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Online

Early Universe, Inflation and Primordial Gravitational Waves

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Lecture IV

Gravitational Waves

Einstein equations:

$$\boxed{R_{\mu\nu} = -8\pi G_N \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}\right)}$$

nonlinear

Perturbed background (weak-field):

$$\boxed{g_{\mu\nu} = g^{(1)}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}}$$

$|h_{\mu\nu}| \ll 1$

Flat metric *Perturbation*

Linearised vacuum Einstein equations: $R^{(1)}_{\mu\nu} = 0$

$$\boxed{\partial_\sigma \partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma \partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} = 0}$$

Linearised Einstein equations
in the presence of matter :

$$\boxed{G^{(1)}_{\mu\nu} = 8\pi G_N T^{(0)}_{\mu\nu}}$$

In an analogous way as for Maxwell theory:

- Linearised Einstein equations are invariant under linearised general coordinate transformations

- In contrast to the Lagrangian of Maxwell theory, the Lagrangian for linearised gravity is **not** strictly gauge invariant, but only invariant up to a local derivative

- Harmonic gauge condition (*analogous to the Lorenz gauge in Maxwell theory*) :

$$\boxed{\partial_\mu h^\mu{}_\lambda - \tfrac{1}{2}\partial_\lambda h = 0}$$

$$\Downarrow$$

Linearised Einstein equations : $\boxed{\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G_N T^{(0)}_{\mu\nu}}$

Linearised vacuum Einstein equations: $\boxed{\Box h_{\mu\nu} = 0}$ *standard relativistic wave equation*

In an analogous way as for Maxwell theory:

Consider linear combination: $\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$

Linearised Einstein equations and harmonic gauge conditions:

$$\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G_N T^{(0)}_{\mu\nu} \\ \partial_\mu \bar{h}^\mu{}_\nu &= 0 \end{aligned}$$

To determine the evolution of a disturbance in a gravitational field in the harmonic gauge

Retarded solution

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int d^3x' \, \frac{T^{(0)}_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}$$

The general solution is then the sum of this particular solution of the inhomogeneous equation and the general solution of the homogeneous equation

Homogeneous equation: the linearised vacuum Einstein equation in the harmonic gauge

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}$$

Many are spurious, they can be eliminated by linearised coordinate transformations and Lorentz rotations

Solution:

$$\boxed{\bar{h}_{\mu\nu} = \epsilon_{\mu\nu} \mathrm{e}^{\,ik_\alpha x^\alpha}}$$

4 parameters

constant, null wave vector

$$k^\alpha k_\alpha = 0$$

constant, symmetric polarisation tensor 10 parameters

Plane waves are solutions to the linearised equations of motjon and the Einstein equations predict the existence of GWs travelling along null geodesics (at the speed of light)

The timelike component of the wave vector is referred to as the frequency ω of the wave:

$$k^\mu = (\omega, k^i)$$

Transverse traceless gauge:

$$\boxed{k^\mu \bar{h}_{\mu\nu} = 0 \quad , \quad \bar{h}_{\mu 0} = 0 \quad , \quad \bar{h}^\mu_{\ \mu} = 0}$$

Only two independent polarisations for a GW

Consider a GW travelling in the x^3 - direction: $k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$

$$\boxed{ds^2 = -dt^2 + (\delta_{ab} + h_{ab})dx^a dx^b + (dx^3)^2}$$

Describes the distortion of spacetime geometry in the direction transverse to GW

$h_{ab} = h_{ab}(t \mp x^3)$ *symmetric and traceless*

Just looking at the gravitational field one cannot determine the physical effect of a passing GW

Consider its **influence on the relative motion of nearby particles** (*geodesic equation*)

A family of nearby test particles, initially at rest, at a separation vector $S^\mu(x)$

A GW passing by will lead, to lowest order in the perturbation $h_{\mu\nu}$, to a 4-velocity:

$$u^\mu = (1,0,0,0) + \mathcal{O}(h)$$

$$\boxed{\ddot{S}^\mu = \frac{1}{2}\ddot{h}^\mu{}_\sigma \, S^\sigma}$$

Geodesic deviation equation

The GW is transversally polarised (*the component* S^3 *of* S^μ *in the longitudinal direction of the wave is unaffected and the particles are only disturbed in directions* $\perp$ *to the wave*)

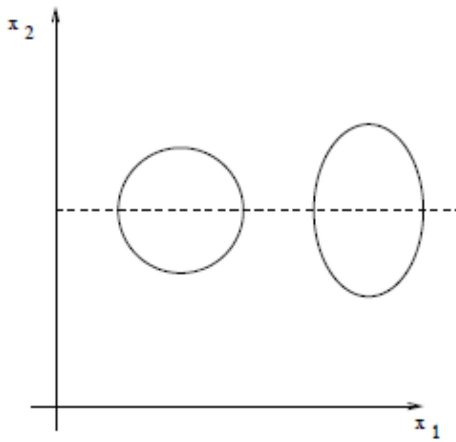
The movement of the particles in the (1-2) plane is governed by:

$$\boxed{\ddot{S}^a = \frac{1}{2}\ddot{h}^a{}_b \, S^b}$$

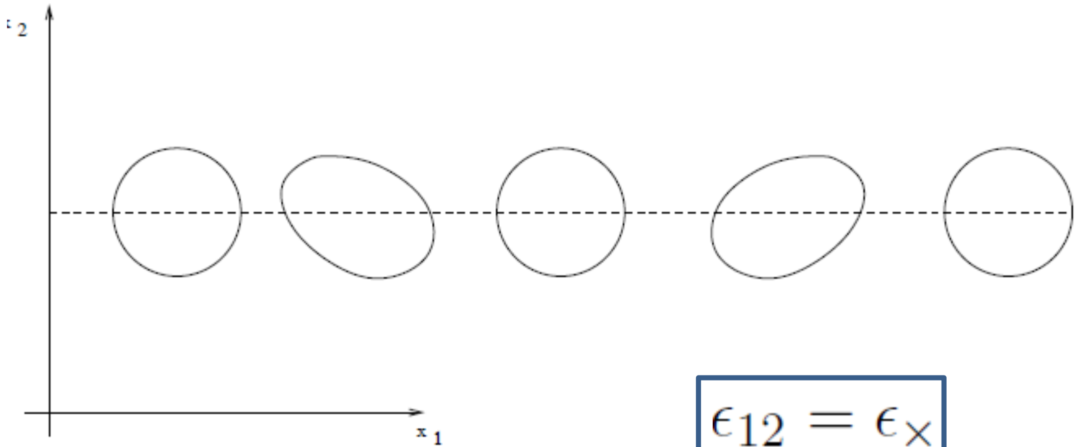
equation of a 2-dim time-dependent harmonic oscillator, leading to oscillations of the test particles in the (1-2) plane

- $\epsilon_{12} = 0$
- $\epsilon_{11} = -\epsilon_{22} = 0$

Effect of a GW with polarisation ϵ_{11} moving in the x^3-direction on a ring of test particles in the ($x^1 - x^2$) -plane



$$\boxed{\epsilon_{11} = \epsilon_{+}}$$



$$\boxed{\epsilon_{12} = \epsilon_{\times}}$$

Effect of a GW with polarisatior ϵ_{12} moving in the x^3-direction on a ring of test particles in the ($x^1 - x^2$) -plane

Circularly polarised waves:

$$\epsilon_{R,L} = \frac{1}{\sqrt{2}}(\epsilon_{11} \pm i\epsilon_{12})$$

- Birkhoff's theorem: there can be no monopole radiation
- Momentum conservation: there can be no dipole radiation

⇒ the lowest possible mode of gravitational radiation is quadrupole radiation

$$\frac{dE}{dt} = -\frac{G_N}{5}(\dddot{Q}^{ret})_{ik}(\dddot{Q}^{ret})^{ik}$$

Reduced (traceless) quadrupole moment tensor of the retarded energy density

$$Q_{ik}^{ret} = \int d^3x \, \rho^{ret}(x_i x_k - \tfrac{1}{3}\delta_{ik} r^2)$$

Maxwell theory: - the leading contribution arises from dipole radiation

- the radiated power of an electric quadrupole is also proportional to the third derivative squared of the quadrupole moments

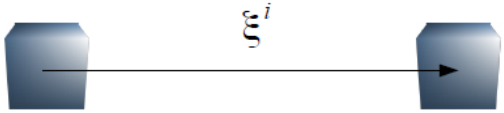
For 2 stars of equal mass M at distance 2r from each other, the prediction of GR is that the power radiated by the binary system is:

$$P = dE/dt = -\frac{2}{5}\frac{G_N^4 M^5}{r^5}$$

This energy loss has actually been observed

Principles of LIGO-Virgo detectors

Consider 2 objects at rest



ξ^i : separation vector

A GW passing by will not modify their positions (they are at rest and stay at rest)
but
it will modify their relative position, i.e., ξ^i

