SIGRAV

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Online

Early Universe, Inflation and Primordial Gravitational Waves

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Lecture II

Phase transitions, Spontaneously Broken Symmetries, Topological Defects

The concept of Spontaneous Symmetry Breaking (SSB) has its origin in condensed matter physics

In field theory (FT), the role of the order parameter is played by scalar fields, the Higgs fields

The symmetry is said to be spontaneously broken if *the ground state is characterised by a nonzero expectation value of the Higgs field and does not exhibit the full symmetry of the Hamiltonian*

The Goldstone model

ϕ a complex scalar field with classical Lagrangian density:

$$\mathcal{L} = (\partial_\mu \bar{\phi})(\partial^\mu \phi) - V(\phi)$$

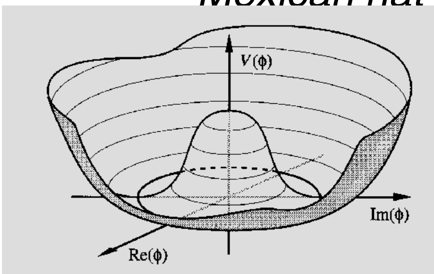
and potential:

$$V(\phi) = \frac{1}{4}\lambda[\bar{\phi}\phi - \eta^2]^2$$

dimensions of mass :
Vacuum Expectation value (VEV)

positive constants

Mexican hat



The Goldstone model is invariant under the U(1) group of global phase transformations

$$\phi(x) \rightarrow e^{i\alpha}\phi(x)$$

constant (independent of spacetime)

The minima of the potential lie on a circle with fixed radius $|\phi| = \eta$

arbitrary phase

The ground state is characterised by: $\langle 0|\phi|0\rangle = \eta e^{i\theta} \neq 0$

Note: At the classical level, the vacuum can be obtained by demanding the energy (the Hamiltonian) to be a minimum.

The minimal energy is reached when the field is invariant w.r.t. space-time transformations $\Longrightarrow$ $\dot{\phi} = \nabla\phi = 0$

Furthermore, the V should minimised $\Longrightarrow$ $\bar{\phi}(\bar{\phi}\phi - \eta^2) = 0$

$\Rightarrow$ $\phi = 0$ **local maximum** $\qquad |\phi| = \eta$ **absolute minimum**

The phase transformation leads to the change $\langle 0|\phi|0\rangle = 0$

$\Rightarrow$ the vacuum state $|0\rangle$ is not invariant under the phase transformation; spontaneously broken symmetry

The state of unbroken symmetry with $\theta \rightarrow \theta + \alpha$ is a local maximum of the Mexican hat potential

All broken symmetry vacua, each with a different value of the phase θ are equivalent

If we select the vacuum with $\theta = 0$, the complex scalar field ϕ can be written in terms of two real scalar fields ϕ_1, ϕ_2, with zero vacuum expectation values as:

$$\phi = \eta + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

⇨ Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}\lambda \eta^2 \phi_1^2 + \mathcal{L}_{\text{int}} \; .$$

massive particle with mass $\sqrt{\lambda}\eta > 0$

massless scalar particle; the Goldstone boson

interaction term; it includes cubic and higher order terms in the real scalar fields

The appearance of Goldstone bosons is a generic feature of models with spontaneously broken global symmetries

Note:

We considered a purely classical potential to determine expectation value of Higgs field

However, the Higgs field is a quantum field which interacts with itself, as well as with other quantum fields

⇨ the classical potential should be modified by radiative corrections, leading to an effective potential; it can be calculated perturbatively as an expansion in powers of coupling constants:

$$V_{\mathrm{eff}}(\phi) = V(\phi) + V_1(\phi) + V_2(\phi) + \ldots$$

classical potential (→ $V(\phi)$)

contribution of Feyman diagrams with n closed loops (→ $V_1(\phi)$, $V_2(\phi)$)

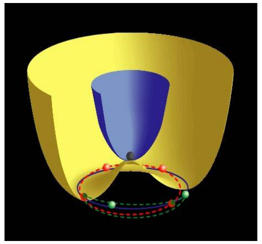
There are models for which radiative corrections can be neglected, while there are others for which they play an important role

The Abelian-Higgs model

Simplest gauge theory with spontaneously broken symmetry

Lagrangian density:

$$\mathcal{L} = \bar{D}_\mu \phi \mathcal{D}^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi)$$



