SIGRAV

14–18 Feb 2022
Online

Early Universe, Inflation and Primordial Gravitational Waves

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Lecture I

Standard Cosmology

Einstein's equations

Get second order equations for the metric tensor $g_{\mu\nu}$: thus the Lagrangian depends on the metric and its first order derivatives only

Einstein-Hilbert action:

$$S = \frac{1}{2\kappa}\int(R - 2\Lambda)\sqrt{-g}\,d^4x + \int L_{\mathrm{m}}\sqrt{-g}\,d^4x$$

Ricci scalar

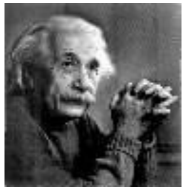
cosmological constant

matter Lagrangian
(anything not the gravitational field)

K : *a coefficient to be obtained by requiring the theory to reduce to the Newtonian gravity in the weak-field approximation; this is the only free parameter of the theory*

<u>Principle of extremal action</u> (S should be extremum wrt the choice of geometry, $\delta S = 0$ for arbitrary $\delta g^{\mu\nu}$

Eisntein's equations $$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$



Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

Bianchi identities imply: $$G^{\mu\nu}{}_{;\nu} = 0$$

Moreover, $$g_{\mu\nu;\alpha} = 0$$

Energy conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

satisfied by total energy-momentum tensor (all matter components)

To solve Einstein's eqs. you need to define the background geometry.

Cosmological principle:

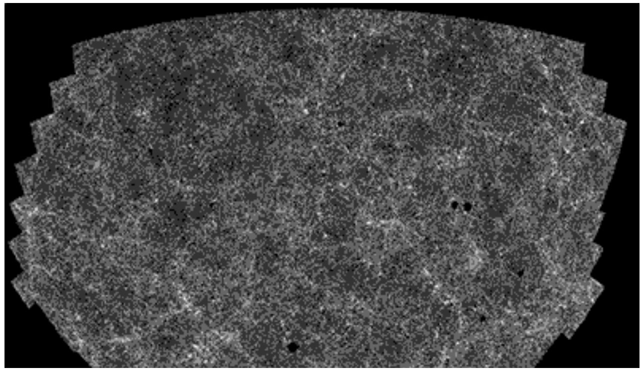
The Universe is homogeneous and isotropic on large scales.

Homogeneity:The physical conditions are the same at every point of any given hypersurface.

Isotropy: The physical conditions are identical in all directions when viewed from a given point on the hypersurface.

Isotropy at every point automatically enforces homogeneity.

Homogeneity does not necessarily imply isotropy.



The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

The only way to preserve the homogeneity and isotropy of space and incorporate time evolution is to allow the curvature scale, characterised by the scale factor a, to be time dependent,

The scale factor $a(t)$ completely describes the time evolution of a homogeneous and isotropic universe.

In relativistic theory, there is no absolute time and spatial distances are not invariant w.r.t. coordinate transformations.
Instead, the infinitessimal space-time interval between events is invariant.
There exist preferred coordinate systems in which the symmetries of the universe are clearly manifest.

General space-time interval:

$$ds^2 = dt^2 - dl_{3d}^2 \quad = \quad g_{\alpha\beta} dx^\alpha dx^\beta$$

where $g_{\alpha\beta}$ is the metric of the spacetime

$x^\alpha \equiv (t, r, \theta, \varphi)$ are the coordinates of events

Einstein convention for summation over repeated indices:

$$g_{\alpha\beta} dx^\alpha dx^\beta \equiv \sum_{\alpha,\beta} g_{\alpha\beta} dx^\alpha dx^\beta$$

Greek indices run from 0 to 3

time-like coordinate

Latin indices run only over spatial coordinates: $i, l, \ldots = 1, 2, 3$

The spatial coordinates introduced before are comoving: every object with zero peculiar velocity has constant coordinates r, θ, φ.

The coordinate t is the proper time measured by a comoving observer.