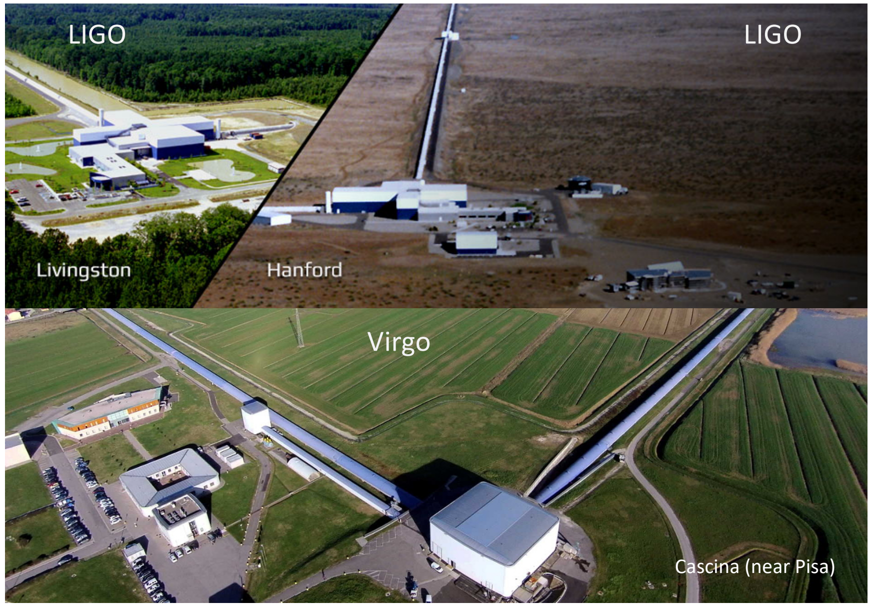
Geodesic deviation equation:

$$\frac{d^2\xi^i}{dt^2} = -c^2 \xi^k R_{0i0k}(h_{\alpha\beta})$$

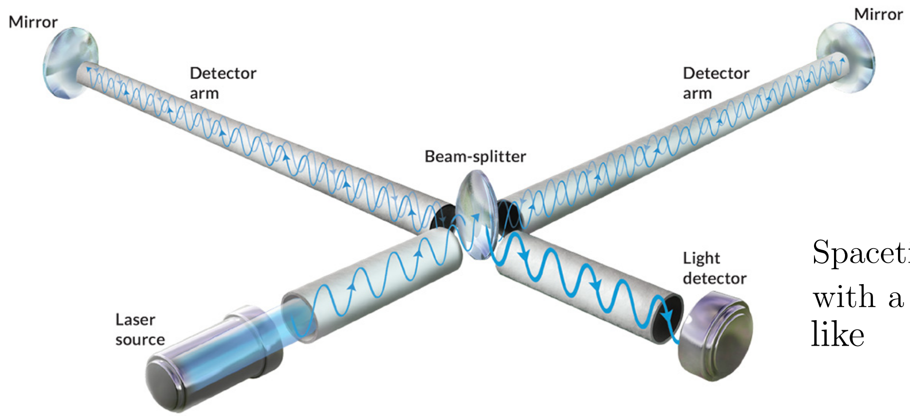
with linear order solution:

$$\xi^i(t) = \xi^i(0) + \frac{1}{2} h^{ij}_{TT}(t)\,\xi_k(0)$$

If a GW passes by, the relative position oscillates



by using a *Michelson interferometer*



$+$ polarization

plane wave moving in the **z**-direction

$$h_{ij}(z,t) = h_+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} e^{i(kz-\omega t)}$$

Spacetime is stretched due to the strain created by the gravitational wave. Starting with a length L_0 along the **x**-axis, the gravitational wave causes the length to oscillate like

$$L(t) = L_0 + \frac{h_+ L_0}{2}\cos(\omega t)$$

There is a change in its length of $\Delta L_x = \frac{h_+ L_0}{2}\cos(\omega t)$

Along the **y**-axis, a similar length L_0 subjected to the same gravitational wave oscillates like $\Delta L_y = -\frac{h_+ L_0}{2}\cos(\omega t)$

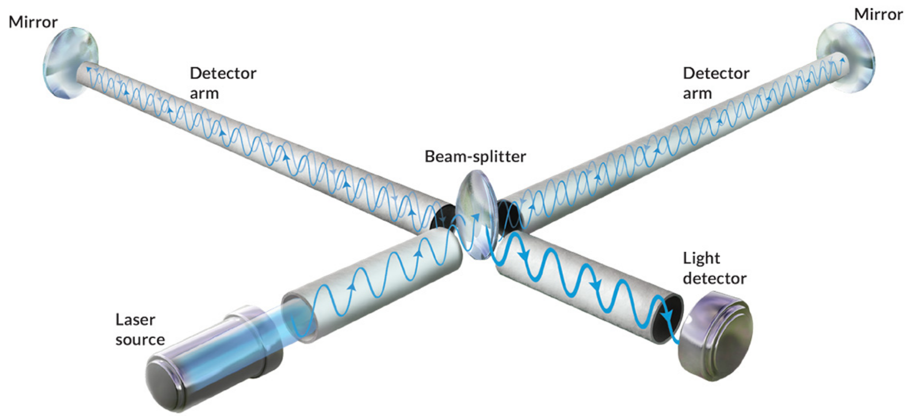
In this example, the **x**-axis stretches while the **y**-axis contracts, and then vice versa as the wave propagates through the region of space. In terms of the relative change of the lengths of the two arms (at $t = 0$) $\Delta L = \Delta L_x - \Delta L_y = h_+ L_0 \cos(\omega t)$ $\Rightarrow$ $h_+ = \frac{\Delta L}{L_0}$

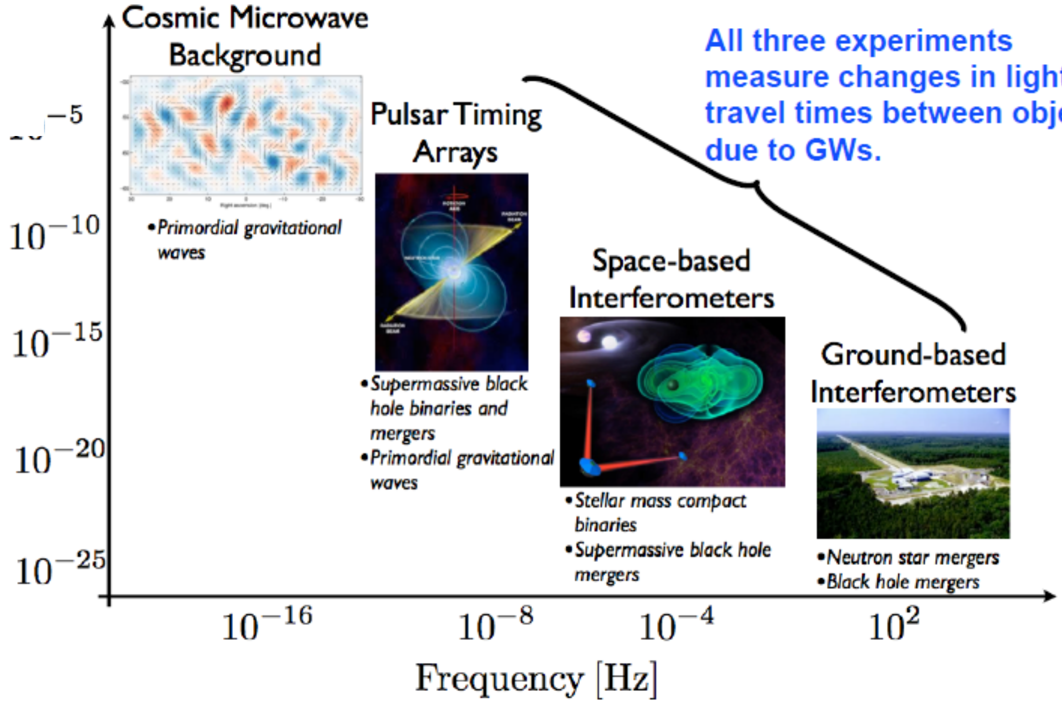
The amplitude of a gravitational wave, h_+, is the amount of strain that it produces on spacetime.

The other gravitational wave polarization ($h_\times$) produces a similar strain on axes 45^o from (**x**,**y**).

The stretching and contracting of space is the physical effect of a gravitational wave, and detectors of gravitational waves are designed to measure this strain on space.

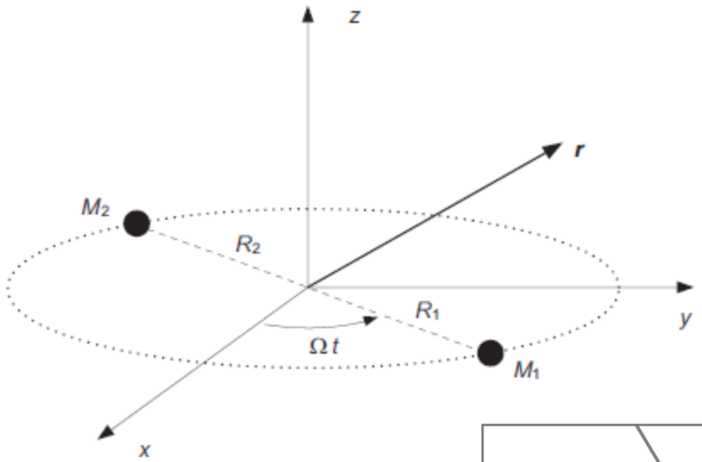
by using a *Michelson interferometer*





All three experiments measure changes in light travel times between objects due to GWs.

Gravitational radiation from Binary systems: effect on the orbit



$$E = E_1^{\text{kinetic}} + E_2^{\text{kinetic}} + E^{\text{grav}}$$

$$= -\frac{M}{4}\left(\frac{4\pi M}{P}\right)^{2/3}$$

bound system

energy ↘ by GW emission ⟹ period P ↘

The emission of GW radiation also tends to circularise elliptical orbits

$$\boxed{\begin{aligned}\frac{dP}{dt} &= -\frac{96\pi}{5}4^{1/3}\left(\frac{2\pi M}{P}\right)^{5/3} \\ &= -3.4\times 10^{-12}\left(\frac{M}{M_\odot}\right)^{5/3}\left(\frac{1\ \mathrm{h}}{P}\right)^{5/3}\end{aligned}}$$

Gravitational radiation from a Binary Pulsar

Binary pulsar: a pulsar and a compact companion (most likely also a neutron star) in orbit around their common centre of mass

Each star has a mass near $1.4\ M_{\odot}$ and the orbital period is 7.75 hours

$$\frac{dP}{dt} \sim -2 \times 10^{-13}$$

Over period of one year: $\Delta P \simeq -6.2 \times 10^{-6}\ \mathrm{s}$

But pulsars are exceptionally accurate celestial clocks

Binary Pulsar PSR 1913+16

$m_1 = 1.4 m_{\odot}$; $m_2 = 1.36 m_{\odot}$; $\varepsilon = 0.617$

Orbit decays by 3mm per orbit
The rate of decrease of orbital period
76.5 microseconds / year

Discovered in 1974 by
Russell Hulse and **Joseph Taylor**

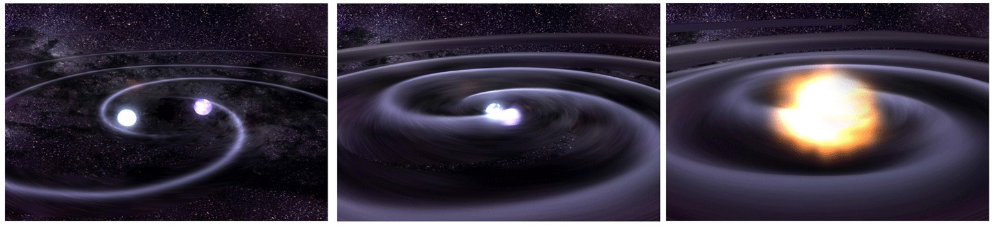
Nobel prize in Physics (1993)