ϕ complex scalar field with mexican hat potential

covariant derivative

field strength tensor

$$\mathcal{D}_\mu = \partial_\mu - ieA_\mu$$

gauge coupling constant

gauge field

The Abelian-Higgs model is invariant under the group U(1) of local gauge transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \quad ; \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

real single-valued function

The minima of the Mexican hat potential lie on a circle of fixed radius $|\phi| = \eta$, so the symmetry is spontaneously broken and the complex scalar field ϕ acquires a nonzero vacuum expectation value

chose to represent ϕ as: $\phi = \eta + \frac{\phi_1}{\sqrt{2}}$

Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}\mu^2 \phi_1^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu + \mathcal{L}_{\rm int}$$

Breaking of a gauge symmetry does not imply a massless Goldstone boson

The particle spectrum contains a scalar particle (Higgs boson) with mass $m_{\rm s} = \sqrt{\lambda}\eta$ and a vector field (gauge boson) with mass $m_{\rm V} = \sqrt{2}e\eta$

Phase Transitions

- in analogy to condensed matter systems, a SSB at low temperatures can be restored at higher temperatures

- in field theory, the expectation value of Higgs field can be considered as a Bose condensate of Higgs particles

- if temperature T is nonzero , consider a thermal distribution of particles/antiparticles, in addition to the condensate

- the equilibrium value of Higgs field is obtained by minimising the free energy $F = E - T S$

- only at high T the free energy is effectively temperature-dependent; at low T the free energy is minimised by the ordered state of the minimum energy

- if the Higgs field becomes smaller, the particle masses decrease, the available phase space becomes larger, and the entropy grows

⇒ there is a tendency for the Higgs field to decrease as a function of temperature and to vanish completely above some critical temperature T_c

If η is the characteristic energy scale of symmetry breaking and the couplings of the Higgs are not too small ⇒ on dimensional grounds: $T_c \sim \eta$

Hot Big Bang model:

Universe starts at a very high T, so initial equilibrium value of Higgs field is at $\phi = 0$

As the universe expands and cools down, it undergoes a phase transition at T_c , when symmetry is spontaneously broken

A GUT model with a sequence of symmetry breakings

$$G \to H \to \ldots \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)_{\text{em}}$$

predicts a series of phase transitions in the early universe with critical temperatures related to corresponding symmetry breaking scales

To describe high-T symmetry restoration, we need the free energy as a function of Higgs field and temperature

$$V_{\text{eff}}(\phi, T) = F(\phi, T)/\mathcal{V}$$

effective potential

free energy per unit volume $\mathcal{V}$

To lowest order in coupling constants, thermal particles can be considered to be non-interacting, so:

$$V_{\text{eff}}(\phi, T) = V(\phi) + \sum_n F_n(\phi, T)$$

zero-T effective potential

summation over particle spin states

Free energy contribution of different spin states

$$F_n = \pm T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 \mp \exp(-\epsilon_k/T)\right)$$

bosons (upper sign), *fermions* (lower sign)

$$\epsilon_k = \left(k^2 + m_n^2\right)^{1/2}$$

- for $T << m_n$ the free energy F_n is exponentially small and can be neglected

- for bosons at high temperatures $T >> m_n$

$$F_n = -\frac{\pi^2}{90}T^4 + \frac{m_n^2 T^2}{24} + \mathcal{O}\left(m_n^4\right)$$

while for fermions

$$F_n = -\frac{7\pi^2}{720}T^4 + \frac{m_n^2 T^2}{48} + \mathcal{O}\left(m_n^4\right)$$

- often, symmetry restoration occurs at a T much higher than all relevant mass thresholds, then

$$V_{\text{eff}}(\phi, T) = V(\phi) + \frac{1}{24}\mathcal{M}^2 T^2 - \frac{\pi^2}{90}\mathcal{N} T^4$$

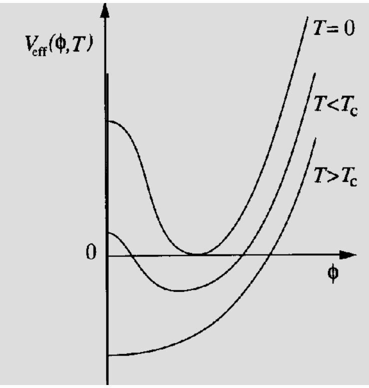
$$\mathcal{N} = \mathcal{N}_{\mathrm{B}} + \tfrac{7}{8}\mathcal{N}_{\mathrm{F}}$$

number of bosonic and fermionic spin states

$$\mathcal{M}^2 = \sum_{\mathrm{B}} m_n^2 + \tfrac{1}{2}\sum_{\mathrm{F}} m_n^2$$

example: **Goldstone model (2nd order pt)**

High temperature effective potential: $V_{\rm eff}(\phi,T) = m^2(T)|\phi|^2 + \frac{\lambda}{4}|\phi|^4$



calculated using perturbation theory and the leading contribution comes from one-loop Feymnam diagrams

for a scalar theory, the main effect is a T-dependent quadratic contribution

Effective mass squared of field $m^2(T) = \frac{\lambda}{12}(T^2 - 6\eta^2)$

This effective mass is equal to zero at the critical temperature: $T_c = \sqrt{6}\eta$