The distance between two comoving observers at a particular moment of time is:

$$\int \sqrt{- \, ds^2_{t=\text{const.}}} \propto a(t)$$

$\Rightarrow$ Increases or decreases in proportion to the scale factor

Isotropy of space: $g_{0i} = 0$ *Otherwise there is a particular direction in space related to the 3vector v_i with components g_{0i}*

In the coordinate system of fundamental observers for whom the Universe appears homogeneous and isotropic, we use the proper time of clocks carried by the observers to label space-like surfaces $\Rightarrow$ $g_{00} = 1$

$$ds^2 = dt^2 - g_{ij}dx^i dx^j \quad = dt^2 - dl_{3d}^2$$

We have to determine the 3metric g_{ij} of a 3space which, at any instant of time, is homogeneous and isotropic.

Isotropy $\Longrightarrow$ spherical symmetry $\Longrightarrow$ line interval:

$$dl_{3d}^2 = a^2(t)\,[\lambda^2(r)dr^2 + r^2 d\Omega^2]$$

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

The scalar curvature for this 3dim space is:

$$^3R = \frac{3}{2a^2r^3}\,\frac{d}{dr}[r^2(1-\frac{1}{\lambda^2})]$$

Homogeneity: all geometrical properties are independent of radial coordinate

$\Longrightarrow$ ${}^3R$ must be constant

Hence, integrating you get: $r^2(1-\frac{1}{\lambda^2}) = c_1 r^4 + c_2$

constants

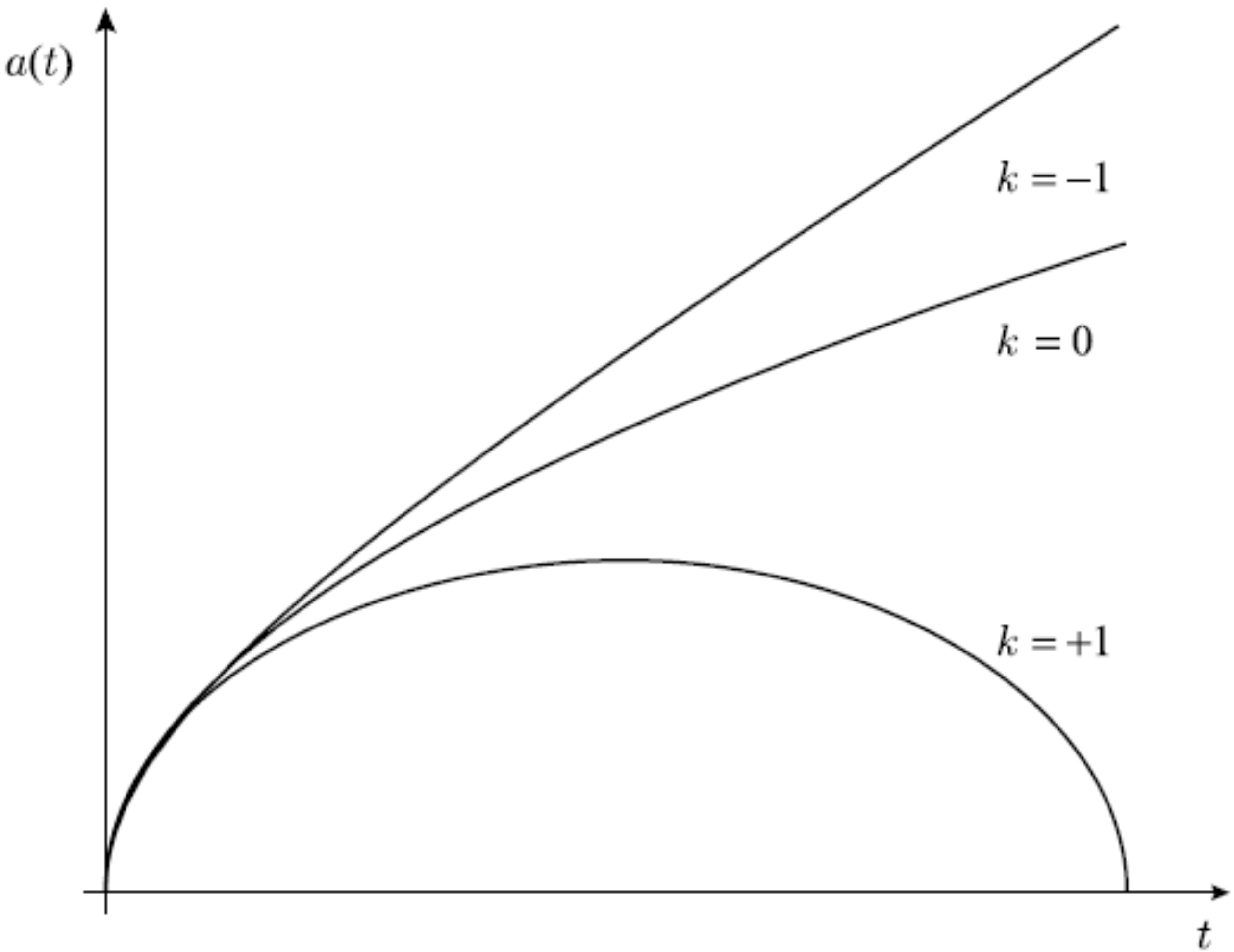
To avoid a singularity at r=0, we set $c_2 = 0$

