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Early Universe, Inflation and Primordial Gravitational Waves

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Lecture III

Linear perturbation theory

first order perturbation theory

all quantities are written as a sum of the background value, corresponding to the homogeneous and isotropic model, and a perturbation, which is the deviation from the background value.

The perturbed metric

line element

$$
\begin{aligned}
ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\
&= (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}) dx^\alpha dx^\beta
\end{aligned}
$$

$\bar{g}_{\alpha\beta}$ is the background metric FLRW

$$\bar{g}_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$$

$\delta g_{\alpha\beta}$ is a perturbation, which we take to be small.

consider spatially flat backgrounds

assume that we have chosen an appropriate coordinate system such that the metric perturbations are small, so we can neglect all terms which are second order or higher in the metric perturbations.

most general linear perturbation around the $\bar{g}_{\alpha\beta}$

density perturbation	Φ, Ψ, B and E are *scalars*
rotation	S_i, F_i are *vectors*
GWs	h_{ij} is a *tensor*

$$f_{,i} \equiv \partial f / \partial x^i$$

$$
\begin{aligned}
ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\
&= \bar{g}_{\alpha\beta} dx^\alpha dx^\beta + \delta g_{\alpha\beta} dx^\alpha dx^\beta \\
&= -(1+2\Phi)dt^2 + 2a(t)(B_{,i} - S_i)dx^i dt \\
&\quad + a(t)^2 \left[(1-2\Psi)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij} \right] dx^i dx^j
\end{aligned}
$$

vector perturbations are transverse, $\delta^{ij} S_{i,j} = \delta^{ij} F_{i,j} = 0$

tensor perturbation is transverse and traceless, $\delta^{ij} h_{ij} = 0, h_{ij,j} = 0$

Since we drop all non-linear terms, the scalar, vector and tensor perturbations evolve independently.

The vector perturbations decay with the expansion, and are expected to be negligible in the linear regime, so we put them to zero, $S_i = F_i = 0$.

For the metric perturbation, we have 10 functions $\delta g_{\alpha\beta}(t, \boldsymbol{x})$. However, four of them are not physical degrees of freedom, they just correspond to the freedom of choosing the four coordinates. So there are 6 physical degrees of freedom.

There are thus different coordinate systems (also called different *gauges*) which describe the same physics.

longitudinal gauge and also as *the conformal Newtonian gauge.*

$E = B = 0$ doing so fixes the coordinate system completely.

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= \bar{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \delta g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= -(1+2\Phi)\mathrm{d}t^2 + a(t)^2\left[(1-2\Psi)\delta_{ij} + h_{ij}\right]\mathrm{d}x^i \mathrm{d}x^j \end{aligned}$$

two scalar degrees of freedom

one transverse traceless symmetric tensor, which has two independent degrees of freedom

metric perturbations $\Phi(t, \boldsymbol{x})$ and $\Psi(t, \boldsymbol{x})$ *Bardeen potentials*

The evolution of the metric perturbations is determined by the Einstein equation, which couples the metric to the matter content as described by the energy-momentum tensor.

The perturbed equations of motion

Einstein equation $G_{\alpha\beta} = 8\pi G_{\rm N} T_{\alpha\beta}$

$$T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta}$$

four-velocity is normalised $g_{\alpha\beta} u^\alpha u^\beta = -1$

⇒ $\delta u^0 = -\Phi$ in linear theory

$$\begin{aligned} \rho(t, \boldsymbol{x}) &= \bar\rho(t) + \delta\rho(t, \boldsymbol{x}) \\ p(t, \boldsymbol{x}) &= \bar p(t) + \delta p(t, \boldsymbol{x}) \\ u^\alpha(t, \boldsymbol{x}) &= \delta^{\alpha 0} + \delta u^\alpha(t, \boldsymbol{x}) \end{aligned}$$