- for $T > T_c$ then $m^2(T) > 0$ ⇨

minimum of $V_{\rm eff}$ at $\phi = 0$, so symmetry restoration

- for $T < T_c$ then $m^2(T) < 0$ ⇨

the symmetric state becomes unstable and the Higgs field develops a non-zero expectation value:

$$|\phi| = \frac{1}{\sqrt{6}}\left(T_c^2 - T^2\right)^{1/2}$$

The defining feature of 2nd order PT is that the order parameter $|\phi|$ grows continuously from zero as the T is decreasing below T_c

- when the universe cools through critical temperature, the field develops an nonzero expectation value

- but the phase θ of ϕ , is not determined only by local physics; its choice depends on random fluctuations, and takes different values in different regions in space

- since the free energy is minimised by a homogeneous field ϕ , the spatial variations in θ will gradually die out

- thermal fluctuations have a Gaussian distribution, so they can be characterised by a 2-point correlation function, which typically decays exponentially with a decay rate characterised by the **correlation length** $\xi(t)$

- above $\xi(t)$ the values of θ are uncorrelated

The rate at which the correlation length grows depends on details of the relaxation process which are involved, but $\xi(t)$ should satisfy the causality bound

$$\xi(t) < d_{\mathrm{H}}(t)$$

causal horizon: the distance travelled by light during the lifetime of the universe $d_{\mathrm{H}} \sim t$

for a power law expansion:

$$\xi(t) \lesssim t$$

Classification of topological defects

a system represented by a high symmetric group G is spontaneously broken to a subgroup H with less symmetry,

$$G \to H \to \cdots SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)_{em}.$$

The system cools down to a critical temperature T_c defined by symmetry breaking scales

Thus, the symmetries of the system are no longer determined by the group G, but by the smaller group H instead.

The choice of the minimum of the system corresponding to some point at the ground states (or vacuum manifold), is randomly determined and can differ for different regions of the space if the regions are separated by a distance greater than some finite correlation length ξ. Such mechanism can lead to formation of defects, and it is known as Kibble mechanism

In order to determine what kind of topological defect emerges for a given SSB transition $G \to H$, one may study the content of homotopy groups $\pi_k(G/H)$ of the vacuum manifold $\mathcal{M} = G/H$, since the defect to arise is strictly determined by the topology of $\mathcal{M}$. When the vacuum manifold $\mathcal{M}$ has a non-trivial topology, is multiply connected, $\pi_k(G/H) \neq 1$ (1 corresponds to the trivial topology), stable topological defects of dimension $2-k$ will appear with a characteristic length scale of the size of the correlation length ξ

Topological defect	Dimension	Classification	Non trivial mappings in $\mathcal{M}$
Domain walls	2	$\pi_0(\mathcal{M})$	Disconnected
Cosmic strings	1	$\pi_1(\mathcal{M})$	Non-contractible loops
Monopoles	0	$\pi_2(\mathcal{M})$	Non-contractible S^2 spheres
Textures	-	$\pi_3(\mathcal{M})$	Non-contractible S^3 spheres

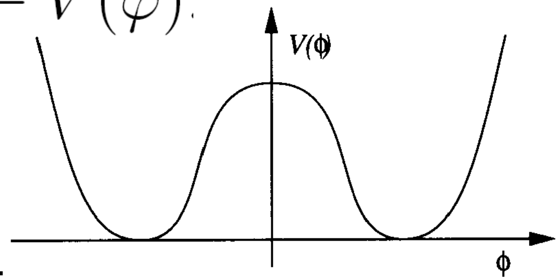
- **domain walls**: associated with breaking of discrete symmetry
  $\mathcal{M}$ consists of several disconnected components