Measured energy losses in orbit (orbital energy converted to gravitational radiation) in agreement with prediction from post-Newtonian theory

Sources of Gravitational Waves

Pretty much anything dynamical that is "powerful enough"
(Though, in General Relativity, only non-spherical dynamics radiate)

e.g. *Compact binary coalescence,*

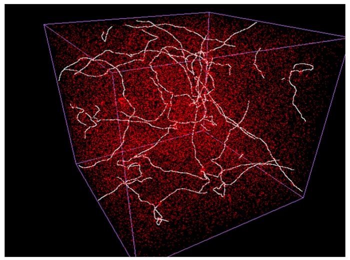


Supernovae

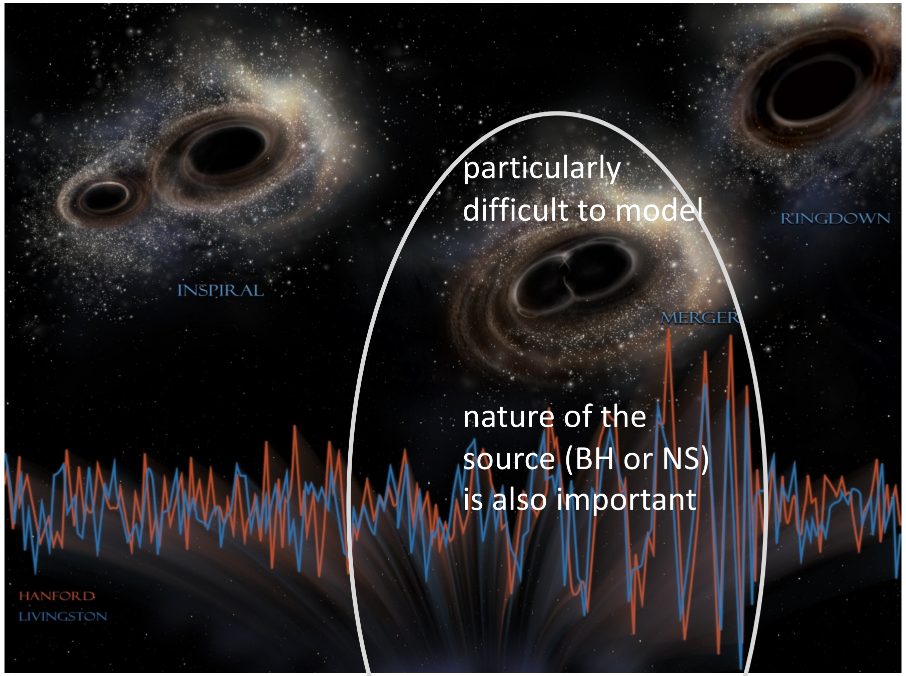
But also more exotic potential sources

e.g. *Electroweak first order Phase Transition,*

Cosmic Strings



GW from strong sources: Binary Black Holes Coalescence



inspiral

a binary system gradually spirals inward because of emission of GWs, and the resulting waveform increases in amplitude and in frequency, producing a characteristic "chirp"

Large separation, low speed, weak gravitational field

Post-Newtonian methods

merger

the two objects plunge into each other and merge

Short separation, speed close to c, strong gravitational field

Numerical Relativity

ringdown

the resulting system, typically a BH, finally settles down to its ground state, radiating away the energy stored in its exited modes

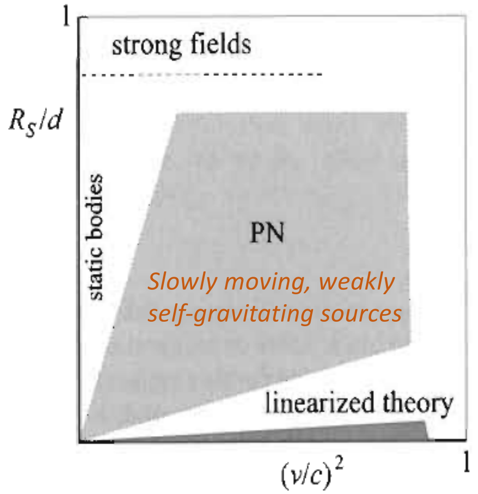
Quasinormal modes

Perturbation theory

Post-Newtonian expansion

The astrophysical sources more interesting for GW detection are held together by gravitational forces

$\Rightarrow$ The assumption that the velocity of the source and the space-time curvature are independent is no longer valid



For a self-gravitating system with total mass M and typical size d: $(v/c)^2 \sim R_S/d$

measure of the strength of gravitational field near the source

As soon as we switch on the v/c corrections, we must also consider deviation of the background from flat spacetime

Assume a non-relativistic and self-gravitating source

Introduce: $\epsilon \sim (R_S/d)^{1/2} \sim v/c$

Demand the source to be weakly stressed: $|T^{ij}|/T^{00} = \mathcal{O}(\epsilon^2)$

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \cdots$$

$$g_{0i} = {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \cdots$$

$$g_{ij} = \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \cdots$$

terms of ϵ^2 in the expansion

$$T^{00} = {}^{(0)}T^{00} + {}^{(2)}T^{00} + \cdots$$

$$T^{0i} = {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \cdots$$

$$T^{ij} = {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \cdots$$

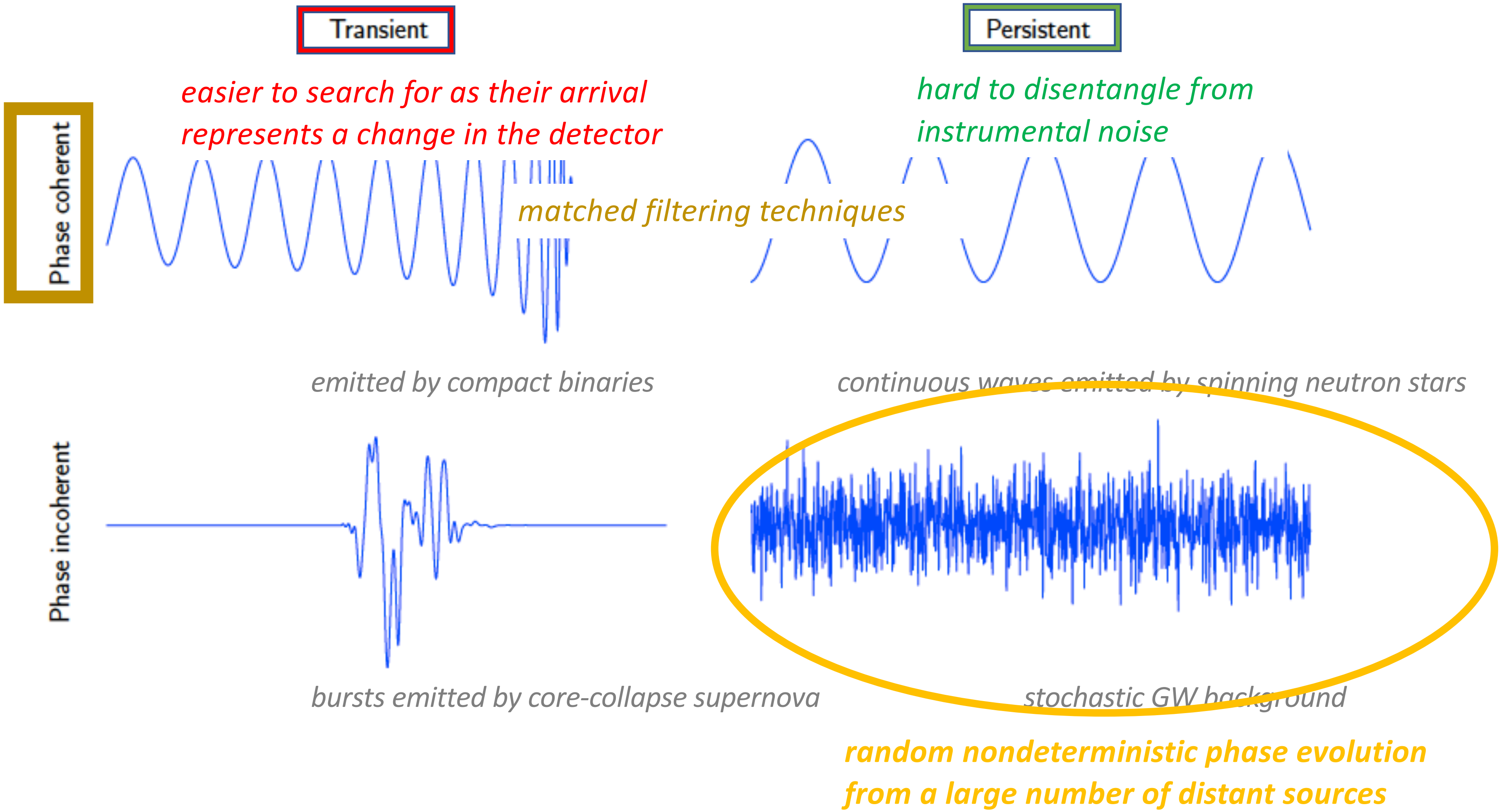
Plug into Einstein equations and equate terms of the same order in ε

Use the harmonic gauge condition

$\Rightarrow$ Get the post-Newtonian (PN) metric

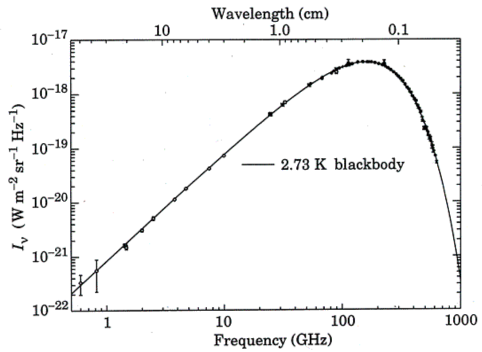
$\Rightarrow$ Obtain the equations of motion of a test particle in the PN metric from the geodesic equation