⇓

for the background

$$\begin{aligned} 3H^2 &= 8\pi G_{\rm N} \bar\rho \\ 3(\dot H + H^2) &= -4\pi G_{\rm N} (\bar\rho + 3\bar p) \end{aligned}$$

$$\ddot a / a = \dot H + H^2$$

For the perturbations

$$\begin{aligned} 4\pi G_{\rm N} \delta\rho &= \frac{1}{a^2} \nabla^2 \Psi - 3H(\dot\Psi + H\Phi) \\ 4\pi G_{\rm N} (\bar\rho + \bar p) \delta u_i &= -(\dot\Psi + H\Phi)_{,i} \\ 4\pi G_{\rm N} \delta p \delta_{ij} &= \left[(2\dot H + 3H^2)\Phi + H\dot\Phi + \ddot\Psi + 3H\dot\Psi + \frac{1}{2}\frac{1}{a^2}\nabla^2 D \right] \delta_{ij} \\ &\quad - \frac{1}{2}\frac{1}{a^2} D_{,ij} \\ 0 &= \ddot h_{ij} + 3H\dot h_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} \, , \end{aligned}$$

$\nabla^2 \equiv \delta^{ij} \partial_i \partial_j$

$D \equiv \Phi - \Psi .$

$D_{,ij} = 0$ for all $i \neq j$.

$D = A(t, x) + B(t, y) + C(t, z)$.

no preferred coordinate axes, $D = D(t)$

$D(t) = 0$ without loss of generality.

⇐ $\Phi = \Psi$

notation:

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \qquad w \equiv \bar{p}/\bar{\rho} \qquad v^2 \equiv \delta p/\delta\rho$$

density contrast background equation of state

$$\begin{aligned}
0 &= \ddot{\Phi} + H(4+3v^2)\dot{\Phi} - v^2\frac{1}{a^2}\nabla^2\Phi + [2\dot{H} + (3+3v^2)H^2]\Phi \\
\delta &= \frac{2}{3}\frac{1}{(aH)^2}\nabla^2\Phi - 2\frac{1}{H}\dot{\Phi} - 2\Phi \\
\delta u^i &= \frac{1}{a^2\dot{H}}\partial_i(\dot{\Phi} + H\Phi) \\
0 &= \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} \ .
\end{aligned}$$

the metric perturbation

Φ is non-zero only if there is matter

So ϕ is generated directly by matter sources, in particular by the density perturbations.

In contrast, the tensor perturbation h_{ij} can be non-zero even if the space is empty: they correspond to *gravity waves*

The procedure for solving the perturbed equations is the following.

1) Give the matter model, i.e. give w and v^2.

2) Solve for the evolution of the background and obtain $a(t)$.

3) Solve the perturbation equations.

Since the equations are linear, they are easily solved in terms of a Fourier transformation

$$\Phi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \Phi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

and $\delta_{\boldsymbol{k}}$, $u^i_{\boldsymbol{k}}$ and $h_{\boldsymbol{k}ij}$ are defined in the same way.

the variable $\boldsymbol{k}$ the *comoving momentum* or *comoving wavenumber*

the physical momentum $\boldsymbol{k}/a$

$k \equiv |\boldsymbol{k}|$

$$\begin{aligned} 0 &= \ddot{\Phi}_{\boldsymbol{k}} + H(4+3v^2)\dot{\Phi}_{\boldsymbol{k}} + v^2\frac{k^2}{a^2}\Phi_{\boldsymbol{k}} + [2\dot{H} + (3+3v^2)H^2]\Phi_{\boldsymbol{k}} \\ \delta_{\boldsymbol{k}} &= -\frac{2}{3}\frac{k^2}{(aH)^2}\Phi_{\boldsymbol{k}} - 2\frac{1}{H}\dot{\Phi}_{\boldsymbol{k}} - 2\Phi_{\boldsymbol{k}} \\ 0 &= \ddot{h}_{\boldsymbol{k}ij} + 3H\dot{h}_{\boldsymbol{k}ij} + \frac{k^2}{a^2}h_{\boldsymbol{k}ij}\ , \end{aligned}$$

give the time evolution of the Fourier components, but the spatial dependence (i.e. dependence on $\boldsymbol{k}$) is left unconstrained