Thus, $\lambda^2 = (1-c_1 r^2)^{-1}$

When $c_1 \neq 0$, we can rescale r and make $c_1 = 1 \text{ or } -1$

$\Downarrow$

$$\boxed{ds^2 = dt^2 - dl_{3d}^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)}$$



a(t)
k = −1
k = 0
k = +1
t

Conformal time

To treat spatial and time coordinates on equal footing, you may replace the physical (cosmological) time t by conformal time η

$$\boxed{\eta \equiv \int \frac{dt}{a(t)}}$$

so that $dt = a(\eta) d\eta$

Then the metric becomes:

$$\boxed{ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - f^2(\chi) d\Omega^2]}$$

Conformal to Minkowski metric

$$ds^2 = d\tau^2 - dr^2 - r^2 d\Omega^2$$

$$f(\chi) = \begin{cases} \sin\chi \ \ (\text{for} \ \ k = +1) \\ \chi \ \ (\text{for} \ \ k = 0) \\ \sinh\chi \ \ (\text{for} \ \ k = -1) \end{cases}$$

In GR, the dynamical variables characterising the gravitational field are the components of the metric $g_{\alpha\beta}(x^\gamma)$ and they obey the Einstein equations:

$$R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R - \Lambda\delta^\alpha_\beta = 8\pi G T^\alpha_\beta$$

Cosmological term (constant)

For a FLRW metric:

the non-vanishing components of the Einstein tensor are:

$$G_{00} = 3(H^2 + \frac{k}{a^2}) \quad \text{and} \quad G_{ij} = -(H^2 + 2\frac{a}{a} + \frac{k}{a^2})a^2 g_{ij}$$

The Hubble parameter relates how fast the most distant galaxies are receding from us to their distance from us via Hubble's law $v \simeq Hd.$

$$\boxed{H \equiv \frac{\dot{a}}{a}}$$

Hubble parameter

where $\dot{} \equiv \frac{d}{dt}$

Matter is incorporated in Einstein's equations through the energy momentum tensor $T^{\alpha}_{\ \beta}$ of rank 2

$$T^{\alpha\beta} \equiv g^{\beta\delta} T^{\alpha}_{\ \delta} = T^{\beta\alpha} \Longrightarrow T^{\alpha\beta} \quad \text{symmetric tensor}$$

Since the Bianchi identities are satisfied by the Einstein tensor, i.e. $G^{\alpha}_{\ \beta;\alpha} = 0$

$$\Longrightarrow T^{\alpha\beta}_{\ \ ;\beta} \equiv \frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\gamma\beta} T^{\gamma\beta} + \Gamma^{\beta}_{\gamma\beta} T^{\alpha\gamma} = 0$$

The possible forms that this tensor can take are reduced by the space-time symmetries, namely the r.h.s. of Einstein's equations must obey the same symmetries as the l.h.s. one.

