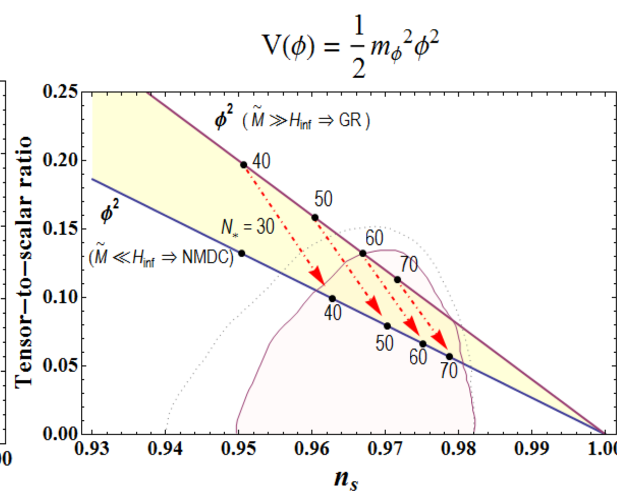
Inflation with Nonminimal Derivative Coupling

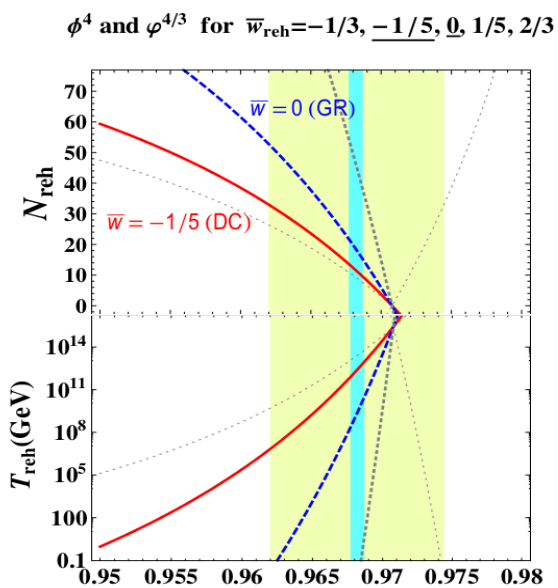
- New Higgs Inflation: $r \approx 0.05$



$$V(\phi) = V_0 \phi^2$$







Dark-Energy in Nonminimal Derivative Coupling

- $$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R-\frac{1}{2}(g_{\mu\nu}-\zeta G_{\mu\nu})\partial^\mu\phi\partial^\nu\phi-V(\phi)\right]+S_m+S_r$$
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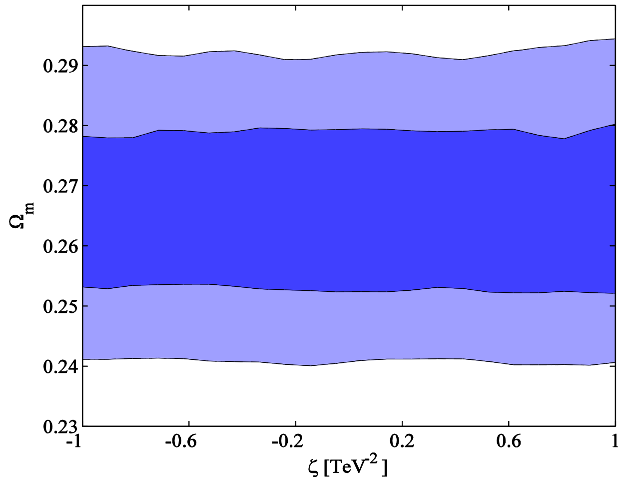
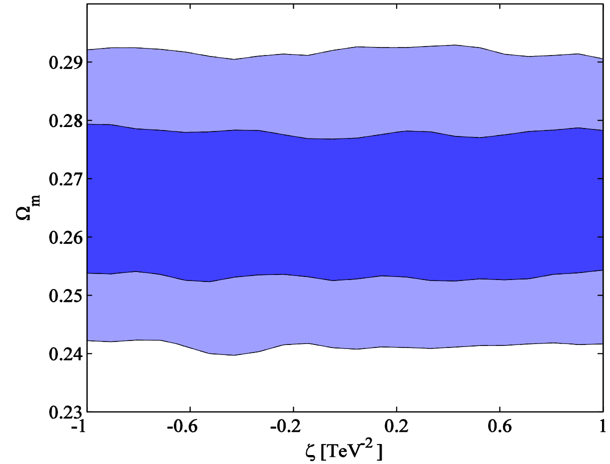
$$2\dot H+3H^2=-8\pi G\left[\frac{\dot\phi^2}{2}\left[1-\zeta\left(2\dot H+3H^2+\frac{4H\ddot\phi}{\dot\phi}\right)\right]-V(\phi)+p_m+p_r\right]$$



$V(\phi)=V_0e^{\lambda\phi}$

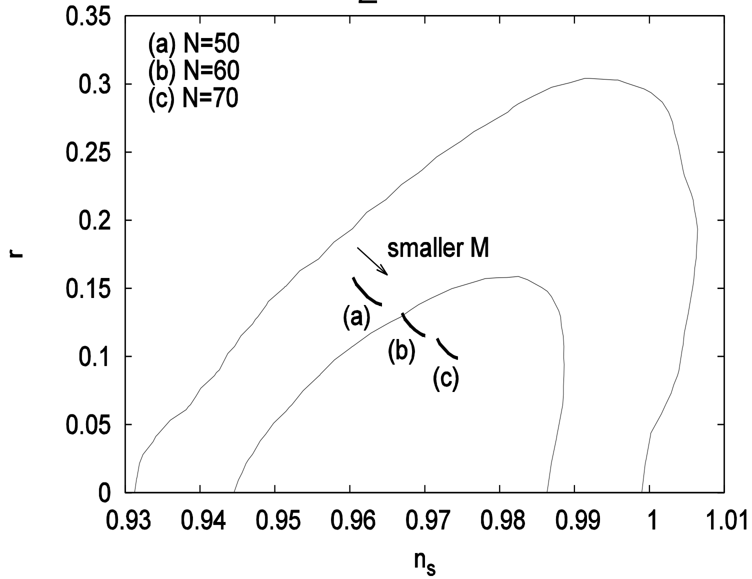


$V(\phi)=V_0\phi^n$

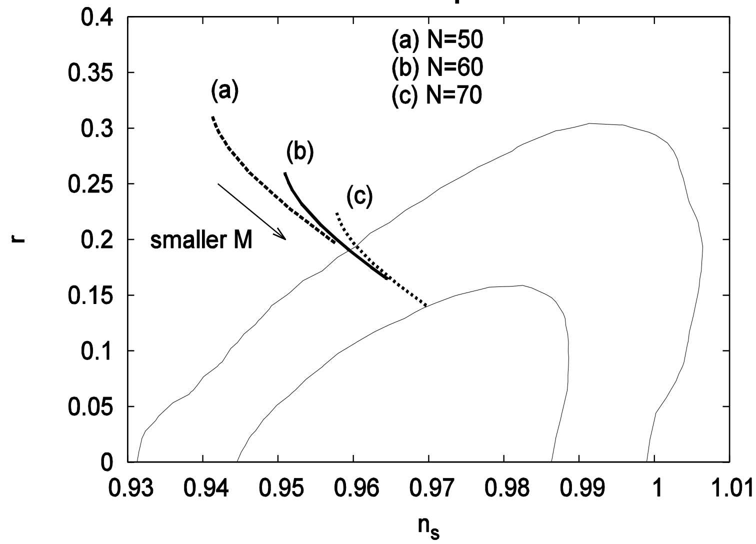
Inflation in Horndeski Theories

- $K(\phi, X) = X - V(\phi),\ G_3(\phi, X) = \frac{c_3}{M^3} X,\ G_4 = G_5 = 0$

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$



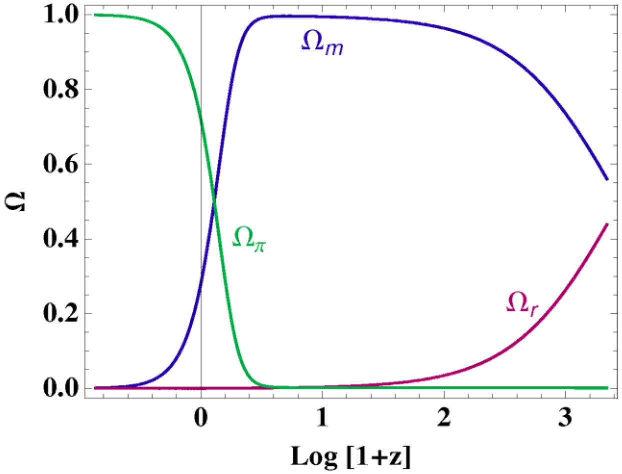
$$V(\phi) = \frac{1}{4} \lambda \phi^4$$