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi).$$

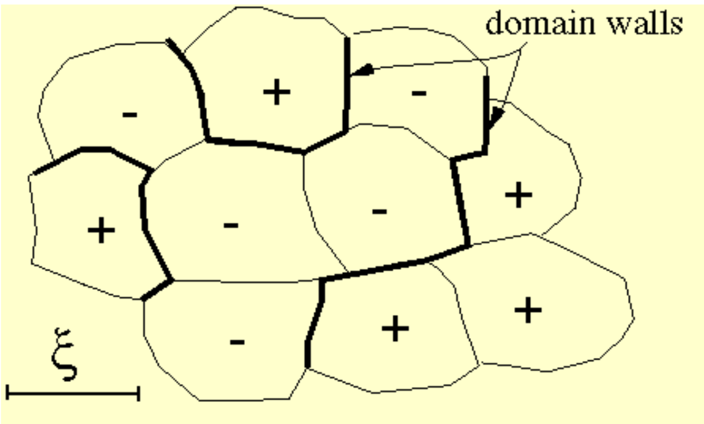
real scalar field

$$V(\varphi) = \frac{1}{4}\lambda(\varphi^2 - \eta^2)^2$$

distributed as:



double-well potential



The states $\phi = \eta$ and $\phi = -\eta$ correspond to two degenerate minima of the potential

during symmetry breaking the field ϕ acquires the values η and $-\eta$ with equal probability.

The phase transition sets the maximum distance over which the scalar field is correlated.

In the early universe the correlating length cannot exceed the size of the causally connected region.

Let us consider two causally disconnected regions A and B, and assume that the field ϕ in region A went to the minimum at η.

The field in region B does not “know” what happened in region A and, with probability ½, goes to the minimum $-\eta$.

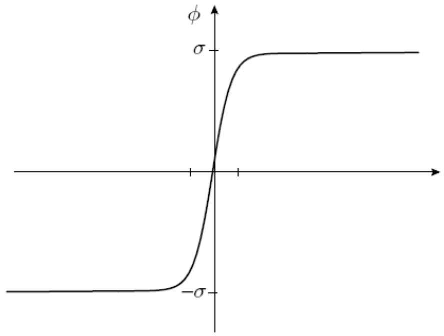
Since the scalar field changes continuously from $-\eta$ to η it must vanish on some 2dim surface separating regions A and B.

This 2dim surface is called a domain wall

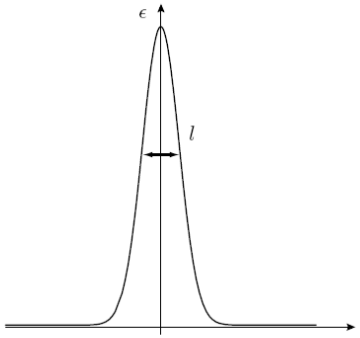
DW

finite thickness l

Assume a static DW which is not curved.



The energy density of the scalar field is: $\rho = \frac{1}{2}(\partial_i \phi)^2 + V$ distributed as:



The total energy per unit surface area is: $E \sim \rho l \sim (\frac{\eta}{l})^2 l + \lambda \eta^4 l$

from the gradient term

The energy is minimised $dE/dl = 0$ for $l \sim \lambda^{-1/2} \eta^{-1}$

and is equal to $E_{\rm DW} \sim \lambda^{1/2} \eta^3$

Domain walls are nonperturbative solutions of the field equation and they are stable w.r.t. small perturbations.

To remove the wall one has to “lift” the scalar field over the potential barrier from $\phi = \eta$ to $\phi = -\eta$ in infinite space.

This costs an infinite amount of energy.

On average, at least one domain wall per horizon volume is formed during the cosmological phase transitions.

The subsequent evolution of the domain wall network is quite complicated and has been investigated numerically.

The result is that one expects at least one domain wall per present horizon scale $\sim t_0$

The mass of the domain wall can be estimated as:

$$M_{\rm DW} \sim E_{\rm DW} t_0^2 \sim 10^{65} \lambda^{1/2} (\eta/100 {\rm GeV})^3 {\rm g}$$

For realistic values of λ and η, the mass of the domain wall exceeds the mass of matter within the present horizon by many orders of magnitude.

Such a domain wall would lead to unacceptably large CMB fluctuations.

Therefore, domain walls are cosmologically admissible only if the coupling constant λ and the symmetry breaking scale η are unjustifiably small.

domain walls & local monopoles are ruled out on cosmological groun

domain walls:

if domain walls formed at a PT in early universe, then from causality there will be even today at least one DW per Hubble volum

$$\rho_{DW}(t) \sim \eta^3 t^{-1}$$

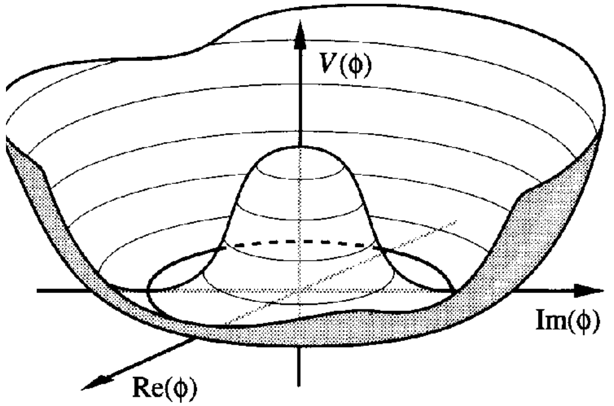
$$\rho_c = H^2 \frac{3}{8\pi G} \sim m_{p\ell}^2 t^{-2}$$