On large scales (so that we can assume homogeneity and isotropy), matter can be approximated as a perfect fluid characterised by energy density ρ , pressure p and 4-velocity u^α

$$T^\alpha_\beta = (\rho + p) u^\alpha u_\beta - p \delta^\alpha_\beta$$

they depend only on time

the equation of state $p = p(\rho)$ depends on the properties of matter

Usually, we will consider $p = w\rho$

constant

For ultra-relativistic gas: $p = \rho/3$

Friedmann equation $H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}$

(*We have included* Λ *in* ρ)

evolution equation $\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \sum_i p_i - \frac{k}{2a^2}$

acceleration equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i)$

energy-conservation equation $\dot{\rho} + 3H(\rho + p) = 0$.

These three equations are not independent; this is a consequence of the Bianchi identities

2 independent eqs. for 3 unknown

$p = w\rho$

equation of state

critical energy density: $$\boxed{\rho_c \equiv \frac{3H^2}{8\pi G}}$$

the spatial sections must be precisely flat ($k = 0$)

density parameter $$\boxed{\Omega_{\text{total}} \equiv \frac{\rho}{\rho_c}}$$

relate the total energy density in the universe to its local geometry

$$\boxed{\begin{array}{lcl} \Omega_{\text{total}} > 1 & \Leftrightarrow & k = +1 \\ \Omega_{\text{total}} = 1 & \Leftrightarrow & k = 0 \\ \Omega_{\text{total}} < 1 & \Leftrightarrow & k = -1 \end{array}}$$

$$\boxed{\begin{array}{c} p = w\rho \\ \text{constant } w \end{array}}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \Longrightarrow \quad \boxed{\rho(a) \propto \frac{1}{a(t)^{3(1+w)}}}$$

pressureless matter, or dust $\quad w = 0 \quad \Longrightarrow \quad \rho(a) \propto a(t)^{-3}$

radiation $\quad w = 1/3 \quad \Longrightarrow \quad \rho(a) \propto a(t)^{-4}$

cosmological constant $\Lambda \quad w_\Lambda = -1 \quad \Longrightarrow \quad \rho_\Lambda = \text{constant}$

$$\begin{aligned} \rho_\Lambda &= \frac{\Lambda}{8\pi G} \\ p_\Lambda &= -\rho_\Lambda \ , \end{aligned}$$

flat spatial sections and a constant equation of state parameter w

Friedmann equation $\Rightarrow$ $$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3(1+w)}$$ unless $w = -1$

scale factor today

$w = -1$ $\longrightarrow$ $a(t) \propto e^{Ht}$

Type of Energy	$\rho(a)$	$a(t)$
Dust	a^{-3}	$t^{2/3}$
Radiation	a^{-4}	$t^{1/2}$
Cosmological Constant	constant	e^{Ht}

age of such a universe $$t_0 = \int_0^1 \frac{da}{aH(a)} = \frac{2}{3(1+w)H_0}$$

Big Bang

$\Rightarrow$ $$t_0 \sim H_0^{-1}$$ *Hubble time*

Unless w is close to -1

focus on what happens if the only energy density in the universe is a cosmological constant, with $w = -1$

$\Lambda > 0$ $\longrightarrow$

$$\frac{a(t)}{a_0} = \begin{cases} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = +1 \\ \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = 0 \\ \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = -1 \end{cases}$$

the Friedmann equation may be solved for any value of the spatial curvature parameter k

$t \to \infty$ limit $\longrightarrow$ all solutions expand exponentially, independently of the spatial curvature

de Sitter space

the universe clearly expands forever irrespective of the value of the spatial curvature

$\Lambda < 0$ $\xrightarrow{\text{Friedmann equation}}$ such a spacetime can only exist in a space with spatial curvature $k = -1$

Anti-de Sitter space (AdS)

$$a(t) = a_0 \sin\left(\sqrt{-\frac{\Lambda}{3}}t\right)$$

Horizons

In SR: space-time interval along trajectory of massless particle propagating with speed of light is

$$ds^2 = 0$$

In GR: it also holds in every local inertial coordinate frame.