- G-Inflation (Shift-symmetric): $K(\phi, X) = X + \frac{X^2}{2M^3 \mu},\ G_3(\phi, X) = \frac{1}{M^3} X,\ G_4 = G_5 = 0$

Dark Energy in Horndeski Theories

$$K(\phi, X) = c_2 X, \; G_3(\phi, X) = c_3, \; G_4 = 1, \; G_5 = c_5$$



Dark Energy in Horndeski Theories

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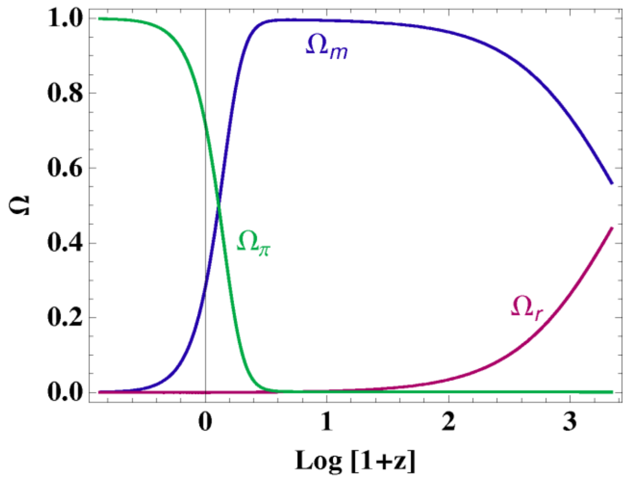


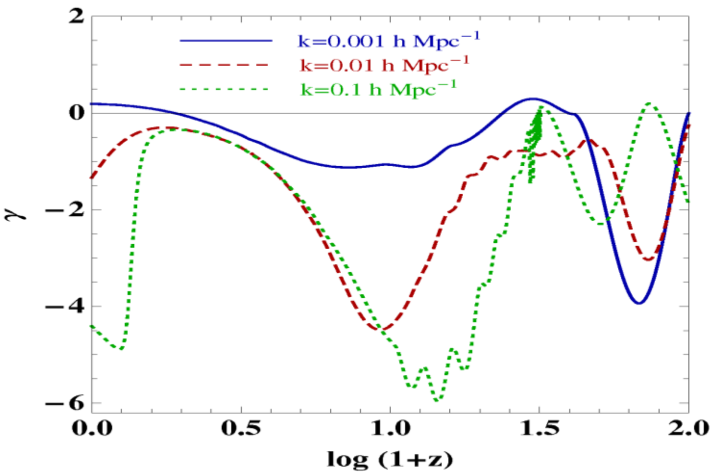
- Background evolution: Universe thermal history

- Perturbations: $\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{eff}\rho_m\delta_m$

  with $G_{eff} = G_{eff}(\phi, K, G_3, G_4, G_5)$

- Clustering growth rate: $\dfrac{d\ln\delta_m}{d\ln a} = \Omega_m^\gamma(a)$

  γ(z): Growth index.



f(R) gravity

- $$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, f(R) + S_m\left(g_{\mu\nu}, \psi\right)$$

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu \nabla_\mu - g_{\mu\nu}\Diamond\right] f'(R) = 8\pi G\, T_{\mu\nu}$$

- Field Equations (metric formalism):
- Conformal transformation: $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu} \equiv \phi g_{\mu\nu},\ d\varphi = \sqrt{\frac{2\omega_0 + 3}{16\pi G}}\frac{d\phi}{\phi}$

$$\Rightarrow_{\omega_0 = 0}\ S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\partial^\alpha\varphi\partial_\alpha\varphi - U(\varphi)\right] + S_m\left(e^{-\sqrt{16\pi G/3}}\tilde{g}_{\mu\nu}, \psi\right) \qquad U(\varphi) = \frac{Rf'(R) - f(R)}{16\pi G\left[f'(R)\right]^2}$$

Conformal transformation in f (R) gravity

In $f(R)$ gravity one can derive the action in the Einstein frame under the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

The Ricci scalars R and $\tilde{R}$ in the two frames have the following relation

$$R = \Omega^2(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega)$$

where $\quad \omega \equiv \ln\Omega\,, \quad \partial_\mu\omega \equiv \frac{\partial\omega}{\partial\tilde{x}^\mu}\,, \quad \tilde{\Box}\omega \equiv \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\omega)$

The action in $f(R)$ gravity $S = \int d^4x\sqrt{-g}f(R)/2 + \int d^4x\mathcal{L}_M$ can be written as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}FR - U\right) + \int d^4x\mathcal{L}_M(g_{\mu\nu}) \qquad \text{where} \quad U = \frac{FR - f}{2}$$

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}F\Omega^{-2}(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U\right] + \int d^4x\mathcal{L}_M(\Omega^{-2}\tilde{g}_{\mu\nu})$$

where we used $\quad \sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$

For the choice $\Omega^2 = F$ we obtain the action linear in $\tilde{R}$.

Introducing a field $\phi \equiv \sqrt{3/2}\,\ln F$, we have $\omega = \phi/\sqrt{6}$.

The Einstein frame action is

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + \int d^4x\mathcal{L}_M(F^{-1}(\phi)\tilde{g}_{\mu\nu}),$$

with the potential $\quad V(\phi) = \frac{U}{F^2} = \frac{FR - f}{2F^2}$

In the Einstein frame the field couples to non-relativistic matter (coupled quintessence).

f(R) cosmology - Inflation

- Firedmann Equations (metric formalism):

$$3FH^2 = \frac{FR - f}{2} - 3H\dot{F} + 8\pi G\,\rho_m$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + 8\pi G\left(\rho_m + p_m\right)$$