The spatial dependence is fixed by the initial conditions

we drop the equation for the velocity

Inflationary perturbations

$$\varphi(t,\boldsymbol{x}) = \bar{\varphi}(t) + \delta\varphi(t,\boldsymbol{x})$$

equation of motion for the scalar field

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - V'(\varphi) = 0$$

Spatially flat FLRW metric

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V'(\varphi) = 0$$

in Minkowski space reduces to

$$\ddot{\varphi} - \nabla^2\varphi + V'(\varphi) = 0$$

For perturbed FLRW metric in longitudinal gauge

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\nabla^2 + V''(\bar{\varphi})\right)\delta\varphi = -2\Phi V'(\bar{\varphi}) + \left(\dot{\Phi} + 3\dot{\Psi}\right)\dot{\bar{\varphi}}$$

Making a Fourier transformation

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = -2\Phi_{\boldsymbol{k}}V'(\bar{\varphi}) + \left(\dot{\Phi}_{\boldsymbol{k}} + 3\dot{\Psi}_{\boldsymbol{k}}\right)\dot{\bar{\varphi}}$$

$$m^2(\bar{\varphi}) \equiv V''(\bar{\varphi})$$

equation of motion of the inflaton perturbations *during slow-roll inflation*

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0$$

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0$$

During inflation, H and m^2 change slowly. → treat them as constants

general solution
$$\delta\varphi_{\boldsymbol{k}}(t) = a^{-3/2}\left[A_{\boldsymbol{k}}J_{-\nu}\left(\frac{k}{aH}\right) + B_{\boldsymbol{k}}J_{\nu}\left(\frac{k}{aH}\right)\right]$$

J_ν is the Bessel function of order ν
$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

time dependence of the scale factor for constant H
$$a(t) \propto e^{Ht}$$

If the slow-roll approximation is valid, the inflaton has negligible mass, $m^2 \ll H^2$,
$$\frac{m^2}{H^2} = 3M_{\rm Pl}^2\frac{V''}{V} = 3\eta \ll 1$$

$$\nu = \frac{3}{2}$$

→ solution
$$\delta\varphi_{\boldsymbol{k}}(t) = A_{\boldsymbol{k}}w_k(t) + B_{\boldsymbol{k}}w_k^*(t)$$
$$w_k(t) = \left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right)$$

→ Well before horizon exit, $k \gg aH$, the argument of the exponent is large, solution oscillates rapidly.
After horizon exit, $k \ll aH$, the solution stops oscillating and approaches the constant value $i(A_{\boldsymbol{k}} - B_{\boldsymbol{k}})$

comoving curvature perturbation $\mathcal{R}$ in the longitudinal gauge

we should calculate the inflaton field perturbation some time after horizon exit, when it has settled to a constant value, calculate $\mathcal{R}$ with

$$\mathcal{R}_{\boldsymbol{k}} = -H\frac{\delta\varphi_{\boldsymbol{k}}}{\dot{\varphi}}$$

gauge-independent and conserved outside the horizon

inflation generates primordial perturbations $\mathcal{R}_{\boldsymbol{k}}$ with the power spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\varphi}}\frac{H}{2\pi}\right)^2_{aH=k}$$

slow-roll inflation

$H^2 = \frac{V}{3M_{\rm Pl}^2}$ and $3H\dot{\varphi} = -V'$

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2}\frac{1}{M_{\rm Pl}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2}\frac{1}{M_{\rm Pl}^4}\frac{V}{\varepsilon}$$

observations of CMB and large-scale structure, $\mathcal{P}_{\mathcal{R}}(k)^{1/2} \approx 5\times10^{-5}$ on cosmological scales.

constraint on inflation $\left(\frac{V}{\varepsilon}\right)^{1/4} \approx 24^{1/4}\sqrt{\pi}\sqrt{5\times10^{-5}}M_{\rm Pl} \approx 0.028M_{\rm Pl} = 6.8\times10^{16}$ GeV

$\varepsilon \ll 1$

upper limit on the energy scale of inflation $V^{1/4} < 0.028M_{\rm Pl}$

$H^2 = V/(3M_{\rm Pl}^2)$ ⇒ $H < 10^{15}$ GeV, $H^{-1} > 10^{-31}$ m slow-roll inflation → expect $\mathcal{P}_{\mathcal{R}}(k)$ to be a slowly varying function of k

describe this small variation with the *spectral index* n of the primordial spectrum

$$n(k) - 1 \equiv \frac{d\ln \mathcal{P}_{\mathcal{R}}}{d\ln k}$$

If the spectral index is independent of k, we say that the spectrum is *scale-free*.