But since the interval is invariant $\Rightarrow$

$ds^2 = 0$ is valid along the light geodesic in any curved space-time

Consider radial propagation of light in isotropic universe in coordinate system where the observer is at the origin.

The metric
$$ds^2 = a^2(\eta)(d\eta^2 - d\chi^2 - \Phi^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2))$$

where
$$\Phi^2(\chi) = \begin{cases} \sinh^2\chi, & k=-1; \\ \chi^2, & k=0; \\ \sin^2\chi, & k=+1. \end{cases}$$

$$\eta \equiv \int \frac{dt}{a(t)}$$

Along the radial trajectory $\theta, \varphi = \mathrm{const.}$ the function $\chi(\eta)$ is determined by:

$$ds^2 = 0, \quad \text{i.e.} \quad d\eta^2 - d\chi^2 = 0$$

Hence, the radial light geodesics are described by: $\chi(\eta) = \pm\,\eta + \mathrm{const.}$

and correspond to straight lines at angles $\pm\,45^0$ in the $\eta - \chi$ plane

Particle horizon: If the universe has a finite age, then the light travels only a finite distance in that time and the volume of space from which we can receive information at a given moment of time is limited.

The max comoving distance that light can propagate

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a}$$

beginning of the universe

Physical size of particle horizon

$$d_p(t) = a(t)\chi_p = a(t) \int_{t_i}^{t} \frac{dt}{a}$$

Event horizon: The event horizon is the complement of the particle horizon. The event horizon encloses the set of points from which signals sent at a given moment of time η will never be received by an observer in the future . These points have

$$\chi > \chi_e(\eta) = \int_\eta^{\eta_{\max}} d\eta = \eta_{\max} - \eta$$

final moment of time

Physical size of event horizon

$$\boxed{d_e(t) = a(t) \int_t^{t_{\max}} \frac{dt}{a}}$$

If the universe expands forever, then $t_{\max}$ is infinite.

Redshift

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}}$$

since $$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a(\eta_{\rm obs})}{a(\eta_{\rm em})}$$

$\Longrightarrow$

The present value of the scale factor

$$\boxed{1 + z = \frac{a_0}{a(t_{\rm em})}}$$

$\Longrightarrow$ The redshift can be used instead of time in order to describe the history of the universe

$$\boxed{t = \int_z^{\infty} \frac{dz}{H(z)(1+z)}}$$

$z \longrightarrow \infty$ corresponds to $t = 0$

Matter content of the universe

The traditional method to estimate the mass density of the universe is to "weigh" a cluster of galaxies, divide by its luminosity, and extrapolate the result to the universe as a whole.

applying the virial theorem to cluster dynamics have typically obtained values $\Omega_{\mathrm{M}} = 0.2 \pm 0.1$

Rather than measuring the mass relative to the luminosity density, which may be different inside and outside clusters, we can also measure it with respect to the baryon density which is very likely to have the same value in clusters as elsewhere in the universe, simply because there is no way to segregate the baryons from the dark matter on such large scales.

$$\Omega_{\mathrm{M}} = \Omega_{\mathrm{B}}/f_{\mathrm{gas}} = 0.3 \pm 0.1$$

by direct observation of X-rays from the gas
or by distortions of the microwave background by scattering off hot electrons

properties of clusters at high redshift → $\Omega_{\mathrm{M}} < 1.0$

advances in very large redshift surveys → $0.1 \leq \Omega_{\mathrm{M}} \leq 0.4$

measurements of the power spectrum of density fluctuations → $\Omega_{\mathrm{M}} \sim 0.31$

Ordinary baryonic matter, it turns out, is not nearly enough to account for the observed matter density.

$$\Omega_{\mathrm{b}} = 0.04 \pm 0.02$$

Most of the matter density must therefore be in the form of non-baryonic dark matter.