$$F(R) \equiv f'(R)$$

$$R = 12H^2 + 6\dot{H}$$

f(R) cosmology - Inflation

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$$F(R) \equiv f'(R)$$

$$R = 12H^2 + 6\dot{H}$$

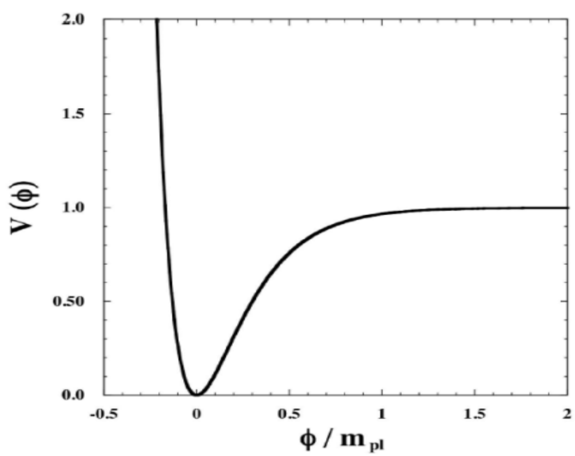
- Inflation: e.g. Starobinsky inflation

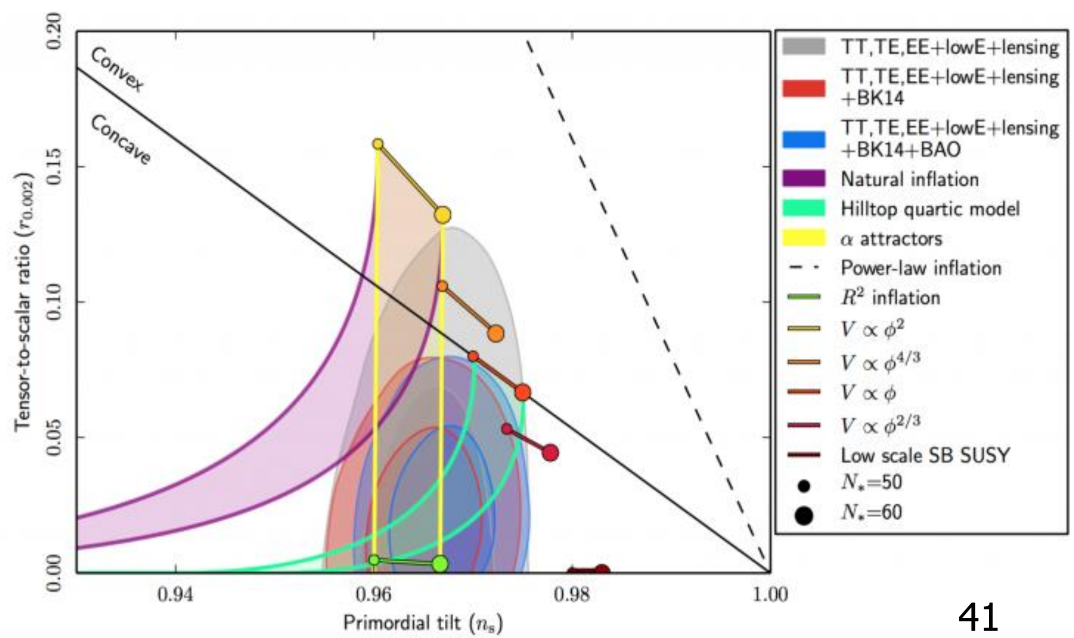
$$f(R) = R + \frac{R^2}{6M^2}$$

$$\Rightarrow V(\phi) = \frac{3M^2}{32\pi G}\left(1 - e^{-\sqrt{2/3}8\pi G\phi}\right)$$

$$H \approx H_i - \frac{M^2}{6}(t - t_i)$$

$$T_{reh} \le 3\times10^{17} g_*^{1/4}\left(\frac{M}{m_{pl}}\right)^{3/2} GeV \qquad M \approx 3\times10^{13}\ GeV$$





f(R) cosmology – Dark energy

$$8\pi G\,\rho_{DE} = \frac{FR-f}{2} - 3H\dot{F} + 3H^2(1-F)$$

$$8\pi G\,p_{DE} = \ddot{F} + 2H\dot{F} - \frac{FR-f}{2} - \left(3H^2 + 2\dot{H}\right)(1-F)$$

for viable: $f_{,R} > 0,\; f_{,RR} > 0,\; for\; R \geq R_0 (> 0)$

Matter perturbations in viable f(R) models

Under the sub-horizon approximation ($k \gg aH$, k is a comoving wavenumber), the matter perturbation δ_m satisfies

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\rm eff}\rho_m\delta_m \simeq 0$$

where $G_{\rm eff} = \frac{G}{F}\frac{1+4\frac{k^2}{a^2R}m}{1+3\frac{k^2}{a^2R}m}$ $\quad F = \frac{\partial f}{\partial R}$

$m = Rf_{,RR}/f_{,R}$ is the deviation parameter from the LCDM.

$(m = 0 \quad \text{for} \quad f = R - 2\Lambda)$

f(R) cosmology – Dark energy

$$8\pi G\,\rho_{DE} = \frac{FR - f}{2} - 3H\dot{F} + 3H^2(1-F)$$

$$8\pi G\,p_{DE} = \ddot{F} + 2H\dot{F} - \frac{FR - f}{2} - \left(3H^2 + 2\dot{H}\right)(1-F)$$

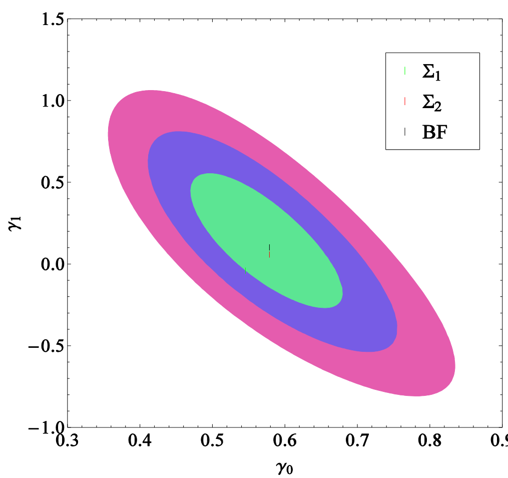
for viable: $f_{,R} > 0,\ f_{,RR} > 0,\ for\ R \geq R_0(>0)$

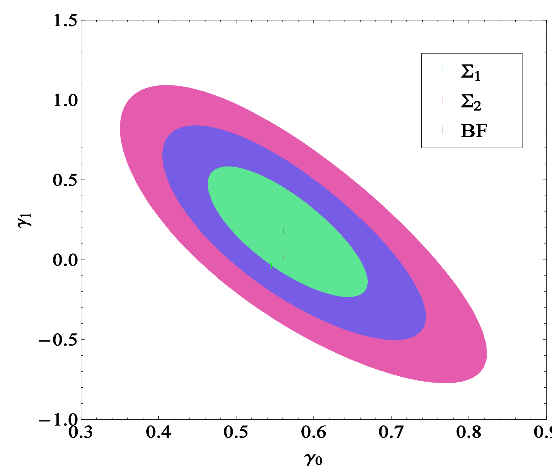
model	$f(R)$	Constant parameters
(i) Hu-Sawicki	$R - \frac{c_1 R_{HS}(R/R_{HS})^p}{c_2(R/R_{HS})^p+1}$	$c_1,\ c_2,\ p(>0), R_{HS}(>0)$
(ii) Starobinsky	$R + \lambda R_S\left[\left(1+\frac{R^2}{R_S^2}\right)^{-n} - 1\right]$	$\lambda(>0),\ n(>0),\ R_S$
(iii) Tsujikawa	$R - \mu R_T \tanh\left(\frac{R}{R_T}\right)$	$\mu(>0),\ R_T(>0)$
(iv) Exponential	$R - \beta R_E\left(1 - e^{-R/R_E}\right)$	$\beta,\ R_E$