→ the primordial spectrum is a *power law*

$$\mathcal{P}_{\mathcal{R}}(k) = A^2 \left(\frac{k}{k_p}\right)^{n-1}$$

the "pivot scale" k_p is some chosen reference scale

A is the amplitude at this pivot scale.

If $\mathcal{P}_{\mathcal{R}}(k) = \text{const.}$ the spectrum is *scale- invariant*

$n = 1$

Harrison–Zel'dovich spectrum

If $n \neq 1$, the spectrum is called *tilted*.

red if $n < 1$ (more power on large scales)

blue if $n > 1$ (more power on small scales).

$dn/dk \neq 0$ *running spectral index*

$\mathcal{P}_{\mathcal{R}}(k)$ is evaluated when $k = aH$

$$\frac{d\ln k}{dt} = \frac{d\ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = (1-\varepsilon)H$$

$$\dot{H} = -\varepsilon H^2$$

$$\frac{d}{d\ln k} = \frac{1}{1-\varepsilon}\frac{1}{H}\frac{d}{dt} = \frac{1}{1-\varepsilon}\frac{\dot{\varphi}}{H}\frac{d}{d\varphi} = -\frac{M_{\rm Pl}^2}{1-\varepsilon}\frac{V'}{V}\frac{d}{d\varphi} \approx -M_{\rm Pl}^2\frac{V'}{V}\frac{d}{d\varphi}$$

scale dependence of the slow-roll parameters

$$\frac{d\varepsilon}{d\ln k} = -M_{\rm Pl}^2\frac{V'}{V}\frac{d}{d\varphi}\left[\frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2\right] = M_{\rm Pl}^4\left[\left(\frac{V'}{V}\right)^4 - \left(\frac{V'}{V}\right)^2\frac{V''}{V}\right] = 4\varepsilon^2 - 2\varepsilon\eta$$

$$\frac{d\eta}{d\ln k} = \ldots = 2\varepsilon\eta - \xi$$

spectral index:

$$n - 1 = \frac{1}{\mathcal{P}_{\mathcal{R}}}\frac{d\mathcal{P}_{\mathcal{R}}}{d\ln k} = \frac{\varepsilon}{V}\frac{d}{d\ln k}\left(\frac{V}{\varepsilon}\right) = \frac{1}{V}\frac{dV}{d\ln k} - \frac{1}{\varepsilon}\frac{d\varepsilon}{d\ln k}$$

$$= -M_{\rm Pl}^2\frac{V'}{V}\cdot\frac{1}{V}\frac{dV}{d\varphi} - 4\varepsilon + 2\eta = -6\varepsilon + 2\eta\ .$$

$\varepsilon \ll 1$ and $|\eta| \ll 1$

the spectrum is predicted to be close to scale invariant.

Example $$V(\varphi) = \frac{1}{2} m^2 \varphi^2$$

$$\varepsilon = \eta = 2 \frac{M_{\rm Pl}^2}{\varphi^2}$$

$$n \ = \ 1 - 6\varepsilon + 2\eta \ = \ 1 - 8 \left(\frac{M_{\rm Pl}}{\varphi} \right)^2$$

$$N(\varphi) = \frac{1}{M_{\rm Pl}^2} \int_{\varphi_{\rm end}}^{\varphi} \frac{V}{V'} d\varphi = \frac{1}{M_{\rm Pl}^2} \int \frac{\varphi}{2} d\varphi = \frac{1}{4M_{\rm Pl}^2} \left(\varphi^2 - {\varphi_{\rm end}}^2 \right)$$