Currently there are 2 preferred candidates: **primordial black holes** and **pseudo-scalar (axion-like) particles**
Modifications to Newton's law (**Modified Newtonian Dynamics – MOND**) and its relativistic extension (**Tensor-Vector-Scalar –TeVeS**) have been proposed but they seem to fail: MOND in addressing the DM issue in all scales and TeVeS in explaining both flat rotation curves and gravitational lensing with the same choice of parameters, while to explain the CMB data it needs contribution from massive neutrinos

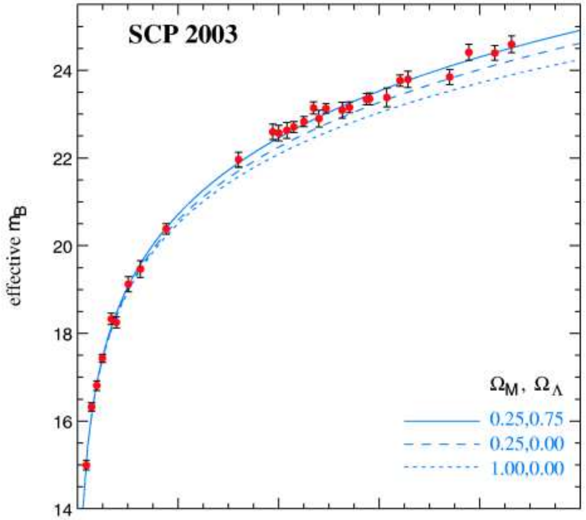
Supernovae and the accelerating universe

Supernovae are rare — perhaps a few per century in a Milky-Way-sized galaxy

Supernovae are also bright, and Type Ia's in particular all seem to be of nearly uniform intrinsic luminosity

The fact that all SNe Ia are of similar intrinsic luminosities fits well with our understanding of these events as explosions which occur when a white dwarf, onto which mass is gradually accreting from a companion star, crosses the Chandrasekhar limit and explodes.

searches for distant supernovae in order to measure cosmological parameters



redshift vs. corrected apparent magnitude from the Supernova Cosmology Project

The data are much better fit by a universe dominated by a cosmological constant than by a flat matter-dominated model.

if $\Omega_{\rm M} \sim 0.3$ ⟹ $\Omega_\Lambda \sim 0.7$ ⟹ $\rho_\Lambda \sim 10^{-8}\ {\rm erg/cm^3} \sim (10^{-3}\ {\rm eV})^4$

vacuum energy density

The cosmological constant problem

In classical general relativity the cosmological constant Λ is a completely free parameter.

Λ has dimensions of $[\text{length}]^{-2}$ (while the energy density ρ_Λ has units [energy/volume])

quantum mechanics → Planck's constant allows us to define the reduced Planck mass $M_p \sim 10^{18}$ GeV

the reduced Planck length $L_{\rm P} = (8\pi G)^{1/2} \sim 10^{-32}$ cm

$$\boxed{\begin{aligned}\Lambda^{(\rm guess)} &\sim L_{\rm P}^{-2} \\ \rho_{\rm vac}^{(\rm guess)} \sim M_{\rm P}^4 &\sim (10^{18}\ \text{GeV})^4 \sim 10^{112}\ \text{erg/cm}^3\end{aligned}}$$

$$\boxed{\rho_{\rm vac}^{(\rm obs)} \sim 10^{-120} \rho_{\rm vac}^{(\rm guess)}}$$

120-orders-of-magnitude discrepancy

$$\rho_{\rm vac} = M_{\rm vac}^4$$

$$M_{\rm vac}^{(\rm obs)} \sim 10^{-3}\ \text{eV}$$

$$\boxed{M_{\rm vac}^{(\rm obs)} \sim 10^{-30} M_{\rm vac}^{(\rm guess)}}$$

discrepancy of 30 orders of magnitude in energy scale.