- $\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{eff}\rho_m\delta_m = 0$

$$G_{eff} = \frac{G}{f'}\frac{1 + 4\frac{k^2}{a^2}\frac{f''}{f'}}{1 + 3\frac{k^2}{a^2}\frac{f''}{f'}}$$

$$\frac{d\ln\delta_m}{d\ln a} = \Omega_m^{\gamma}(a)$$

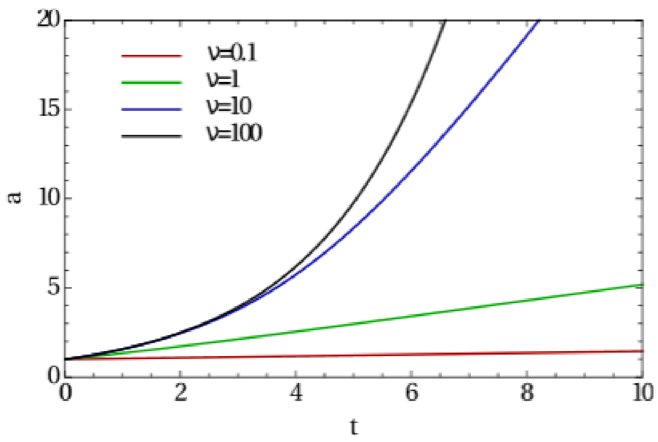


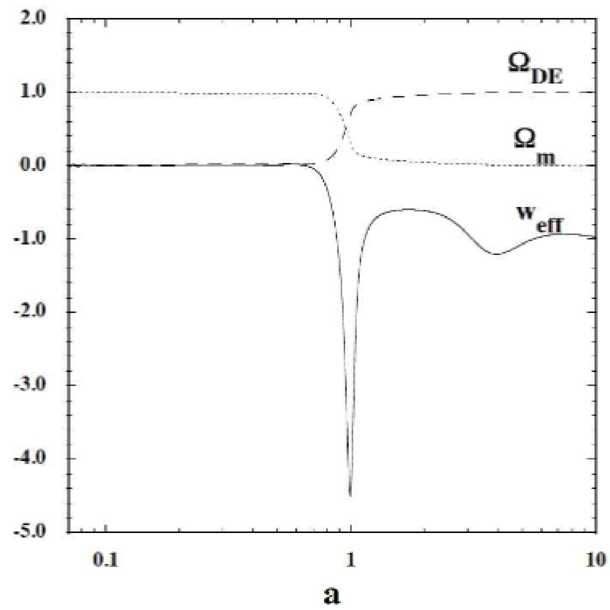


f(G) Theories

- Gauss-Bonnet Invariant: $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

$$S = \int d^4x\sqrt{-g}\,\left[\tfrac{1}{2}R + f(\mathcal{G})\right] + S_m(g_{\mu\nu}, \Psi_m)$$





Bi-scalar Theories

- Modified gravity propagating 2+2 dof's $$S = \int d^4x \sqrt{-g}\, f\left(R, (\nabla R)^2, \Diamond R\right)$$

- For $f\left(R, (\nabla R)^2, \Diamond R\right) = K\left(R, (\nabla R)^2\right) + Q\left(R, (\nabla R)^2\right)\Diamond R$

$$\Rightarrow S = \int d^4x \sqrt{-g}\left[\frac{1}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{\sqrt{6}}e^{-\sqrt{2/3}\,\chi}\hat{g}^{\mu\nu}Q\partial_\mu\chi\partial_\nu\phi + \frac{1}{4}e^{-2\sqrt{2/3}\,\chi}K + \frac{1}{2}e^{-\sqrt{2/3}\,\chi}Q\hat{\Diamond}\phi - \frac{1}{4}e^{-\sqrt{2/3}\,\chi}\phi\right]$$

$$K = K(\phi, B),\; G = G(\phi, B),\; B = 2e^{\sqrt{2/3}\,\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

Bi-scalar Theories

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- For $f\left(R,(\nabla R)^2,\Diamond R\right)=K\left(R,(\nabla R)^2\right)+Q\left(R,(\nabla R)^2\right)\Diamond R$

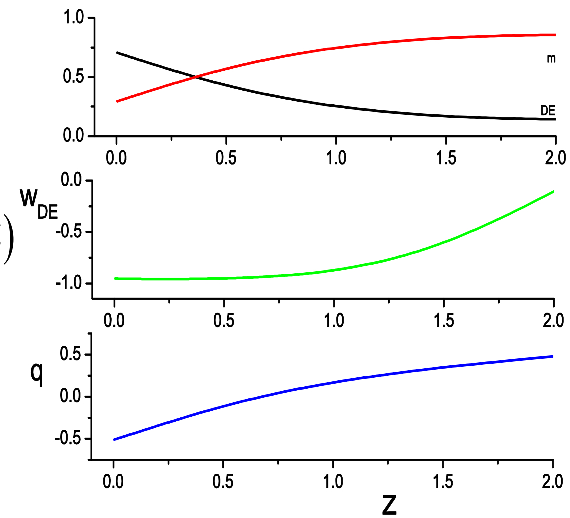
$$\Rightarrow S=\int d^4x\sqrt{-g}\left[\frac{1}{2}\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi-\frac{1}{\sqrt{6}}e^{-\sqrt{2/3}\chi}\hat{g}^{\mu\nu}Q\partial_\mu\chi\partial_\nu\phi+\frac{1}{4}e^{-2\sqrt{2/3}\chi}K+\frac{1}{2}e^{-\sqrt{2/3}\chi}Q\hat{\Diamond}\phi-\frac{1}{4}e^{-\sqrt{2/3}\chi}\phi\right]$$

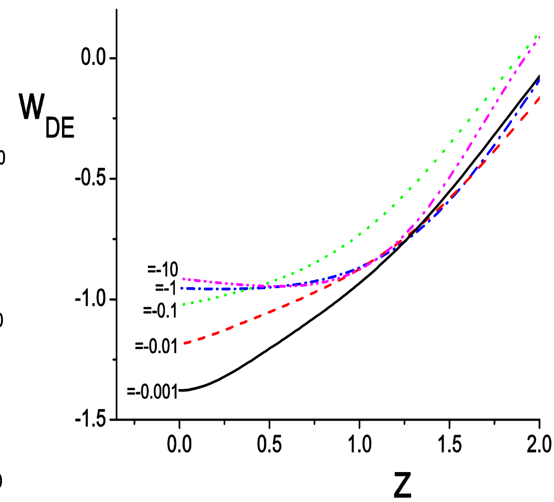
$$K=K(\phi,B),\ G=G(\phi,B),\ B=2e^{\sqrt{2/3}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

- eg.: $K(\phi,B)=\frac{\phi}{2},\ G(\phi,B)=\xi B$

$$\rho_{DE}=\frac{1}{2}\dot{\chi}^2-\frac{1}{8}e^{-2\sqrt{2/3}\chi}\left(1-2e^{\sqrt{2/3}\chi}\right)\phi-\xi\dot{\phi}^3\left(\sqrt{6}\dot{\chi}-6H\right)$$

$$p_{DE}=\frac{1}{2}\dot{\chi}^2+\frac{1}{8}e^{-2\sqrt{2/3}\chi}\left(1-2e^{\sqrt{2/3}\chi}\right)\phi-\frac{1}{3}\xi\dot{\phi}^2\left(\sqrt{6}\dot{\phi}\dot{\chi}+6\ddot{\phi}\right)$$





MASSIVE GRAVITY

Introduction

- Massive Gravity, i.e adding mass to a spin-2 particle, goes back to 1939
- Motivation: i) Theoretical (we know the answer for scalars and vectors)

  ii) Cosmological (explain acceleration)
- Indeed it is the most reasonable modified gravity

  (not the simplest one, since you add 3 dof's)
- It is promising, but…

  [Hinterbichler, Rev.Mod.Phys.84]

Introduction

- 1939: Fierz and Pauli add a linear mass-term to GR $\propto m^2(h_{\mu\nu} - h^2)$
- 1970: van Dam, Veltman, Zakharov: When the linear theory couples to a source, the limit $\boxed{m \to 0}$ does not give GR

  (vDVZ discontinuity)

- 1972: Vainstein: The non-linearities become stronger and stronger as m decreases. They must be taken into account and they do cure vDVZ discontinuity
- 1972: Boulware, Deser: Nonlinearities bring a ghost!

Introduction

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- 1972: Vainstein: The non-linearities become stronger and stronger as m decreases. They must be taken into account and they do cure vDVZ discontinuity
- 1972: Boulware, Deser: Nonlinearities bring a ghost!
- 2010: de Rham, Gabadadze, Tolley: Adding higher-order graviton self-interaction systematically removes the BD ghost
- 2011 and on: The cosmology has severe problems.