$$N \approx 50 \ldots 60.$$

$$\varepsilon(\varphi_{\rm end}) = 2M_{\rm Pl}^2 / {\varphi_{\rm end}}^2 = 1 \quad \Rightarrow \quad \varphi_{\rm end} = \sqrt{2} M_{\rm Pl}$$

$$\varphi^2 = {\varphi_{\rm end}}^2 + 4M_{\rm Pl}^2 N = 2M_{\rm Pl}^2 + 4M_{\rm Pl}^2 N \approx 4M_{\rm Pl}^2 N$$

$$\left(\frac{M_{\rm Pl}}{\varphi} \right)^2 = \frac{1}{4N}$$

$$n \ = \ 1 - \frac{2}{N} \approx 0.96$$

$$m \approx \frac{9}{N} 10^{14} \text{ GeV} \approx 2 \times 10^{13} \text{ GeV} \approx 8 \times 10^{-6} M_{\rm Pl}$$

for $N = 50$.

energy scale of inflation $$V_{\rm end}^{1/4} = \left(\frac{1}{2} m^2 {\varphi_{\rm end}}^2 \right)^{1/4} = \sqrt{\frac{m}{M_{\rm Pl}}} M_{\rm Pl} \approx 3 \times 10^{-3} M_{\rm Pl} \approx 7 \times 10^{15} \text{ GeV}$$

Cosmic Microwave Background Temperature Anisotropies

The observed temperature anisotropy is due to two contributions, an *intrinsic* temperature variation at the surface of last scattering and a variation in the redshift the photons have suffered during their journey to us,

$$\left(\frac{\delta T}{T}\right)_{\rm obs} = \left(\frac{\delta T}{T}\right)_{\rm intr} + \left(\frac{\delta T}{T}\right)_{\rm jour}$$

$(\delta T/T)_{\rm intr}$ and $(\delta T/T)_{\rm jour}$ depending on the gauge, but their sum $(\delta T/T)_{\rm obs}$ is gauge-independent,

angular average of the temperature field $\bar{T} \equiv T_0 \equiv \frac{1}{4\pi}\int d\Omega T$ mean

$$\delta T = T - T_0 \qquad \text{anisotropy}$$

$$T_0 = 2.725 \text{ K}$$

The CMB temperature anisotropy is a function on a sphere. In analogy with Fourier expansion in three-dimensional flat space, we separate out the contributions of different angular scales by doing a multipole expansion,

$$\frac{\delta T}{T_0}(\theta,\phi) = \sum a_{\ell m} Y_{\ell m}(\theta,\phi)$$

$l = 1, 2, \ldots \infty$ and $m = -l, \ldots, l$,

functions $Y_{\ell m}(\theta,\phi)$ are the *spherical harmonics*

$2\ell + 1$ values of m for each ℓ

orthonormal functions on the sphere

$$\int d\Omega\, Y_{\ell m}(\theta,\phi) Y^*_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

closure relation $\sum_m |Y_{\ell m}(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$

$$a_{\ell m} = \int Y^*_{\ell m}(\theta,\phi)\frac{\delta T}{T}(\theta,\phi) d\Omega$$

multipole coefficients $a_{\ell m}$

The theoretical angular power spectrum

Primordial fluctuations through standard inflationary mechanism: Gaussian spectrum

$a_{\ell m}$ Gaussian random variables $\langle a_{\ell m} \rangle = 0$

they represent deviation from the average temperature

the (theoretical) *angular power spectrum* $C_\ell \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell+1} \sum_m \langle |a_{\ell m}|^2 \rangle$ (independent of m)

The $a_{\ell m}$ are independent random variables, so

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell$$

C_ℓ contains all the statistical information about the CMB temperature anisotropy. This is all we can predict from theory.