GW signals

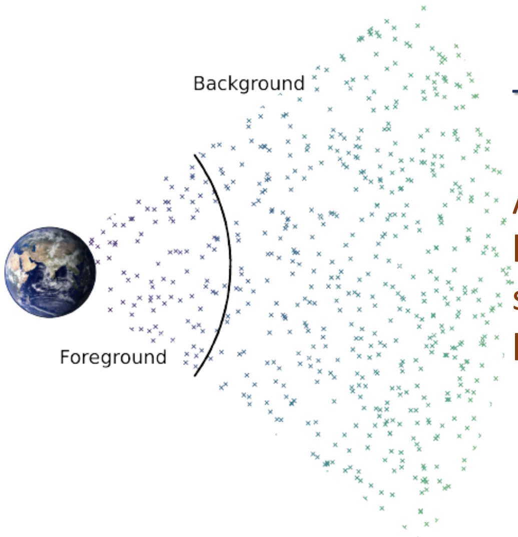


Transient
easier to search for as their arrival represents a change in the detector
Persistent
hard to disentangle from instrumental noise
Phase coherent
matched filtering techniques
emitted by compact binaries
continuous waves emitted by spinning neutron stars
Phase incoherent
bursts emitted by core-collapse supernova
stochastic GW background
random nondeterministic phase evolution from a large number of distant sources

Gravitational-Wave Background (GWB)



Penzias and Wilson (1965) discovered that the Universe is permeated by the CMB electromagnetic radiation



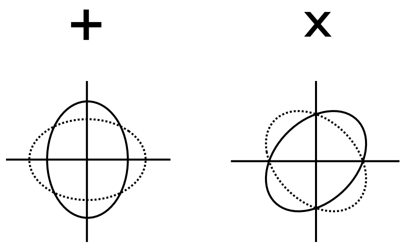
The Universe is permeated by a stochastic GWB generated in the early Universe

A **background of GWs** can also emerge from the incoherent superposition of a large number of astrophysical sources, too weak to be detected separately, and such that the number of sources that contribute to each frequency bin is much larger than one

Mathematical characterization of the GWB

Since the individual signals comprising a GWB background are either too weak or too numerous to individually detect, the combined signal for the background is for all practical purposes *random*, similar to noise in an single detector. Hence, we need to describe the GWB *statistically*, in terms of moments (i.e., ensemble averages) of the metric perturbations describing the GWB.

gravitational waves are time-varying perturbations to the geometry of space-time, which propagate away from the source at the speed of light . In transverse-traceless coordinates the metric perturbations corresponding to a plane wave have two degrees of freedom, corresponding to the amplitudes of the plus $(+)$ and cross $(\times)$ polarizations of the gravitational wave



The two orthogonal polarizations of a gravitational wave. A circular ring of test particles in the plane orthogonal to the direction of propagation of the wave are alternately deformed into ellipses, as space is "squeezed" and "stretched" by the passing of the wave.

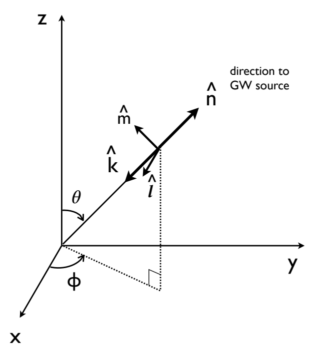
The metric perturbation for the most general GWB can thus be written as a superposition of such waves:

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k}) e^A_{ab}(\hat{k}) e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)} ,$$

where f denotes the frequency of the component waves, $\hat{k}$ their direction of propagation, and $A=+,\times$ their polarization. (The direction to a particular GW source is given by $\hat{n}=-\hat{k}$.) The quantities $e^A_{ab}(\hat{k})$ are polarization tensors, given by

$$\begin{aligned} e^+_{ab}(\hat{k}) &= \hat{l}_a\hat{l}_b - \hat{m}_a\hat{m}_b , \\ e^\times_{ab}(\hat{k}) &= \hat{l}_a\hat{m}_b + \hat{m}_a\hat{l}_b , \end{aligned}$$

where $\hat{l}$, $\hat{m}$ are any two orthogonal unit vectors in the plane orthogonal to $\hat{k}$. Typically, for stochastic background analyses, we take $\hat{l}$, $\hat{m}$ to be proportional to the standard angular unit vectors tangent



$$\begin{aligned} \hat{k} &= -\sin\theta\cos\phi\,\hat{x} - \sin\theta\sin\phi\,\hat{y} - \cos\theta\,\hat{z} = -\hat{r} , \\ \hat{l} &= +\sin\phi\,\hat{x} - \cos\phi\,\hat{y} = -\hat{\phi} , \\ \hat{m} &= -\cos\theta\cos\phi\,\hat{x} - \cos\theta\sin\phi\,\hat{y} + \sin\theta\,\hat{z} = -\hat{\theta} . \end{aligned}$$

The quantities $h_A(f,\hat{k})$ are the Fourier coefficients of the plane wave expansion. Since the metric perturbations for a stochastic background are random variables, so too are the Fourier coefficients.

assume that the expected value of the Fourier coefficients $\langle h_A(f,\hat{k})\rangle = 0\,,$

ensemble average over different realizations of the background.

if the background is *unpolarized, stationary, and isotropic*, then

no preferred origin of time

stationarity

$$\langle h_A(f,\hat{k})h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi}S_h(f)\delta(f-f')\delta_{AA'}\boxed{\delta^2(\hat{k},\hat{k}')}$$

$S_h(f)$ is the *strain power spectral density* of the background, having units of strain^2 Hz^{-1}.

background being unpolarized

exact isotropy

If we drop the last assumption, allowing the background to be either *anisotropic* or *statistically isotropic*, then the quadratic expectation values become

$$\langle h_A(f,\hat{k})h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{4}\mathcal{P}(f,\hat{k})\delta(f-f')\delta_{AA'}\delta^2(\hat{k},\hat{k}')\,.$$

$\mathcal{P}(f,\hat{k})$ is the strain power spectral density per unit solid angle, with units strain^2 Hz^{-1} sr^{-1}.

$$S_h(f) = \int d^2\Omega_{\hat{k}}\,\mathcal{P}(f,\hat{k})$$

For statistically isotropic backgrounds, the angular power spectra C_l are the coefficients of a series expansion of the two-point function $C(\theta) \equiv \langle \mathcal{P}(f,\hat{k})\mathcal{P}(f,\hat{k}')\rangle_{\text{sky avg}}$, for all $\hat{k}$, $\hat{k}'$ having $\cos\theta = \hat{k}\cdot\hat{k}'$

$$C(\theta) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}C_l P_l(\cos\theta)$$

$S_h(f)$ is the strain power spectral density of the GWB

(normalized) *energy density spectrum*

$$\boxed{\Omega_{\rm gw}(f) \equiv \frac{1}{\rho_{\rm c}} \frac{{\rm d}\rho_{\rm gw}}{{\rm d}\ln f} = \frac{f}{\rho_{\rm c}} \frac{{\rm d}\rho_{\rm gw}}{{\rm d}f}}$$

$$\rho_{\rm gw} = \frac{c^2}{32\pi G} \langle \dot{h}_{ab}(t,\vec{x}) \dot{h}^{ab}(t,\vec{x}) \rangle$$

$$\boxed{S_h(f) = \frac{3H_0^2}{2\pi^2} \frac{\Omega_{\rm gw}(f)}{f^3}}$$

In addition to $S_h(f)$ and $\Omega_{\rm gw}(f)$, one sometimes describes the strength of a GWB in terms of the (dimensionless) *characteristic strain* $h_c(f)$ defined by

$$h_c(f) = \sqrt{f S_h(f)}$$

For backgrounds described by a power-law dependence on frequency

$$h_c(f) = A_\alpha \left(\frac{f}{f_{\rm ref}} \right)^\alpha \quad \Leftrightarrow \quad \boxed{\Omega_{\rm gw}(f) = \Omega_\beta \left(\frac{f}{f_{\rm ref}} \right)^\beta}$$

where α and β are spectral indices, and A_α and Ω_β are the amplitudes of the characteristic strain and energy density spectrum, respectively, at some reference frequency $f = f_{\rm ref}$.

$$\Omega_\beta = \frac{2\pi^2}{3H_0^2} f_{\rm ref}^2 A_\alpha^2 \,, \qquad \beta = 2\alpha + 2$$

standard inflationary backgrounds, $\Omega_{\rm gw}(f) = {\rm const}$, for which $\beta = 0$ and $\alpha = -1$

GWBs associated with binary inspiral, $\Omega_{\rm gw}(f) \propto f^{2/3}$ for which $\beta = 2/3$ and $\alpha = -2/3$

$\Omega_{\rm gw}(f)$ for an astrophysically-generated background

comoving rate density $R(z)$

$$E(z) \equiv \sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_\Lambda}$$

$$\boxed{\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm c} H_0} \int_0^\infty {\rm d}z\, R(z) \frac{1}{(1+z)E(z)} \left(\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}}\right)\Bigg|_{f_{\rm s}=f(1+z)}}$$

$\Omega_{\rm gw}(f)$ for binary inspiral

consider two masses, m_1 and m_2, in circular orbits around their common center of mass

$$r \equiv r_1 + r_2\,, \quad M \equiv m_1 + m_2\,, \quad \mu \equiv \frac{m_1 m_2}{m_1+m_2}$$

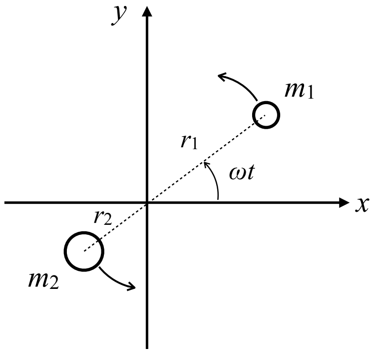
$$f_s = (1+z)f$$

$$\omega^2 r^3 = GM$$

Kepler's third law

$$E_{\rm orb} = -\frac{GM\mu}{2r}$$

total orbital energy of the system



$$\omega \equiv 2\pi f_{\rm orb}$$

orbital angular frequency

$$\frac{{\rm d}E_{\rm gw}}{{\rm d}t} = -\frac{{\rm d}E_{\rm orb}}{{\rm d}t}$$

power emitted in GWs comes from the orbital energy

$$\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}} = \frac{{\rm d}t}{{\rm d}f_{\rm s}}\frac{{\rm d}E_{\rm gw}}{{\rm d}t} = -\frac{{\rm d}t}{{\rm d}f_{\rm s}}\frac{{\rm d}E_{\rm orb}}{{\rm d}t}$$

energy spectrum

$\mathcal{M}_{\rm c}$ is the *chirp mass*

$$\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}} \sim \mathcal{M}_{\rm c}^{5/3} f_{\rm s}^{-1/3}\,, \qquad \mathcal{M}_c^{5/3} \equiv M^{2/3}\mu$$

$$\boxed{\Omega_{\rm gw}(f) \propto f^{2/3}}$$

$f_{\rm s} = 2f_{\rm orb}$ factor of 2 arising for quadrupolar radiation in general relativity