Fierz-Pauli linear theory

- Put source $T^{\mu\nu}$ with coupling $\kappa\, h_{\mu\nu} T^{\mu\nu}$. Eoms':

$$\Diamond h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_\nu - \partial_\lambda \partial_\nu h^\lambda_\mu + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Diamond h \boxed{- m^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h^2 \right)} = -\kappa\, T_{\mu\nu}$$

- Note: For $m = 0 \Rightarrow \partial^\mu T_{\mu\nu} = 0$ (conservation)

  For $m \neq 0$ no such condition (but we assume it, otherwise obvious discontinuity)

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  For $m \neq 0$ no such condition (but we assume it otherwise, obvious discontinuity)

- GR is NOT recovered in the massless limit (vDVZ discontinuity)

Massless gravity: 2 spin states

2 helicity states of a massless graviton

Massive gravity: 5 spin states

2 helicity states of a massless graviton

2 helicity states of a massless vector

1 single massive scalar

no 6[th] dof since the time components h_{00} appear as Lagr. multiplier

- The scalar (longitudinal graviton) maintains a coupling to T even in the massless limit
- I.e, the massless limit does not describe a massless graviton, but a massless graviton plus a coupled scalar
- The gauge symmetry of GR, that kills the extra dof appears ONLY for $m = 0$ and NOT for $m \to 0$

[van Dam, Veltman 1970], [Zakharov 1970]

Nonlinear theory and the BD ghost

- Nonlinearities become stronger as $m \to 0$, need to be taken into account.

$$S = \frac{1}{2\kappa^2}\underbrace{\int d^4x \ \ [\sqrt{-g}R}_{\text{Full nonlinear EH action}} \underbrace{-\sqrt{-g^{(0)}}\frac{1}{4}m^2 g^{(0)\mu\alpha}g^{(0)\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta}-h_{\mu\alpha}h_{\nu\beta}\right)\ ]}_{\text{Fierz-Pauli mass term}}$$

Full nonlinear EH action

Fierz-Pauli mass term

$g^{(0)}_{\mu\nu}$ the fixed metric on which the massive graviton propagates

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Full nonlinear EH action

Fierz-Pauli mass term

$g^{(0)}_{\mu\nu}$ the fixed metric on which the massive graviton propagates

- The nonlinearities re-bring the 6th dof (no Lagrange multiplier anymore)
- The Hamiltonian constraint analysis shows that it is a ghost!  [Boulware,Deser 1972]

- But this ghost cures the vDVZ discontinuity! (it provides a repulsive force that counteracts the attractive force of the longitudinal scalar mode) [Vainstein 1972]

- But it could still make sense, if quantum effects push the ghost above a cutoff Λ, and see the whole story as an effective theory [Arkani-Hamed, Georgi, Schwartz 2002]

dRGT nonlinear massive gravity

- The 6^{th} dof (ghost) survives since the lapse function N is not a Lagrange multiplier in the nonlinear case, as it was in the linear one.
- Idea: Specially design nonlinear terms, so that N becomes again a Lagrange multiplier

dRGT nonlinear massive gravity

- Finally:

$$S_{MG} = M_p^2 \int d^4x \sqrt{-g} \left[\frac{R}{2} + m_g^2 \left(L_2 + \alpha_3 L_3 + \alpha_4 L_4 \right) \right]$$

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2] \right)$$

$$L_3 = \frac{1}{6}\left([K]^3 - 3[K][K^2] + 2[K^3] \right)$$

$$L_4 = \frac{1}{24}\left([K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4] \right)$$

$$[K] = tr\left(K_\mu^\nu\right)$$

$$K_\nu^\mu \equiv \delta_\nu^\mu - \sqrt{g^{\mu\sigma} f_{ab}(\phi) \partial_\nu \phi^a \partial_\sigma \phi^b}$$

[de Rham, Gabadadze, PRD 82],
[de Rham, Gabadadze, Tolley PRL 106]

fiducial metric (f_{ab}) — Stückelberg fields (ϕ^b)

- **Free of BD ghost! Free of vDVZ discontinuity!**
- Vainstein mechanism: extra dof's are suppressed at small scales due to non-linearities

Cosmological applications

- Simplest Example: Physical metric: flat FRW: $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$

  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$

  Stückelberg scalars: $\phi^0 = b(t),\ \phi^i = x^i$

Variation wrt ϕ : $m^2\partial_0(a^3 - a^2) = 0 \Rightarrow \dot{a} = 0$ NO nontrivial solution (same for closed)

[dRGT et al, PRD 84]

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  [dRGT et al, PRD 84]

- Next: Physical metric open FRW: $ds^2 = -N^2dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2 - \frac{|K|(xdx + ydy + zdz)^2}{1+|K|(x^2+y^2+z^2)}\right]$

  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$

  Stückelberg scalars: $\phi^0 = b(t)\sqrt{1+|K|(x^2+y^2+z^2)},\ \phi^i = \sqrt{|K|}\,b(t)x^i$

  Variation wrt ϕ gives a constraint for b(t): $\frac{b(t)}{a(t)} = \frac{X_\pm}{\sqrt{|K|}} = const.,\ X_\pm = \frac{1+2\alpha_3+\alpha_4 \pm \sqrt{1+\alpha_3+\alpha_3^2-\alpha_4}}{\alpha_3+\alpha_4}$

- Friedmann equations:

$$3H^2 - 3\frac{|K|}{a^2} = \rho_m + m_g^2 c_\pm$$

$$-2\dot{H} - 2\frac{|K|}{a^2} = \rho_m + p_m$$

We get an Effective Cosmological Constant: $\Lambda_\pm = m_g^2 c_\pm(\alpha_3, \alpha_4)$

Self-acceleration for $c_\pm(\alpha_3, \alpha_4) > 0$

[Gumrukcuoglu, Lin, Mukohyama, JCAP1111]

Perturbations

- Unfortunately, there is ALWAYS a ghost instability (it's frequency tends to vanish at low scales so it always remain in the low-energy effective theory)

- This instability is related to the FRW structure of the physical metric, and in particular from the high symmetries (isotropy).

  [Gumrukcuoglu, Lin, Mukohyama, JCAP1203]

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- Unfortunately, there is ALWAYS a ghost instability (it's frequency tends to vanish at low scales so it always remain in the low-energy effective theory)

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[Gumrukcuoglu, Lin, Mukohyama, JCAP1203]

- In order to construct a healthy model we must insert anisotropies:

Physical metric: axisymmetric Bianchi I: $ds^2 = -N^2 dt^2 + a(t)^2\left(e^{4\sigma(t)}dx^2 + e^{-2\sigma(t)}dy^2 + e^{-2\sigma(t)}dz^2\right)$

Fiducial metric: FRW: as before

Stückelberg scalars: as before

$$\Longrightarrow \rho_{MG}(t) = \cdots$$

[Gumrukcuoglu, Lin, Mukohyama, PLB717]

- The only healthy model. Disadvantage: There is NO isotropic limit!