for $\eta \sim 10^{16}\ \mathrm{GeV}$

$$\frac{\rho_{DW}}{\rho_c}(t) \sim \left(\frac{\eta}{m_{p\ell}}\right)^2 (\eta t) \sim 10^{52}$$

the above argument depends essentially on dimensionality of defect

Goldstone model



complex scalar field

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$$

$$V(\phi) = \frac{\lambda}{4}(\varphi\bar{\varphi} - \eta^2)^2$$

$$\varphi(x) \rightarrow \exp^{ia} \varphi(x)$$

minima: $|\varphi| = \eta$

$$\mathcal{M} = \{\varphi : V(\varphi) = V_{min} = 0\}$$

ground state (vacuum): $< 0|\varphi|0 >= \eta \exp^{i\theta}$

nonzero

the phase transf. changes θ into $\theta + a$ and SSB

independent of position, i.e. constant

Global strings

At a large distance r from the core of the string, the derivative $\partial_i \varphi$ can be estimated on dimensional grounds as η/r

⇒ The gradient term gives a logarithmically divergent contribution to the energy per unit length:

$$\mu_{\mathrm{CS}} \sim \eta^2 \int (1/r^2) d^2x$$

the natural regularization factor is the distance to the nearest string.

The Abelian-Higgs model

The Abelian-Higgs model is the simplest model which admits string solutions, the Nielsen-Olesen vortex lines

Simplest gauge theory with spontaneously broken symmetry

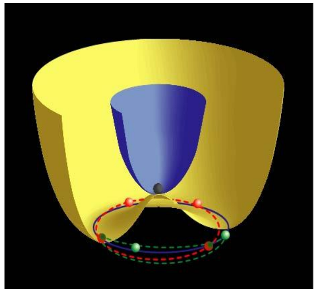
Lagrangian density:

$$\mathcal{L} = \bar{D}_\mu \phi \mathcal{D}^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi)$$

ϕ complex scalar field with mexican hat potential

field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$



covariant derivative $\mathcal{D}_\mu = \partial_\mu - ieA_\mu$

gauge coupling constant

gauge field

Local (gauge) strings

The Abelian-Higgs model is invariant under the group U(1) of local gauge transformations

$$\phi(x) \rightarrow e^{i\alpha(x)}\phi(x) \quad ; \quad A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

real single-valued function

represent ϕ as: $\phi = \eta + \frac{\phi_1}{\sqrt{2}}$

⇨ Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}\mu^2\phi_1^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu + \mathcal{L}_{\rm int}$$

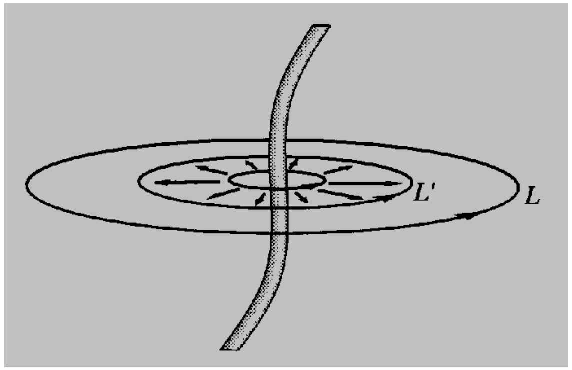
The particle spectrum contains a scalar particle (Higgs boson) with mass $m_{\rm s} = \sqrt{\lambda}\eta$ and a vector field (gauge boson) with mass $m_{\rm v} = \sqrt{2}e\eta$

Breaking of a gauge symmetry does not imply a massless Goldstone boson

The width of the string is determined by the Compton wavelength of the Higgs $\sim m_{\rm s}^{-1}$ and gauge boson $\sim m_{\rm v}^{-1}$

Going around any closed path L in physical space, the phase θ of the Higgs field ϕ develops a nontrivial winding, $\Delta\theta = 2\pi$

This closed path can be shrunk continuously to a point, only if the field ϕ is lifted to the top of its potential where it takes the value $\phi = 0$



Within a closed path for which the total change of the Higgs field ϕ is 2π ,a string is trapped

A string must be either a closed loop or an infinitely long (no ends) string; otherwise one could deform the closed path L and avoid to cross a string

The Goldstone model is an example of a second-order phase transition leading to the formation of global strings, vortices