Search for a GWB

standard search techniques like *matched filtering* which correlate the data against known, deterministic waveforms (e.g., BBH chirps) won't work when trying to detect a GWB

cross-correlation

$$d_1 = h + n_1\,,$$
$$d_2 = h + n_2\,.$$

a single sample of data from two colocated and coaligned detectors

common GW signal component

instrumental noise components

$$\langle \hat{C}_{12} \rangle = \langle d_1 d_2 \rangle = \langle h^2 \rangle + \cancel{\langle h n_2 \rangle}^{0} + \cancel{\langle n_1 h \rangle}^{0} + \langle n_1 n_2 \rangle$$

$\langle h n_2 \rangle = 0 = \langle n_1 h \rangle$, since the GW signal and instrumental noise are not correlated

assume that the noise in the two detectors is *uncorrelated* $\langle n_1 n_2 \rangle = 0,$

$$\langle \hat{C}_{12} \rangle = \langle h^2 \rangle \equiv S_h$$

the variance (i.e., power) in the GW signal

To handle the case of physically-separated and misaligned detectors, we need to include the non-trivial response of a GW detector to a GWB.

trivial response of a GW detector to a GWB. $\Gamma_{12}(f)$

the transfer function relating the strain power in the GWB, $S_h(f)$, to the cross-correlated signal power in the two detectors

$$C_{12}(f) \equiv \Gamma_{12}(f) S_h(f)$$

$$\boxed{\langle \tilde{h}_1(f) \tilde{h}_2^*(f') \rangle = \frac{1}{2} \delta(f - f') \Gamma_{12}(f) S_h(f)}$$

$\tilde{h}_1(f)$, $\tilde{h}_2(f)$ denote the Fourier transforms of the GW signal components $h_1(t)$, $h_2(t)$

(auto-correlated) power spectra of the detector noise $P_{n_1}(f)$, $P_{n_2}(f)$ in terms of the noise components $\tilde{n}_1(f)$, $\tilde{n}_2(f)$

$$\langle \tilde{n}_1(f) \tilde{n}_1^*(f') \rangle = \frac{1}{2} \delta(f - f') P_{n_1}(f) \,,$$

$$\langle \tilde{n}_2(f) \tilde{n}_2^*(f') \rangle = \frac{1}{2} \delta(f - f') P_{n_2}(f) \,,$$

the cross-correlated noise is assumed to be zero: $\langle \tilde{n}_1(f) \tilde{n}_2^*(f') \rangle = 0$

GWs offer a new window for exploring late and early stages of the Universe

The importance of the recent direct direction of GWs from BH and NS mergers can hardly be over-emphasised

Even a non-detection of GWs allowed us to gain important information on cosmology, particle physics models and gravity

GWs offer a novel and powerful way to test

- astrophysical models and *large-scale-structure of the universe*
- beyond ΛCDM cosmological model
- physics beyond the Standard Model of particle physics
- *dark matter candidates (PBHS, axions, …)*
- modified gravity models
- quantum gravity theories

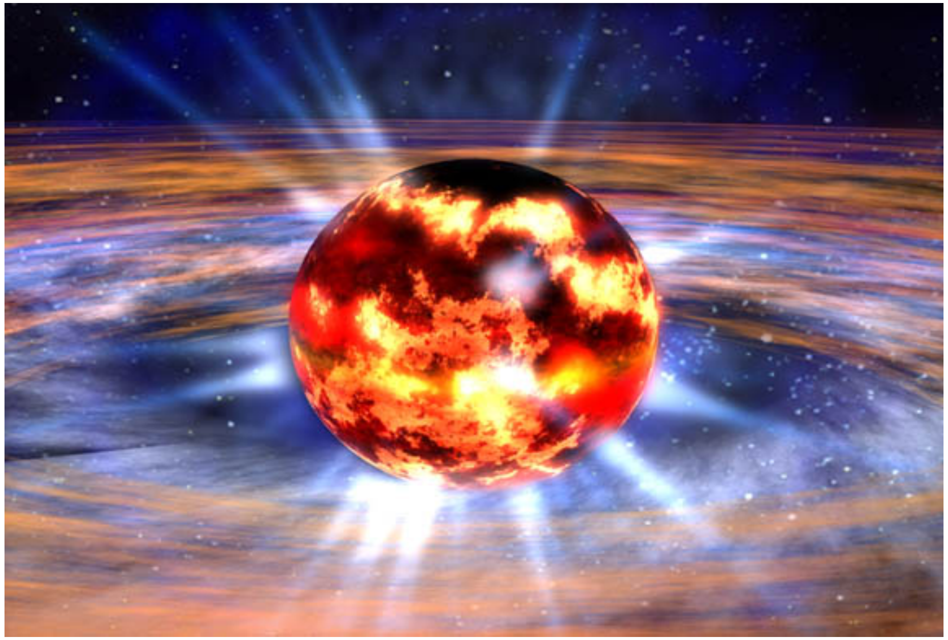
Neutron Stars

Unique natural laboratories for studying behavior of cold, high-density nuclear matter.

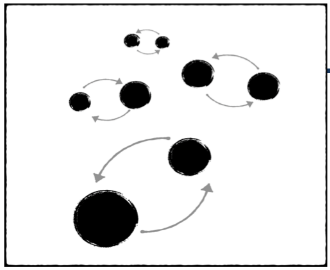
Behavior is governed by **equation of state (EoS)**, relationship between pressure and density:

determines relation between NS mass and radius
determines stellar moment of inertial
determines tidal deformability

Thus measurement of NS masses, radii, moments of inertia and tidal effects provide information about EoS.



GWB from compact binary coalescence (CBC)

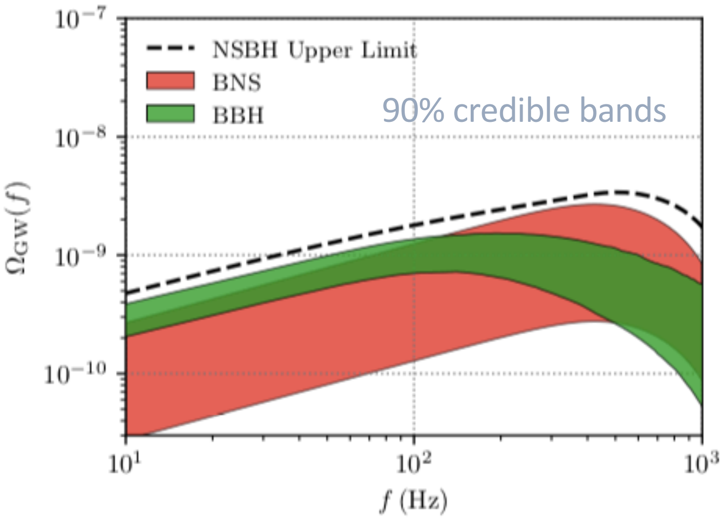


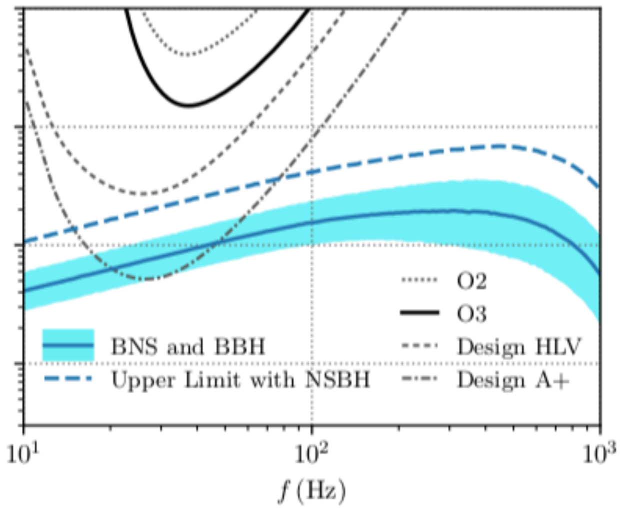
$$\Omega_{\rm GW}(\nu) = \Omega_{\rm ref}\left(\frac{\nu}{\nu_{\rm ref}}\right)^{\alpha}$$

α=2/3

$$\nu_{\rm ref} = 25\text{Hz}$$

$$\frac{dE_{\rm GW}}{d\nu} = \frac{(G\pi)^{2/3}}{3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\nu^{-1/3}$$





$$\Omega_{\rm GW} \ll \Omega_{\rm CMB} \approx 10^{-5}$$

$$\boxed{\Omega_{\rm GW}(f) \leq 3.4\times 10^{-9} \ \text{ at } \ 25 \text{ Hz}}$$

90% credible bands for GWB contributions from BNS/BBH mergers

Consider BBH and BNS populations using the BBH and BNS merger rates derived from LIGO/Virgo detections. A power-law GWB tangent in one of the power-law integrated curves (sensitivity curves) is detectable with 2σ significance

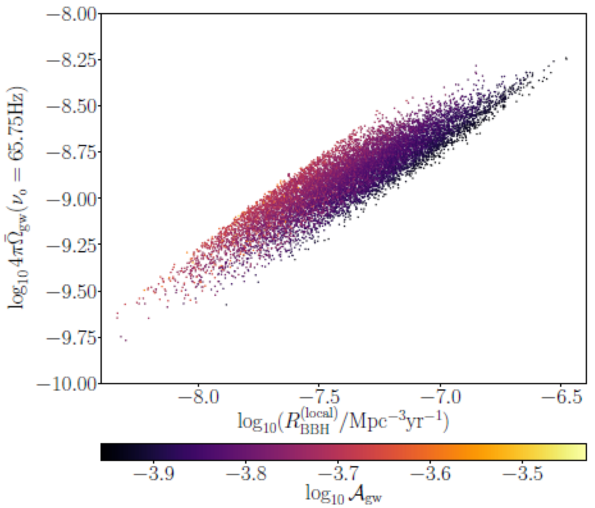
GWB from CBC: info about Compact Binaries

$$\boxed{\Omega_{\rm GW}(\nu,\theta) = \frac{\nu}{\rho_{\rm c} H_0}\int_0^{z_{\rm max}} {\rm d}z \frac{R_{\rm m}(z;\theta)\frac{{\rm d}E_{\rm GW}(\nu_{\rm s};\theta)}{{\rm d}\nu_{\rm s}}}{(1+z)E(\Omega_{\rm M},\Omega_\Lambda,z)}}$$

$$E(\Omega_{\rm M},\Omega_\Lambda,z) = \sqrt{\Omega_{\rm M}(1+z)^3+\Omega_\Lambda}$$

$$\nu_{\rm s} = (1+z)\nu$$

Most important quantities describing each BBH are the **masses** and **spins** of each component BH