Extension 1: Varying mass massive gravity

- Need to find extensions of nonlinear massive gravity where FRW solutions are stable.

$$S_{MG} = M_p^2 \int d^4x \sqrt{-g} \left[\frac{R}{2} + V(\psi)\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right) - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - W(\psi) \right]$$

[Huang, Piao, Zhou PRD86]

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  Stückelberg scalars: $\phi^0 = b(t), \;\; \phi^i = a_{ref} x^i$

$$\Rightarrow 3M_p^2 H^2 = \rho_m + \rho_{MG}$$

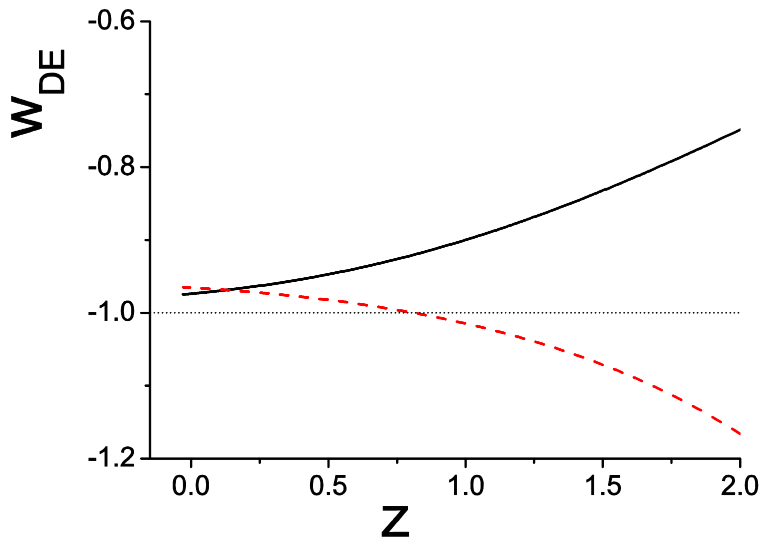
$$-2M_p^2 \dot{H} = \rho_m + p_m + \rho_{MG} + p_{MG}$$

$$\rho_{MG} = \frac{1}{2}\dot{\psi}^2 + W(\psi) + V(\psi)\left(\frac{a_{ref}}{a} - 1\right)\left[f_3(a) + f_1(a)\right]$$

$$p_{MG} = \frac{1}{2}\dot{\psi}^2 - W(\psi) - V(\psi)\left[f_4(a) + \dot{b} f_1(a)\right]$$

$$w_{DE} = \frac{p_{MG}}{\rho_{MG}}$$

$$\rho_{MG} + p_{MG} = \dot{\psi}^2 - V(\psi)\left(\dot{b} - \frac{a_{ref}}{a}\right) f_1(a)$$



[Saridakis CQG 30]

Extension 2: Quasi-dilaton massive gravity

$$S_{MG} = M_p^2 \int d^4x \sqrt{-g}\left[\frac{R}{2} + m_g^2\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right) - \frac{\omega}{2M_p^2} g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]$$

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2]\right)$$

$$L_3 = \frac{1}{6}\left([K]^3 - 3[K][K^2] + 2[K^3]\right)$$

$$L_4 = \frac{1}{24}\left([K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4]\right)$$

$$[K] = tr\left(K_\mu^\nu\right)$$

$$K_\nu^\mu \equiv \delta_\nu^\mu - e^{\sigma/M_p}\sqrt{g^{\mu\sigma}\eta_{ab}(\phi)\partial_\nu\phi^a\partial_\sigma\phi^b}$$

quasi-dilaton (e^{σ/M_p}) fiducial metric (η_{ab}) Stückelberg fields (ϕ^b)

[D'Amico, Gabadadze, Hui, Pirtskhalava PRD 87]

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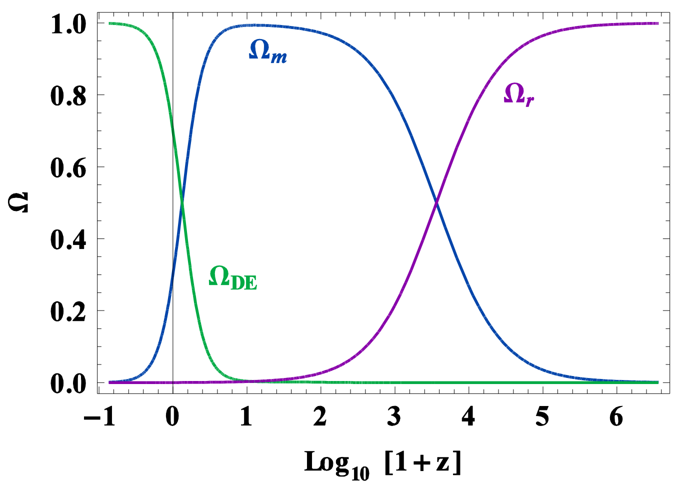
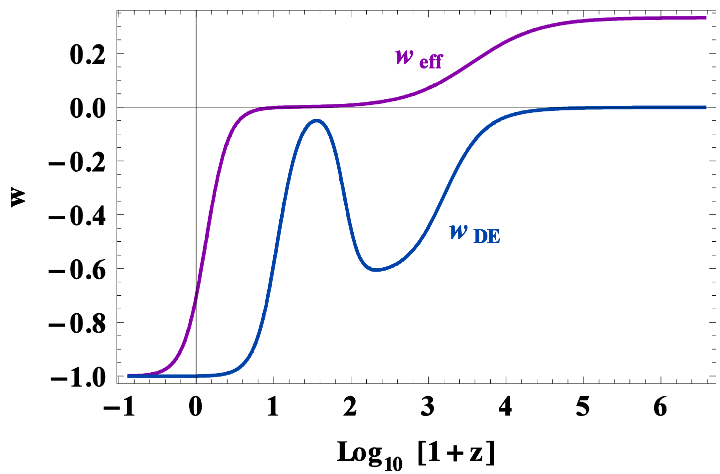
  Stückelberg scalars: $\phi^0 = b(t), \;\; \phi^i = x^i$

$$\Rightarrow 3M_p^2 H^2 = \rho_m + \rho_r + \rho_{DE}$$

$$\rho_{DE} = \frac{\omega}{2}\dot{\psi}^2 - 3M_p^2 m_g^2 \left[(2+\alpha_3+\alpha_4) - \left(3+\frac{9}{4}\alpha_3+3\alpha_4\right)\frac{e^{\frac{\sigma}{M_p}}}{a} + \left(1+\frac{3}{2}\alpha_3+3\alpha_4\right)\frac{e^{\frac{2\sigma}{M_p}}}{a^2} - \frac{1}{4}(\alpha_3+4\alpha_4)\frac{e^{\frac{3\sigma}{M_p}}}{a^3} \right]$$

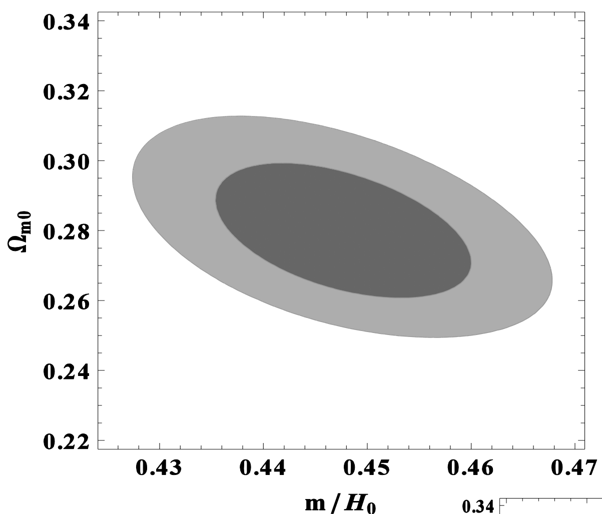


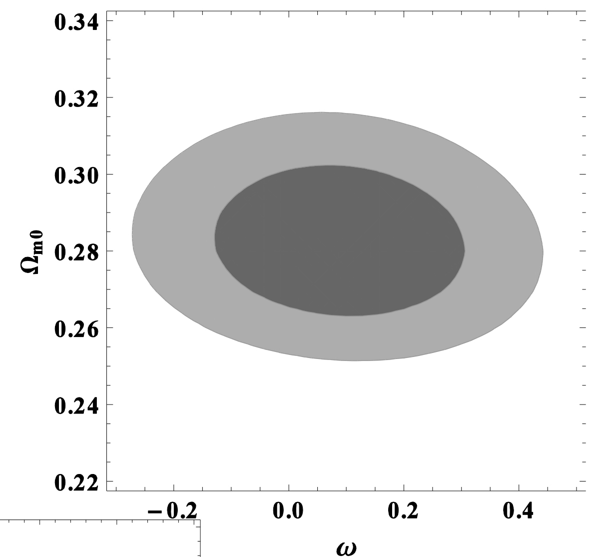


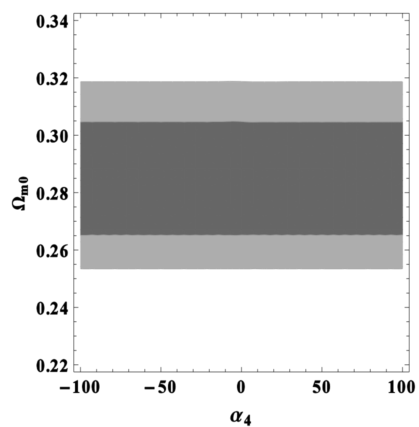
[Gannouji, Hossain, Sami, Saridakis PRD 87]