the angular power spectrum C_ℓ is related to the contribution of multipole ℓ to the temperature variance,

$$\left\langle \left(\frac{\delta T(\theta,\phi)}{T} \right)^2 \right\rangle = \left\langle \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta,\phi) \sum_{\ell' m'} a^*_{\ell' m'} Y^*_{\ell' m'}(\theta,\phi) \right\rangle$$

$$= \sum_{\ell \ell'} \sum_{m m'} Y_{\ell m}(\theta,\phi) Y^*_{\ell' m'}(\theta,\phi) \langle a_{\ell m} a^*_{\ell' m'} \rangle$$

$$= \sum_\ell C_\ell \sum_m |Y_{\ell m}(\theta,\phi)|^2 = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \,,$$

$$\sum_m |Y_{\ell m}(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$$

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell$$

Observed angular power spectrum

Theory predicts expectation values $\langle |a_{\ell m}|^2 \rangle$ from the random process responsible for the CMB anisotropy, but we can observe only one realisation of this random process, the set $\{a_{\ell m}\}$ of our CMB sky.

$$\widehat{C}_\ell \equiv \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2$$

observed angular power spectrum

the average of these observed values.

The variance of the observed temperature anisotropy is the average of $\left(\frac{\delta T(\theta,\phi)}{T}\right)^2$ over the celestial sphere,

$$\begin{aligned}
\frac{1}{4\pi}\int \left[\frac{\delta T(\theta,\phi)}{T}\right]^2 d\Omega &= \frac{1}{4\pi}\int d\Omega \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta,\phi) \sum_{\ell' m'} a^*_{\ell' m'} Y^*_{\ell' m'}(\theta,\phi) \\
&= \frac{1}{4\pi}\sum_{\ell m}\sum_{\ell' m'} a_{\ell m} a^*_{\ell' m'} \underbrace{\int Y_{\ell m}(\theta,\phi) Y^*_{\ell' m'}(\theta,\phi) d\Omega}_{\delta_{\ell\ell'}\delta_{mm'}} \\
&= \frac{1}{4\pi}\sum_{\ell} \underbrace{\sum_{m} |a_{\ell m}|^2}_{(2\ell+1)\widehat{C}_\ell} \\
&= \sum_\ell \frac{2\ell+1}{4\pi}\widehat{C}_\ell \,.
\end{aligned}$$

Cosmic variance

The expectation value of the observed spectrum $\widehat{C}_\ell$ is equal to C_ℓ, the *theoretical* spectrum

$$\langle \widehat{C}_\ell \rangle = C_\ell \quad \Rightarrow \quad \langle \widehat{C}_\ell - C_\ell \rangle = 0$$

but its actual, realised, value is not, although we expect it to be close. The expected squared difference between $\widehat{C}_\ell$ and C_ℓ is called the *cosmic variance*.

$$\langle (\widehat{C}_\ell - C_\ell)^2 \rangle = \frac{2}{2\ell + 1} C_\ell^2 \,.$$

We see that the expected relative difference between $\widehat{C}_\ell$ and C_ℓ is smaller for higher ℓ. This is because we have a larger (size $2\ell + 1$) statistical sample of $a_{\ell m}$ available for calculating the $\widehat{C}_\ell$.

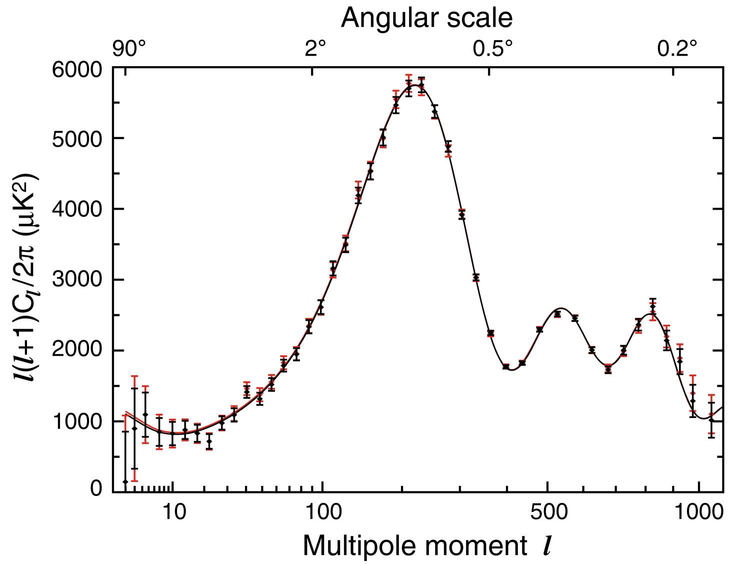
The cosmic variance limits the accuracy of comparison of CMB observations with theory, especially for large scales (low ℓ).