Truncated power-law BH mass distribution:

$$\boxed{p(m_1,m_2) \propto \begin{cases} \frac{m_1^{-\alpha_m}}{m_1-m_{\rm min}}, & m_{\rm min}\le m_2\le m_1\le m_{\rm max} \\ & m_1+m_2\le M_{\rm max} \\ 0, & \text{otherwise}\end{cases}}$$

$$m_{\rm min} = 5M_\odot$$

$$M_{\rm max} = 200M_\odot$$

Beta distribution for the BH spins:

$$\boxed{p(\chi_i) \propto \chi_i^{\alpha_\chi-1}(1-\chi_i)^{\beta_\chi-1}}$$

$$\left.\begin{array}{l}\alpha_m \\ m_{\rm max} \\ \alpha_\chi,\beta_\chi\end{array}\right\}$$ *infrerred from observed BBHs*

The **total energy density varies over nearly two orders of magnitude**

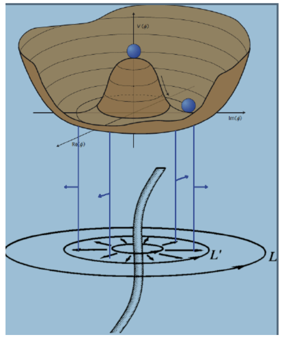
→ **a new probe of population of compact objects**

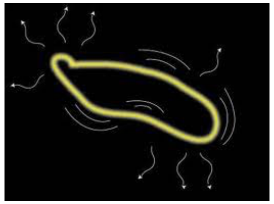
SGWB from cosmic strings: info beyond Standard Model

1dim topological defects formed in the early universe as a result of a PT followed by SSB, characterised by a vacuum manifold with non-contractible closed curves

$$G \to \cdots \to G_{\rm SM} \qquad \pi_1(\mathcal{M}) \neq 0$$

Generically formed in the context of GUTs

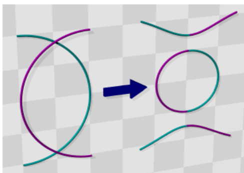


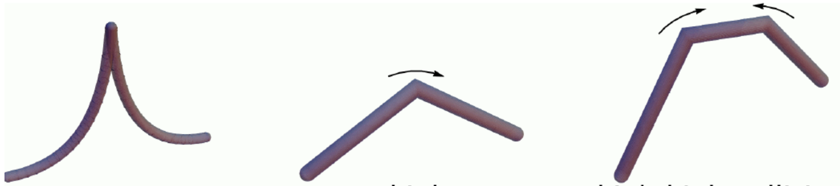


CS loops (length ℓ) oscillate periodically ($T = \ell/2$) in time emitting GWs (fundamental frequency $\omega = 4\pi/\ell$)

$$\boxed{\tau \sim \frac{\ell}{G\mu}}$$

$$G\mu \sim T_{\rm SSB}^2$$

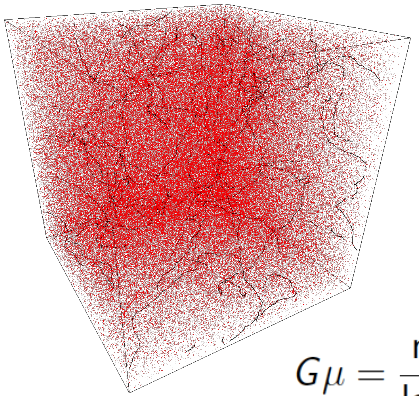




cusp	kink	kink-kink collision
$\tilde{h} \sim f^{-4/3}$	$\tilde{h} \sim f^{-5/3}$	$\tilde{h} \sim f^{-2}$
beamed	beamed	isotropic

SGWB from cosmic strings: info beyond Standard Model

$$\bar{\Omega}_{\rm gw} = \frac{2(G\mu)^2}{3\pi^2 H_{\rm o}^2 \nu_{\rm o}} \int_0^{t_*} \frac{{\rm d}t}{t^4} a^5 \int_0^{\gamma_*} \frac{{\rm d}\gamma}{\gamma} \bar{\mathcal{F}} \Theta\left(\gamma - \frac{2a}{\nu_{\rm o} t}\right) \left[N_{\rm k}^2 + 4AN_{\rm k}\left(\frac{\nu_{\rm o}\gamma t}{a}\right)^{1/3} + A^2 N_{\rm c}\left(\frac{\nu_{\rm o}\gamma t}{a}\right)^{2/3} \right]$$



$$G\mu = \frac{\text{mass}}{\text{length}} \sim \left(\frac{\text{new physics scale}}{\text{Planck scale}}\right)^2 \ll 1$$

$$\text{Energy scale} \approx \sqrt{\frac{G\mu}{10^{-10}}}\, 10^{14}\,\text{GeV}$$

Energy scale	Width	Linear density
GUT : 10^{16} GeV	2×10^{-32} m	$G\mu \approx 10^{-6}$
3×10^{10} GeV	5×10^{-27} m	$G\mu \approx 10^{-17}$
10^{8} GeV	2×10^{-24} m	$G\mu \approx 10^{-22}$
EW : 100 GeV	2×10^{-18} m	$G\mu \approx 10^{-34}$

Excluded regions:

Model A: $G\mu \gtrsim (9.6 \times 10^{-9} - 10^{-6})$

strongest limit from PTA $G\mu \gtrsim 10^{-10}$

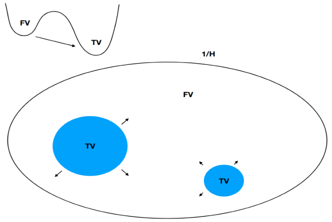
Model B: $G\mu \gtrsim (4.0 - 6.3) \times 10^{-15}$
strongest limit from LVK stochastic

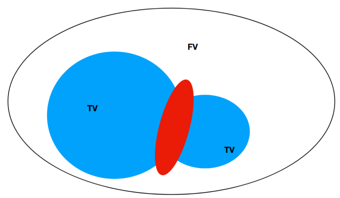
Model C1: $G\mu \gtrsim (2.1 - 4.5) \times 10^{-15}$
strongest limit from LVK stochastic

Model C2: $G\mu \gtrsim (4.2 - 7.0) \times 10^{-15}$
strongest limit from LVK stochastic

SGWB from first order phase transition(FOPT): info Beyond the Standard Model

False vacuum separated by true vacuum by a barrier.
The unstable vacuum decays through bubble nucleation.
After nucleation bubbles grow until they collide, eventually converting the whole Hubble volume into the new phase



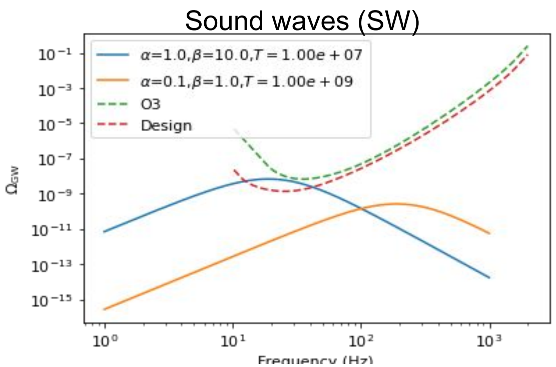


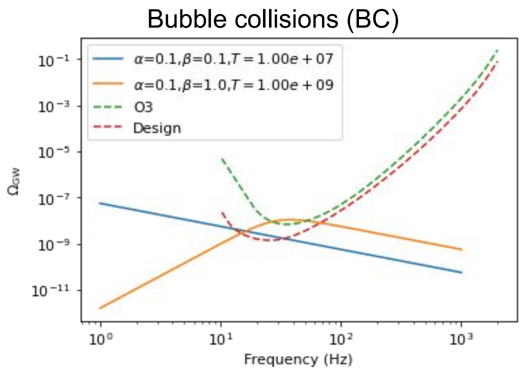
Sources of GWs:

- **Sound waves** (coupling between scalar field and thermal bath)
- **Bubble collisions**
- Magnetohydrodynamic turbulence

SGWB: broken power law with peak frequency mainly determined by temperature of FOPT

If $T_{\rm pt} \sim (10^7 - 10^9)$ GeV (not accessible by LHC) : **SGWB is within aLIGO/aVIRGO**





O1+O2+O3:

$$\Omega_{\rm CBC} < 6.1 \times 10^{-9}$$
$$\Omega_{\rm BPL} < 4.4 \times 10^{-9}$$

Complex phenomenological models
T> 10^8 GeV

$\Omega_{\rm pt} < 5.0 \times 10^{-9}$ Bubble collisions

$\Omega_{\rm pt} < 5.8 \times 10^{-9}$ Sound waves

α : strength of FOPT
β : inverse duration of FOPT

Tests of General Relativity

In GR, GWs far from their source propagate along null geodesics with energy E and momentum p related by the dispersion relation $E^2 = p^2c^2$

Extensions to GR may violate this, e.g. by giving a mass to the graviton.