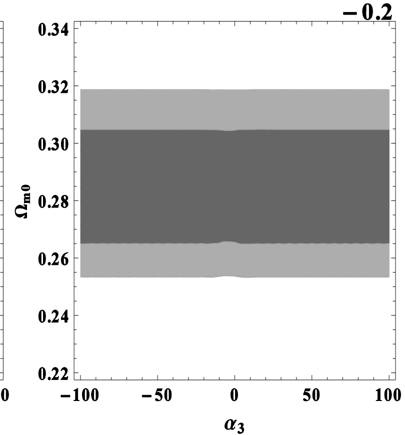
Observational constraints on quasi-dilaton massive gravity

- Use observational data (SNIa, BAO, CMB) to constrain the parameters of the theory. We fit $\Omega_{M0}, \Omega_{DE0}, m_g, \omega, \alpha_3, \alpha_4$









[Gannouji, Hossain, Sami, Saridakis PRD 87]

Extension 3: F(R) nonlinear massive gravity

$$S = M_p^2 \int d^4x \sqrt{-g}\left[\frac{F(R)}{2} + m_g^2\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right)\right]$$

UV modification

IR modification

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2]\right)$$

$$L_3 = \frac{1}{6}\left([K]^3 - 3[K][K^2] + 2[K^3]\right)$$

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$$K_\nu^\mu \equiv \delta_\nu^\mu - \sqrt{g^{\mu\sigma} f_{ab}(\phi)\partial_\nu \phi^a \partial_\sigma \phi^b}$$

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Extension 3: F(R) nonlinear massive gravity

- Einstein frame: $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega^2 = F_{,R} = \exp\left(\sqrt{\frac{2}{3}}\frac{\varphi}{M_p}\right)$

$$S = \int d^4x \sqrt{-g}\left[M_p^2 \frac{\tilde{R}}{2} + M_p^2 m_g^2 \left(\tilde{L}_2 + \alpha_3 \tilde{L}_3 + \alpha_4 \tilde{L}_4\right) - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi)\right]$$

with $U(\varphi) = M_p^2 \frac{RF_{,R} - F}{2F_{,R}^2}$

- Hamiltonian constraint analysis: the BD ghost is removed similar to usual nonlinear massive gravity

- Much more general than other massive gravity extensions.

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- Physical metric: open FRW: $ds^2 = -N^2 dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2 - \frac{|K|(xdx + ydy + zdz)^2}{1+|K|(x^2+y^2+z^2)}\right]$
  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$
  Stückelberg scalars: $\phi^0 = b(t)\sqrt{1+|K|(x^2+y^2+z^2)},\ \phi^i = \sqrt{|K|}b(t)x^i$

- Variation wrt b provides the constraint equation with solution: $\frac{b(t)}{a(t)} = const.$

$$3M_p^2\left(H^2 - \frac{|K|}{a^2}\right) = \rho_m + \rho_{MG} + \rho_{F_R}$$

$$\left.\begin{aligned} \rho_{MG} &= m_g^2 c_\pm \\ \rho_{F_{,R}} &= M_p^2\left[\frac{RF_{,R} - F}{2} - 3H\dot{R}F_{,RR}\right] \end{aligned}\right\}$$

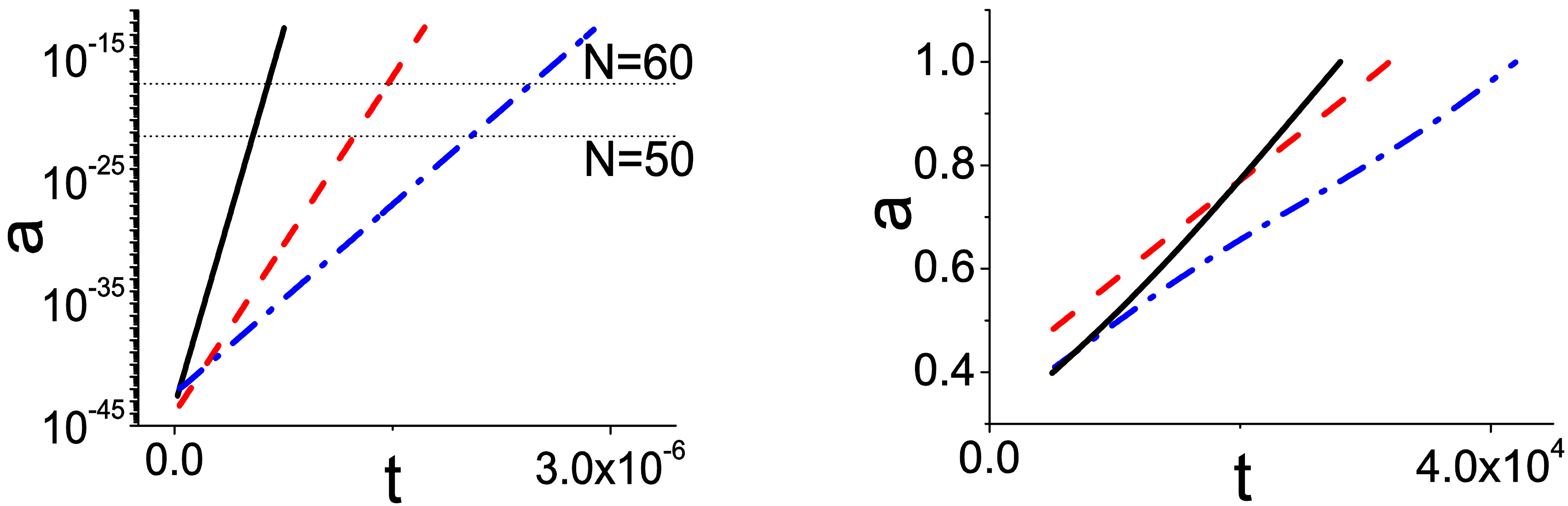
$$\boxed{\rho_{DE} \equiv \rho_{MG} + \rho_{F_R}}$$

- Both IR and UV gravity modifications play a role in universe evolution.
- Huge capabilities.

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- 1) $$F(R) = R + \frac{\xi}{M_p^2} R^2$$