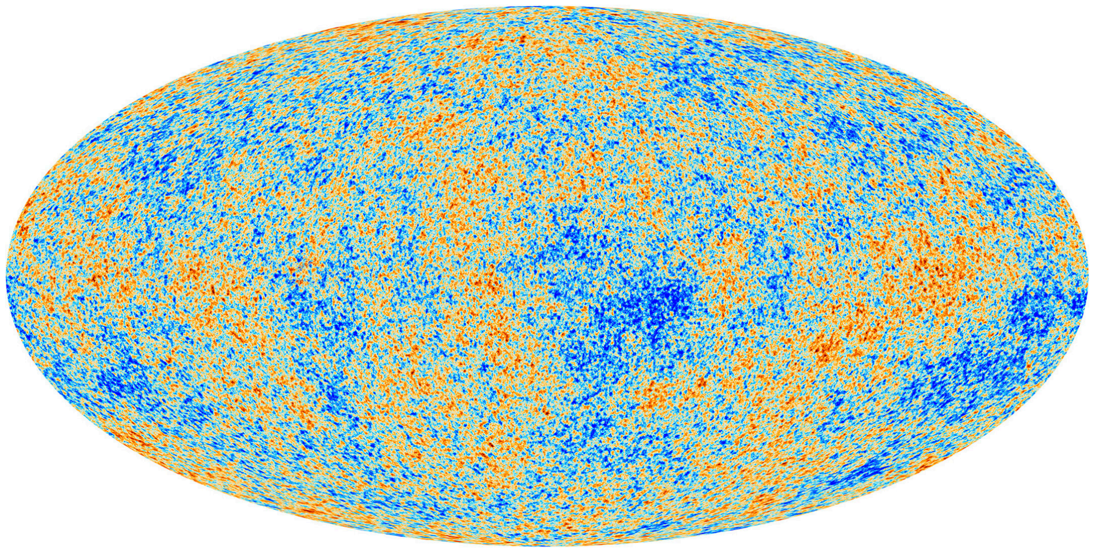
Relation between angular scales and multipole moments

$$\theta_{\rm res} = \frac{\pi}{\ell} = \frac{180^\circ}{\ell}$$

the angular resolution required of the microwave detector for it to be able to resolve the angular power spectrum up to this ℓ.

For example, COBE had an angular resolution of 7° allowing a measurement up to $\ell = 180/7 = 26$, WMAP had resolution 0.23° reaching to $\ell = 180/0.23 = 783$, and the European Planck satellite has resolution $5'$, which allows to measure C_ℓ up to $\ell = 2160$





CMB anisotropy from perturbation theory

$$\left(\frac{\delta T}{T}\right)_{\rm obs} = \left(\frac{\delta T}{T}\right)_{\rm intr} + \left(\frac{\delta T}{T}\right)_{\rm jour}$$

$$\begin{aligned}\left(\frac{\delta T}{T}\right)_{\rm jour} &= -\int_{\rm dec}^{o} d\Phi + \boldsymbol{v}_{\rm obs}\cdot\hat{\boldsymbol{n}} + \int_{\rm dec}^{o} {\rm d}t\left(\dot\Phi + \dot\Psi - \frac{1}{2}\dot h_{ij}\hat n^i \hat n^j\right) \\ &= \Phi(t_{\rm dec}, \boldsymbol{x}_{\rm ls}) - \Phi(t_0, \boldsymbol{0}) + \boldsymbol{v}_{\rm obs}\cdot\hat{\boldsymbol{n}} + \int_{\rm dec}^{o} {\rm d}t\left(\dot\Phi + \dot\Psi - \frac{1}{2}\dot h_{ij}\hat n^i \hat n^j\right) \\ &\overset{\Psi\approx\Phi}{=} \Phi(t_{\rm dec}, \boldsymbol{x}_{\rm ls}) - \Phi(t_0, \boldsymbol{0}) + \boldsymbol{v}_{\rm obs}\cdot\hat{\boldsymbol{n}} + 2\int_{\rm dec}^{o} {\rm d}t\dot\Phi - \frac{1}{2}\hat n^i \hat n^j \int_{\rm dec}^{o} {\rm d}t \dot h_{ij}\end{aligned}$$