Lorentz gauge: $\partial_\mu A^\mu = 0$

The Higgs field has the same form as in the case of a global string at large distances from the string core:

$$\phi \approx \eta e^{in\theta}$$

integer denoting the string winding number

The gauge field asymptotically approaches: $A_\mu \approx \frac{1}{ie}\partial_\mu \ln \phi$

Far from the string core:

$$\mathcal{D}_\mu \phi \approx 0 \quad , \quad F_{\mu\nu} \approx 0$$

⇨ far from string core: energy density vanishes exponentially, while the total energy per unit length is finite

String linear mass density for a local (gauge) cosmic string: $\mu \sim \eta^2$

the gauge field compensates the leading term in $\partial_i\varphi$ and that the covariant derivative $\mathcal{D}_i\varphi$ decays faster than $1/r$ as $r \to \infty$.

As a result the energy per unit length converges.

Global U(1) string: there is no gauge field to compensate variation of phase at large distances for string core, so the linear mass density diverges at long distances from string

String linear mass density for a local (gauge) cosmic string:

$$\mu \sim \eta^2$$

String linear mass density for global U(1) string:

$$\mu \sim \eta^2 + \int_\delta^R \left[\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right]^2 2\pi r dr \approx 2\pi\eta^2 \ln\left(\frac{R}{\delta}\right)$$

cut-off radius at some large distance from string

width of string core

$$\mu \approx 2\pi\eta^2 \ln\left(\frac{R}{\delta}\right)$$

There are long-range interactions between global u(1) string segments, with a force $\sim \eta^2/R$

R *could be the curvature radius of the string, or the distance to the nearest string segment in the case of a string network*

local/global defects

spontaneous broken Local (gauge)/global symmetry may lead to local/global defects

textures are only relevant for theories with global symmetry

(all energy in spatial gradients, so for a local theory the gauge fields can re-orient themselves such as to cancel energy)

local defects have a well defined core outside of which the field contains no energy density in spite of nonvanishing gradients

Global defects have long range density fields and forces

local/global strings, local/global monopoles, global textures

global defects can decay through long-range interactions, so they do not contradict observations

Local defects may be undesirable for cosmology

cosmic string dynamics

- the world history of the string can be exprssed by a 2-dim surface in 4-dim space-time: the string world-sheet

$$x^\mu = x^\mu(\zeta^a) \quad , \quad a = 0, 1$$

the world-sheet coordinates ζ^0, ζ^1 are arbitrary parameters:

ζ^0 timelike and $\zeta^1 \equiv \sigma$ spacelike

- the string eqs. of motion, in the limit of zero thickness string are derived from the Goto-Nambu (GN) effective action which, up to an overall factor, corresponds to the surface area swept out by the sring in space-time

$$S_0[x^\mu] = -\mu \int \sqrt{-\gamma}\, d^2\zeta$$

2-dim world-sheet metric

$$\gamma = \det(\gamma_{ab}) = \frac{1}{2}\epsilon^{ac}\epsilon^{bd}\gamma_{ab}\gamma_{cd} \quad , \quad \gamma_{ab} = g_{\mu\nu}x^\mu_{,a}x^\nu_{,b}$$

vary GN action w.r.t. $x^\mu(\zeta^a)$ & use $d\gamma = \gamma\gamma^{ab}d\gamma_{ab}$ ⇒ string equations of motion:

$$x^{\mu\ ;a}_{,a} + \Gamma^\mu_{\nu\sigma}\gamma^{ab}x^\nu_{,a}x^\sigma_{,b} = 0$$

covariant laplacian $x^{\mu\ ;a}_{,a} = \frac{1}{\sqrt{-\gamma}}\partial_a(\sqrt{-\gamma}\gamma^{ab}x^\mu_{,b})$

4-dim Christoffel symbol $\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\tau}(g_{\tau\nu,\sigma} + g_{\tau\sigma,\nu} - g_{\nu\sigma,\tau})$

cosmic strings in flat space-time

string e.o.m. in flat space-time: $\partial_a(\sqrt{-\gamma}\gamma^{ab}x^\mu_{,b}) = 0$

impose **conformal gauge**: $\dot{x}\cdot x' = 0 \quad , \quad \dot{x}^2 + x'^2 = 0$ *overdot are derivatives wrt* ζ^0 *and prime are derivatives wrt* ζ^1

⇨ string e.o.m is a 2-dim wave equation: $\ddot{\mathbf{x}} - \mathbf{x}'' = 0$

to fix entirely the gauge, also impose: $t \equiv x^0 = \zeta^0$

which allows us to write the string trajectory as the 3-dim vector $\mathbf{x}(\sigma, t)$