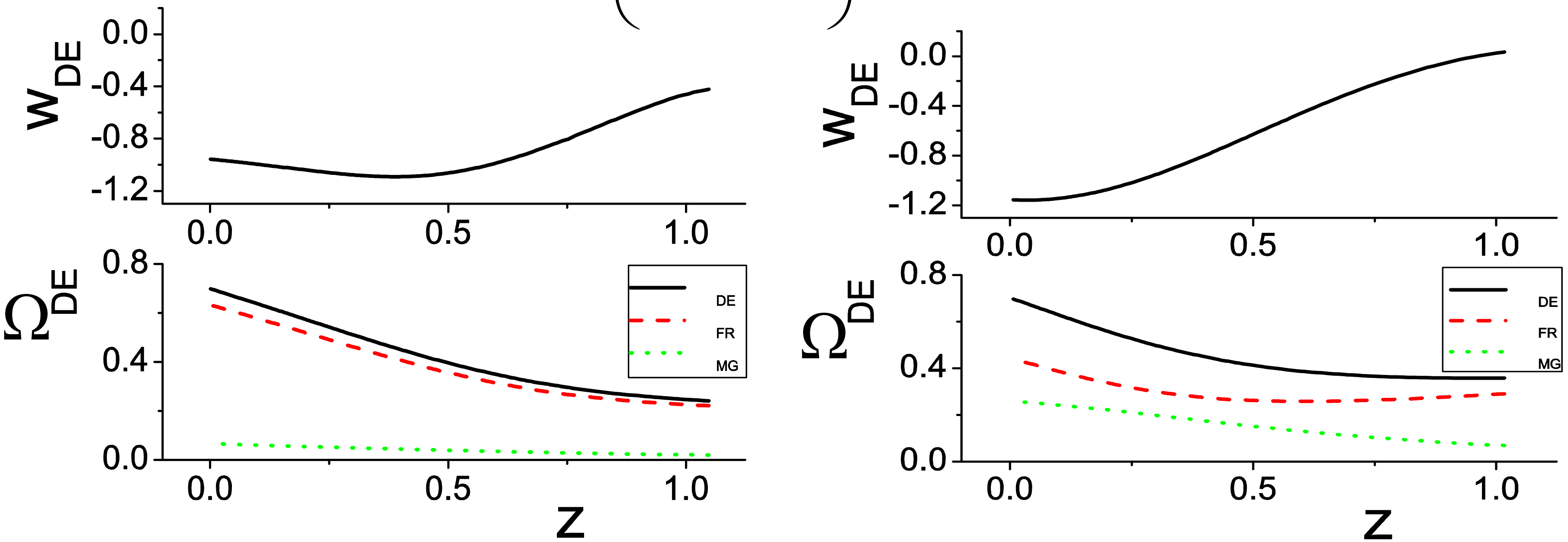


- Early times: F(R) sector drives inflation
- Late times: MG sector drives late-time acceleration

[Cai, Duplessis, Saridakis PRD 90a]

Cosmology of F(R) nonlinear massive gravity

- 2) $$F(R) = R - \beta R_S\left(1 - e^{-\frac{R}{R_S}}\right)$$

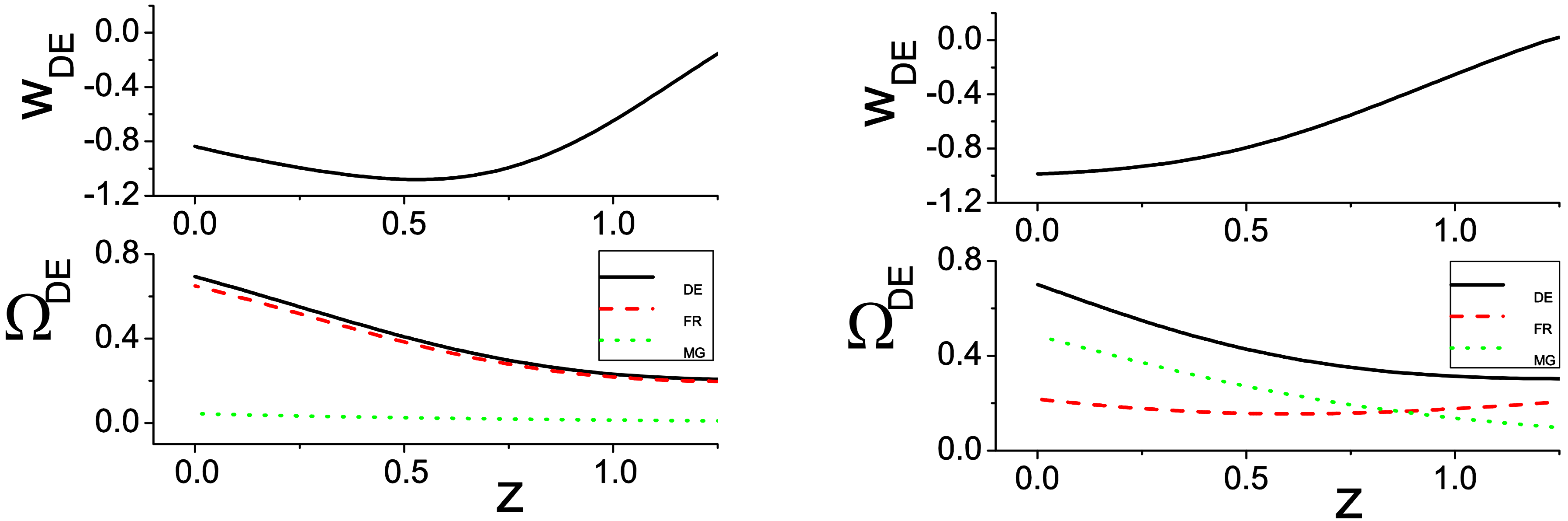


- Both F(R) sector and MG sector constitute Dark Energy $\rho_{DE} \equiv \rho_{MG} + \rho_{F_R}$
- w_{DE} can lie in the phantom regime.

[Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- 3) $F(R) = R - \lambda R_C \left[1 - \left(1 + \frac{R^2}{R_C^2} \right)^{-n} \right]$



- Both F(R) sector and MG sector constitute Dark Energy $\rho_{DE} \equiv \rho_{MG} + \rho_{F_R}$
- w_{DE} can lie in the phantom regime.

[Cai, Saridakis PRD 90b]

Cosmological Perturbations

- $$S=\int d^4x\sqrt{-\tilde{g}}\left[M_p^2\frac{\tilde{R}}{2}+M_p^2m_g^2\left(\tilde{L}_2+\alpha_3\tilde{L}_3+\alpha_4\tilde{L}_4\right)-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-U(\varphi)\right]$$

$$\delta\tilde{g}_{00}=-2N^2\phi,\ \ \delta\tilde{g}_{0i}=Na\partial_iB,\ \ \delta\tilde{g}_{ij}=a^2\left[2\tilde{\gamma}_{ij}^K\psi+\left(\nabla_i\nabla_j-\frac{1}{3}\tilde{\gamma}_{ij}^K\nabla_k\nabla^k\right)\right]E,\ \ \delta\varphi$$

$$\Rightarrow\cdots\ \ \cdots$$

- Integrate out non-dynamical dof's $\phi,\ B,\ E$
- Since ϕ is non-dynamical at the linear level on the self-accelerating solution, we introduce the Bardeen potential ψ_B and Mukkanov-Sasaki variable

$$Q\equiv\delta\varphi+\frac{\dot{\varphi}\psi_B}{H}$$

$$\Rightarrow\ddot{Q}_k+3H\dot{Q}_k+\left[\frac{k^2}{a^2}+U_{,\varphi\varphi}-\frac{1}{M_p^2a^3}\left(\frac{a^3}{H}\dot{\varphi}^2\right)^{\bullet}\right]Q_k=\frac{2m_g^2\tilde{Y}_Q}{3\Omega^4}Q_k-2\frac{k^2}{a^2H^2}\left(\ddot{\varphi}-\frac{\dot{H}\dot{\varphi}}{H}\right)\psi_B$$

GR + scalar (left-hand side) — MG contribution (first term on the right-hand side)

- $\tilde{Y}_Q(\alpha_3,\alpha_4)<0\Rightarrow$ Stability!

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Status of massive gravity

- i) Massive gravity is a reasonable modification to describe acceleration.
- ii) The simplest linear model has the vDVZ discontinuity.
- iii) Non-linearities cure it but bring the BD ghost.
- iv) New nonlinear MG uses suitable graviton self-interactions in order to be free of BD ghosts and vDVZ discontinuity.
- v) But simple FRW cosmology is impossible (cosmological instabilities).
- vi) One should go to anisotropic geometry.
- vii) Or other extensions: Varying mass massive gravity, quasi-dilaton massive gravity.
- viii) F(R) nonlinear massive gravity is the most promising. It is free of BD ghost and vDVZ discontinuity. It exhibits good and rich cosmology, free of instabilities!

Re-parametrization of our ignorance? (instead to explain why Λ is small, we have to explain why m_g is small).