Effect of GWs, usually not considered

Same for photons coming from all directions, so it does not contribute to observed anisotropy

Doppler effect from observer's motion, it causes a dipole pattern in the CMB, usually removed from CMB maps

where the integral is from $(t_{\rm dec}, \boldsymbol{x}_{\rm ls})$ to $(t_0, \boldsymbol{0})$ along the path of the photon (a null geodesic) and $\hat{\boldsymbol{n}}$ is a unit vector pointing in the direction the observer is looking at. The observer's location has been chosen as the origin $\boldsymbol{0}$. The term $\boldsymbol{v}_{\rm obs}\cdot\hat{\boldsymbol{n}}$ is the Doppler effect from the observer's motion (which is assumed nonrelativistic, $|\boldsymbol{v}_{\rm obs}| \ll 1$), where $\boldsymbol{v}_{\rm obs}$ is the observer's velocity.

$$\rho_\gamma = \frac{\pi^2}{15} T^4$$

the local temperature perturbation is directly related to the relative perturbation in the photon energy density,

$$\left(\frac{\delta T}{T}\right)_{\rm intr} = \frac{1}{4}\delta_\gamma - \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,.$$

$$\left(\frac{\delta T}{T}\right)_{\rm obs} = \frac{1}{4}\delta_\gamma - \boldsymbol{v}\cdot\hat{\boldsymbol{n}} + \Phi(t_{\rm dec}, \boldsymbol{x}_{\rm ls}) + 2\int_{\rm dec}^{o} \dot{\Phi}\, dt$$

ordinary Sachs–Wolfe effect

integrated Sachs-Wolfe effect (ISW)

Both the density perturbation δ_γ and the fluid velocity $\boldsymbol{v}$ are gauge dependent

Choose gauge and select initial conditions (adiabatic versus isocurvature)

The *adiabatic mode* is defined as a perturbation affecting all the cosmological species such that the relative ratios in the number densities remain unperturbed, i.e., such that

$$\delta(n_X/n_Y) = 0.$$

It is associated with a curvature perturbation, via Einstein's equations, since there is a global perturbation of the matter content. This is why the adiabatic perturbation is also called *curvature* perturbation. In terms of the energy density contrasts, defined by

$$\delta_X \equiv \frac{\delta\rho_X}{\rho_X}$$

the adiabatic perturbation is characterized by the relations

$$\boxed{\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = \frac{1}{3}\delta_b = \frac{1}{3}\delta_c}$$

Since there are several cosmological species, it is also possible to perturb the matter components without perturbing the geometry. This corresponds to *isocurvature* perturbations, characterized by variations in the particle number ratios but with vanishing curvature perturbation. The variation in the relative particle number densities between two species can be quantified by the so-called *entropy perturbation*

$$S_{A,B} \equiv \frac{\delta n_A}{n_A} - \frac{\delta n_B}{n_B}$$

When the equation of state for a given species is such that $w \equiv p/\rho = \text{Const}$, then one can re-express the entropy perturbation in terms of the density contrast, in the form

$$S_{A,B} \equiv \frac{\delta_A}{1+w_A} - \frac{\delta_B}{1+w_B}$$

choose a species of reference, for instance the photons,

$$S_b \equiv \delta_b - \frac{3}{4}\delta_\gamma,$$ *baryon isocurvature mode*

$$S_c \equiv \delta_c - \frac{3}{4}\delta_\gamma$$ *CDM isocurvature mode*

$$S_\nu \equiv \frac{3}{4}\delta_\nu - \frac{3}{4}\delta_\gamma$$ *neutrino isocurvature mode*

In terms of the entropy perturbations, the adiabatic mode is

$$S_b = S_c = S_\nu = 0$$