$\zeta^1 \equiv \sigma$ *the space-like parameter along string*

constraint equations and string equations of motion:

$$\dot{\mathbf{x}}\cdot\mathbf{x}' = 0$$
$$\dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1$$
$$\ddot{\mathbf{x}} - \mathbf{x}'' = 0$$

- string moves perpendicular to itself with velocity $\dot{\mathbf{x}}$
- σ is proportional to the string energy
- string acceleration in the string rest frame is inversely proportional to the local string curvature radius

a curved string segment tends to straighten itself, resulting to string oscillations

general solution to string e.o.m. in flat space-time:

$$\mathbf{x} = \frac{1}{2}\Big[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)\Big]$$

continuous arbitrary functions which satisfy: $\mathbf{a}'^2 = \mathbf{b}'^2 = 1$

so, σ *is the length parameter along the 3-dim curves* $\mathbf{a}(\sigma), \mathbf{b}(\sigma)$

cusps

an interesting property of loop solutions is that particular points along the string can reach the velocity of light during each period

$$\dot{\mathbf{x}}^2(\sigma,t) = \tfrac{1}{4}[\mathbf{a}'(\sigma - t) - \mathbf{b}'(\sigma + t)]^2$$

the vectors $\mathbf{a}'(\sigma)$ and $-\mathbf{b}'(\sigma)$ describe closed curves on a units sphere as σ runs from $\mathbf{0}$ to $\mathbf{L}$

these functions should satisfy

$$\int_0^L \mathbf{a}' \mathrm{d}\sigma = \int_0^L \mathbf{b}' \mathrm{d}\sigma = 0$$

but otherwise are arbitrary

if the two curves intersect then: $\dot{\mathbf{x}}^2(\sigma,t) = 1$

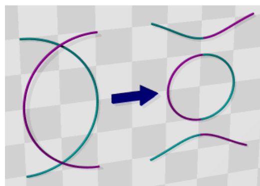
smooth loops will in general have such luminal points: cusps

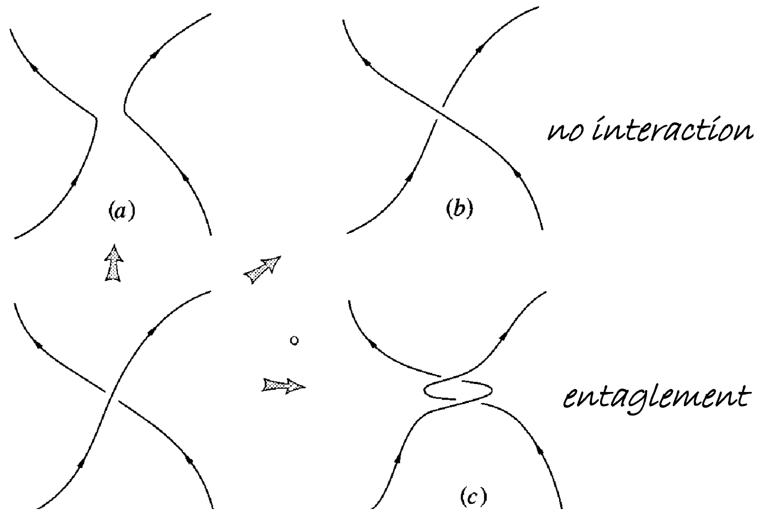
string inter-commutations

GN action describes to a good approximation cosmic string segments which are separated, but it leaves unanswered what happens when strings cross

numerical simulations have shown that strings exchange partners, inter-commute, with probability (almost) equal to 1

string-string and self-string intersections lead to the formation of new long strings and loops