To probe generalized dispersion relations, adopt a phenomenological modification to GR:

$$E^2 = p^2c^2 + A_\alpha p^\alpha c^\alpha \text{ , with } \alpha = 0, 0.5, 1.0, 1.5, 2.5, 3.0, 3.5, 4.0 \text{ ,}$$

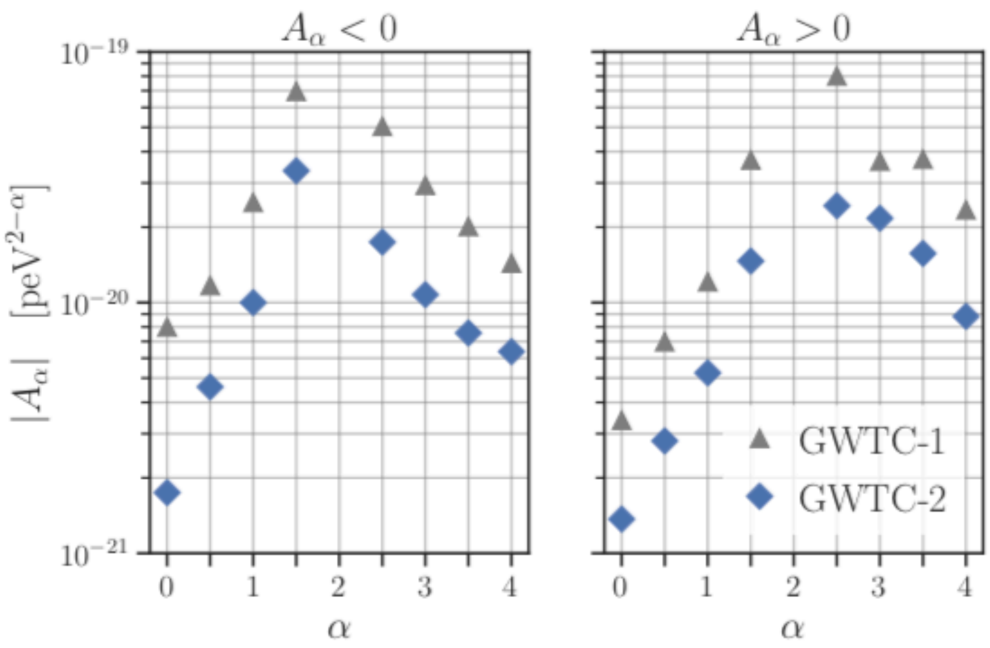
phenomenological parameters

All cases apart $\alpha = 0$ correspond to a Lorentz-violating dispersion relation.

$\alpha = 0$, $A_\alpha > 0$ represents massive gravity with graviton mass $m_g = A_0^{1/2} c^{-2}$

$$\tilde{h}(\nu) = A(\nu)e^{i\Phi(\nu)} \Rightarrow \boxed{\tilde{h}(\nu) = A(\nu)e^{i(\Phi(\nu)+\delta\Phi_\alpha(\nu))}}$$

A non-zero A_α will lead to a frequency-dependent dephasing of the GW signal, $\delta\Phi_\alpha(f)$, building up as the GW propagates towards Earth. For a given model (i.e., given the values of A_α, α) the dephasing $\delta\Phi_\alpha(f)$ depends on the binary's luminosity distance, the binary's detector-frame chirp mass, and the effective wavelength parameter used in the sampling, defined in terms of binary's redshift, and a distance parameter for a given cosmological model.



90% credible upper bounds on the absolute value of the modified dispersion relation parameter A_α as a function of α

no evidence for GW dispersion,

constraining the Lorentz-violating dispersion parameters

$$m_g \leq 1.76 \times 10^{-23} \text{eV}/c^2$$ with 90% credibility

graviton mass

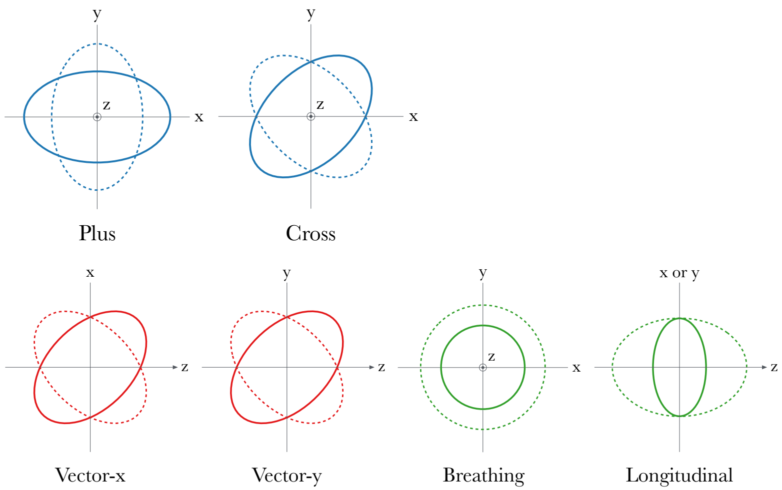
improvement of 1.8 over Solar System bounds

Tests of General Relativity

Deformation of a ring of freely-falling test particles under the influence of GW in z-direction

GR

Alternate gravity theories



The three-detector Advanced LIGO-Virgo network is generally unable to distinguish the polarization of transient GW signals, like those from BBHs

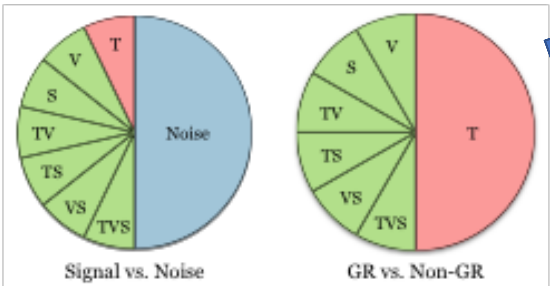
- Two LIGO detectors are nearly **co-oriented**, leaving LIGO sensitive to only a single polarization mode
- Even if the LIGO detectors were more favourably-oriented, a network of **at least six detectors** is generically required to uniquely determine the polarization content of a GW transient

Bayesian search

$$\Omega_{\rm TVS}(\nu) = \Omega_0^{\rm T}\left(\frac{\nu}{\nu_0}\right)^{\alpha_{\rm T}} \boxed{+\,\Omega_0^{\rm V}\left(\frac{\nu}{\nu_0}\right)^{\alpha_{\rm V}} + \Omega_0^{\rm S}\left(\frac{\nu}{\nu_0}\right)^{\alpha_{\rm S}}}$$

7 hypotheses: TVS, TV, TS, VS, T, V, S

Equal prior probability to noise and signal models, as well as equal prior probability to the seven signal sub-hypotheses



consistency with GR-polarization modes

Equal prior probability to the non-GR and GR models and identically weight the six non-GR sub-hypotheses

Tests of modified gravity

In the context of General Relativity, gravitational waves travelling on a four-dimensional Friedmann-Lemaître-Roberson-Walker background, obey the linearised evolution equation

$$h''_A + 2\mathcal{H}h'_A + k^2 h_A = \Pi_A \ ,$$

where $A = +, \times$ stands for the two polarisation plus and cross modes, primes denote derivatives with respect to conformal time η, related to the cosmological time through $\mathrm{d}\eta = \mathrm{d}t/a(t)$ with $a(t)$ the scale factor, $\mathcal{H}$ is the Hubble parameter in conformal time η, and Π_A denotes the source term related to the anisotropic stress tensor. The GW propagation equation above, gets modified in a generic modified gravity model into

$$h''_A + 2[1-\delta(\eta)]\mathcal{H}h'_A + [c_\mathrm{T}^2(\eta)k^2 + m_\mathrm{T}^2(\eta)]k^2 h_A = \Pi_A \ ,$$

where three new quantities have been introduced. The function $\delta(\eta)$ modifies the friction term and hence affects the amplitude of a GW propagating across cosmological distances. The tensor velocity c_T can be in general time and scale dependent; in General Relativity it is equal to the speed of light c. The mass of the tensor mode m_T, can be non-zero in the context of a modified gravity theory. These three quantities are in principle testable with GW data.

The modification in the tensor sector leads to the gravitational-wave luminosity distance $d_L^{(gw)}(z)$, which is different from the standard electromagnetic luminosity distance

$$d_{\mathrm{L}}^{(\mathrm{em})}(z) = (1+z)\int_0^z \frac{\mathrm{d}\tilde{z}}{H(\tilde{z})}$$

$$H(z) = H_0\sqrt{\Omega_{\mathrm{m}}(1+z)^3 + \Omega_{\mathrm{r}}(1+z)^4 + \Omega_{\mathrm{DE}}(z)}$$

A simple phenomenological parametrisation

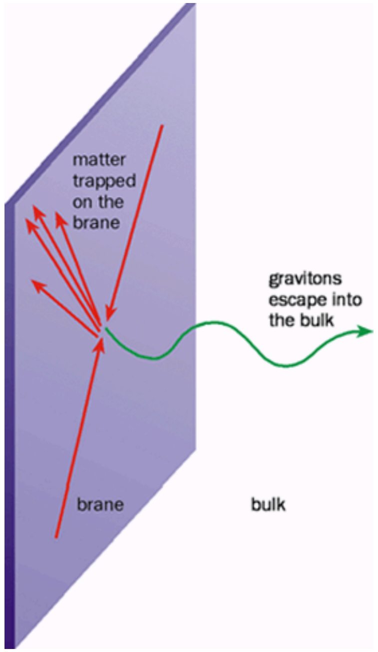
$$\Xi(z) \equiv \frac{d_{\mathrm{L}}^{(\mathrm{gw})}(z)}{d_{\mathrm{L}}^{(\mathrm{em})}(z)} = \Xi_0 + \frac{1-\Xi_0}{(1+z)^n}$$

which depends on the (positive) parameters Ξ_0 and n (with $\Xi_0 = 1$ in General Relativity)

supermassibe black hole mergers binaries detectable with LISA can provide a powerful probe of modified gravity and dark energy.