kinks

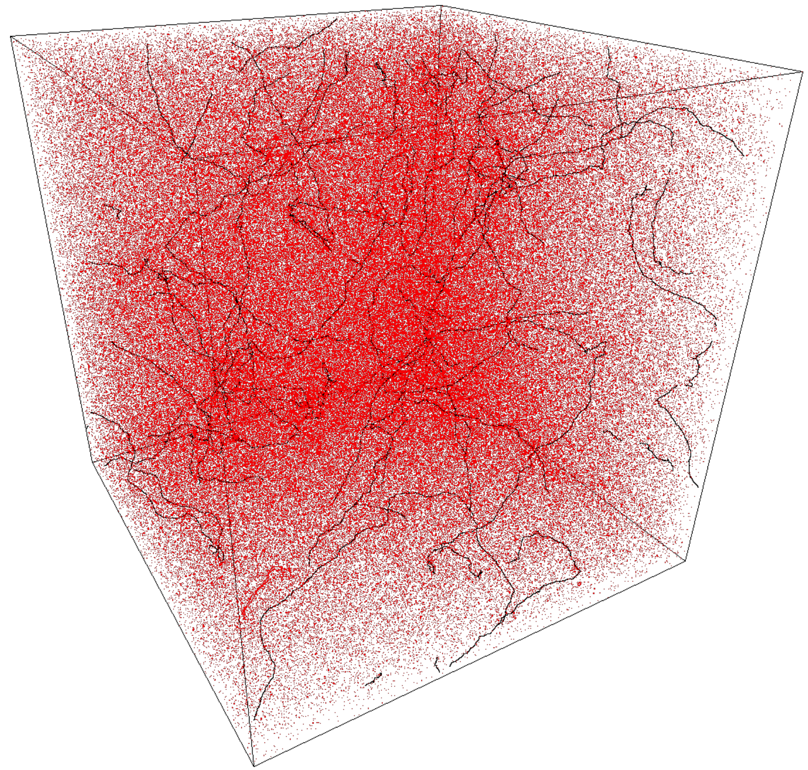
string inter-commutations lead to discontinuities in $\dot{\mathbf{x}}$ and $\mathbf{x}'$ on the new string segments at the intersection point

these discontinuities, kinks, are composed of right- and left-moving pieces travelling along the string at the speed of light

loops with kinks are described by discontinuous functions $\mathbf{a}'$ and $\mathbf{b}'$, leading to gaps in the two curves on the unit sphere, and like that it is much easier to avoid intersections of $\mathbf{a}'$ and - $\mathbf{b}'$

as a result the number of cups per loop oscillation in an actual network of cosmic strings will be diminished

the number of cusps remain still an unknown number, to be evaluated by numerical experiments with string networks



Strings and gravity

the geometry around straight cosmic string is locally identical to that of flat spacetime

however, this geometry is not globally Euclidean: the angle θ' varies in the range

$$0 \leq \theta' < 2\pi(1 - 4G\mu)$$

⇨ the effect of the string is to introduce an azimuthal *deficit angle* $\Delta = 8\pi G\mu$

implying that a surface of constant t and z has the geometry of a cone rather that of a plane

the dimensionless parameter $G\mu$ plays an important role in the physics of cosmic strings

since $\mu \sim \eta^2$

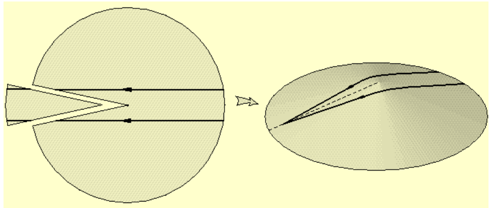
$$G\mu \sim \left(\frac{\eta}{m_{\rm Pl}}\right)^2$$

symmetry breaking scale at string formation (→ η)

propagation of particles and light

the string metric

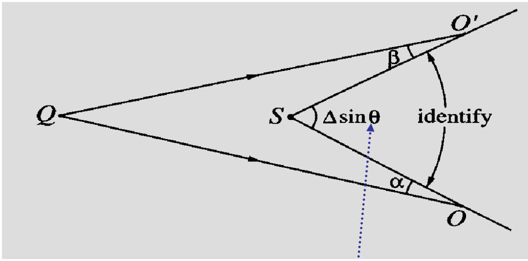
$$ds^2 = dt^2 - dz^2 - dr'^2 - (1-8G\mu)r'^2 d\theta^2$$



describes a conical space, i.e. a flat space with a wedge of angular size $\Delta = 8\pi G\mu$ removed and the two faces of the wedge identified

- double images

double images of light sources located behind the string



the angle that the string makes with the plane

if $l = QS$ and $d = OS$ then for $G\mu << 1$ the angular separation ($\alpha+\beta$) between the two images is:

$$\delta\varphi = \frac{l\Delta\sin\theta}{l+d}$$

gravitational radiation from a loop

the lifetime of a non-intersecting loop depends upon the rate at which it radiates away its energy

the gravitational radiation power from a loop of length L can be roughly estimated using the quadrupole formula:

$$\dot{E} \sim G\left(\frac{d^3D}{dt^3}\right)^2 \sim GM^2L^4\omega^6$$

$D \sim ML^2$ — *quadrupole moment*

$M \sim \mu L$ — *loop's mass*

$\omega \sim L^{-1}$ — *characteristic frequency*

⇨ $$\dot{E} = \Gamma G\mu^2$$

coefficient independent of loop size; it depends on loop shape and its trajectory

⇨ lifetime of a loop $\tau \sim \frac{M}{\dot{E}} \sim \frac{L}{\Gamma G\mu}$