Model	$\Xi_0 - 1$	n
HS $f(R)$ gravity	$\frac{1}{2}f_{R0}$	$\frac{3(\tilde{n}+1)\Omega_m}{4-3\Omega_m}$
Designer $f(R)$ gravity	$-0.24\Omega_m^{0.76}B_0$	$3.1\Omega_m^{0.24}$
Jordan–Brans–Dicke	$\frac{1}{2}\delta\phi_0$	$\frac{3(\tilde{n}+1)\Omega_m}{4-3\Omega_m}$
Galileon cosmology	$\frac{\beta\phi_0}{2M_{\mathrm{Pl}}}$	$\frac{\dot{\phi}_0}{H_0\phi}$
$\alpha_M = \alpha_{M0}a^{\tilde{n}}$	$\frac{\alpha_{M0}}{2\tilde{n}}$	$\tilde{n}$
$\alpha_M = \alpha_{M0}\frac{\Omega_\Lambda(a)}{\Omega_\Lambda}$	$-\frac{\alpha_{M0}}{6\Omega_\Lambda}\ln\Omega_m$	$-\frac{3\Omega_\Lambda}{\ln\Omega_m}$
$\Omega = 1 + \Omega_+ a^{\tilde{n}}$	$\frac{1}{2}\Omega_+$	$\tilde{n}$
Minimal self-acceleration	$\lambda\left(\ln a_{acc} + \frac{C}{2}\chi_{acc}\right)$	$\frac{C/H_0-2}{\ln a_{acc}^2 - C\chi_{acc}}$

Constraints on the number of spacetime dimensions



Damping of the waveform due to gravitational leakage into extra dim

GR: $h_{\rm GR} \propto d_L^{-1}$ $\qquad$ $d_L^{\rm EM} \simeq \dfrac{z(1+z)}{H_0} \overset{z\ll 1}{\simeq} \dfrac{z}{H_0}$

Deviation depends on the number of dimensions D and would result to a systematic **overestimation of the source** $d_L^{\rm EM}$ **inferred from GW data**

Extra dim models: assume that light and matter propagate in 4 ST dim

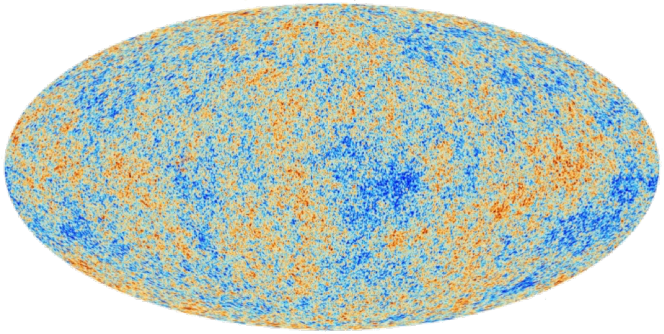
$$\boxed{h \propto \frac{1}{d_L^{\rm GW}} = \frac{1}{d_L^{\rm EM}}\left[1+\left(\frac{d_L^{\rm EM}}{R_c}\right)^n\right]^{-(D-4)/(2n)}}$$

$\mathrm{GW170817}$

For higher-dim theories with characteristic length scale of the order of the Hubble radius ~4Gpc (e.g. DGP model of dark energy), small steepnesses (~0.1) are excluded by the data.

Anisotropies in the GW Background: info about large-scale-structure

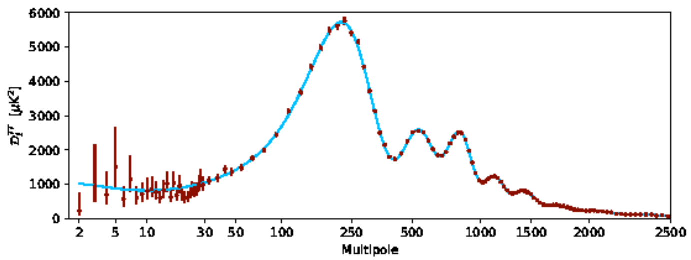
To a first approximation, the SGWB is assumed to be isotropic (analogous to the CMB)



The afterglow radiation left over from the Hot Big Bang

- its temperature is extremely uniform all over the sky
- **tiny temperature fluctuations** (one part 100,000)

$$C_\ell = \int \mathrm{d}^2\hat{\boldsymbol{n}} P_\ell(\cos\theta)\,\langle \delta T_\gamma \delta T_\gamma \rangle_\theta$$



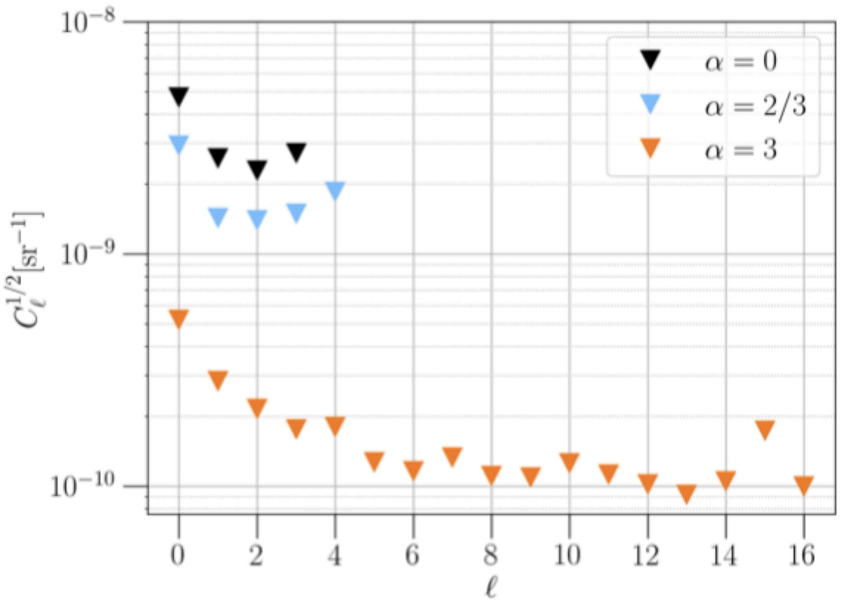
LSS

SGWB

$$\boxed{C_\ell = \int \mathrm{d}^2\hat{\boldsymbol{n}} P_\ell(\cos\theta)\,\langle \delta\Omega_{\rm GW} \delta\Omega_{\rm GW} \rangle_\theta}$$

Anisotropies in the GW Background: info about large-scale-structure

Gravitational wave sources with an anisotropic spatial distribution lead to a GWB characterised by preferred directions, and hence anisotropies



95% upper limits on C_ℓ for different α using combined O1+O2+O3 data

$$\Omega_{\rm GW}(\nu) = \Omega_{\rm ref}\left(\frac{\nu}{\nu_{\rm ref}}\right)^{\alpha}$$

Diffraction-limited angular resolution Θ on the sky:

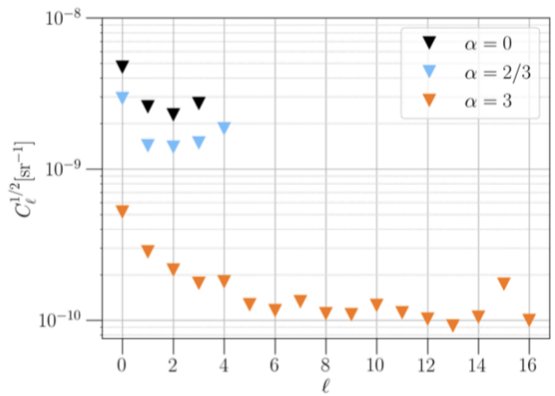
$$\theta = \frac{c}{2Df} \qquad \ell_{\rm max} = \frac{\pi}{\theta}$$

distance between detectors

most sensitive frequency

Anisotropies in the GW Background: info about large-scale-structure

Gravitational wave sources with an anisotropic spatial distribution lead to a GWB characterised by preferred directions, and hence anisotropies



$$\boxed{\alpha = 2/3,}$$

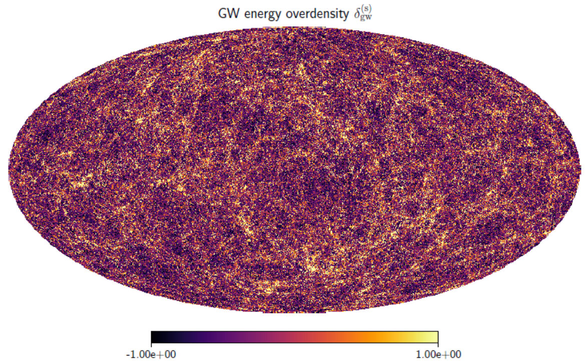
$$\boxed{C_\ell^{1/2} < 1.9\times10^{-9}\,\mathrm{sr}^{-1},}$$

Anisotropies largely independent from astrophysical model of CBCs

But theoretical studies:

$$\boxed{C_\ell^{1/2} \sim 10^{-12}\,\mathrm{sr}^{-1} \;\; \text{for} \;\; 1 \le \ell \le 4,}$$

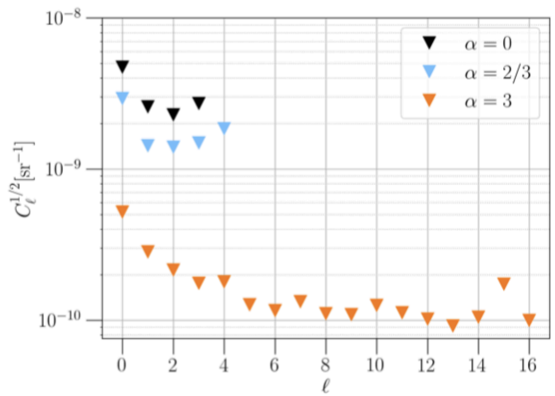
if the normalised GW energy density due to an isotropic GWB of CBC is $\sim 10^{-9}$



Angular resolution: 13.7 arcminutes ---- 7.3 galaxies per pixel

Anisotropies in the GW Background: info about large-scale-structure

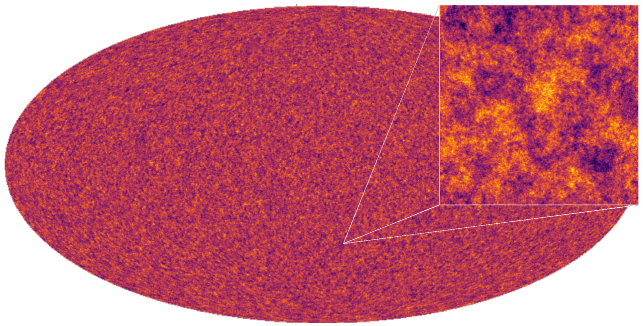
Gravitational wave sources with an anisotropic spatial distribution lead to a GWB characterised by preferred directions, and hence anisotropies



$$\boxed{\alpha = 0}$$

$$\boxed{C_1^{1/2} < 2.6 \times 10^{-9}\,\mathrm{sr}^{-1}}$$

Anisotropies largely independent of the cosmic string loop distribution model



But theoretical studies:

$$\boxed{C_1^{1/2} \lesssim 10^{-12}\mathrm{sr}^{-1}}$$

using that the isotropic component

$$\boxed{C_1^{1/2} \lesssim 10^{-12}\mathrm{sr}^{-1}} \lesssim 4 \times 10^{-15}$$

The dipole is kinematically caused by the peculiar motion of the Earth

GWs offer a new window for exploring late and early stages of the Universe

The importance of the recent direct direction of GWs from BH and NS mergers can hardly be over-emphasised

Even a non-detection of GWs allowed us to gain important information on cosmology, particle physics models and gravity

GWs offer a novel and powerful way to test

- astrophysical models and *large-scale-structure of the universe*
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