In addition to the fact that it is very small compared to its natural value, the vacuum energy presents an additional puzzle: the coincidence between the observed vacuum energy and the current matter density.

$\Omega_{\rm M} =$ 0.31

Ω_Λ = 0.69

$$\boxed{\frac{\Omega_\Lambda}{\Omega_{\rm M}} = \frac{\rho_\Lambda}{\rho_{\rm M}} \propto a^3}$$

As a consequence, at early times the vacuum energy was negligible in comparison to matter and radiation, while at late times matter and radiation are negligible.

If general relativity is correct, cosmic acceleration implies there must be a dark energy density which diminishes relatively slowly as the universe expands.

$$\boxed{\dot{a}^2 \propto a^2 \rho + \text{constant}}$$

the only way to get acceleration ($\dot{a}$ increasing) in an expanding universe is if ρ falls off more slowly than a^{-2};

$\rho_{\rm M} \propto a^{-3}$ $\rho_{\rm R} \propto a^{-4}$ $\rho_{\rm vac}$ is constant

the data are consistent with smoothly-distributed sources of dark energy that vary slowly with time.

$\boxed{\text{dynamical dark energy}}$

Thermodynamics of the early universe

matter as a perfect fluid $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$

energy-momentum tensor is covariantly conserved $\nabla_\mu T^{\mu\nu} = 0$

number flux density N^μ $N^\mu = nU^\mu$ also conserved $\nabla_\mu N^\mu = 0$

entropy flux density S^μ $\nabla_\mu S^\mu \geq 0$

obeys a covariant version of the second law of thermodynamics

The conservation law for the energy-momentum tensor yields $\dot{\rho} + 3H(\rho + p) = 0$

which can be thought of as the first law of thermodynamics $dU = TdS - pdV$ with $dS = 0$

The various particles inhabiting the early universe can be usefully characterized according to three criteria: in equilibrium vs. out of equilibrium (decoupled), bosonic vs. fermionic, and relativistic (velocities near c) vs. non-relativistic.

A given species remains in thermal equilibrium as long as its interaction rate is larger than the expansion rate of the universe.

A particle species for which the interaction rates have fallen below the expansion rate of the universe is said to have *frozen out* or *decoupled.*

If the interaction rate of some particle with the background plasma is Γ, it will be decoupled whenever $\Gamma \ll H$ where the Hubble constant H sets the cosmological timescale.

the expansion rate in the early universe is "slow," and particles tend to be in thermal equilibrium (unless they are very weakly coupled).

interaction rate Γ $\boxed{\Gamma = n\langle\sigma v\rangle}$ n is the number density and v a typical particle velocity

cross- section σ

Since $n \propto a^{-3}$, the density of particles will eventually dip so low that equilibrium can no longer be maintained.

focus on particles in equilibrium

gas of weakly-interacting particles

distribution function $f(\mathbf{p})$

$$f(\mathbf{p}) = \frac{1}{e^{E(\mathbf{p})/T} \pm 1}$$

plus sign is for fermions and the minus sign for bosons

The distribution function characterizes the density of particles in a given momentum bin.

$$E^2(\mathbf{p}) = m^2 + |\mathbf{p}|^2$$

number density
$$n_i = \frac{g_i}{(2\pi)^3} \int f_i(\mathbf{p}) d^3p$$

energy density
$$\rho_i = \frac{g_i}{(2\pi)^3} \int E(\mathbf{p}) f_i(\mathbf{p}) d^3p$$

pressure
$$p_i = \frac{g_i}{(2\pi)^3} \int \frac{|\mathbf{p}|^2}{3E(\mathbf{p})} f_i(\mathbf{p}) d^3p$$

species labeled i

g_i is the number of spin states of the particles highly relativistic $(T \gg m)$

highly non-relativistic $(T \ll m)$

	Relativistic Bosons	Relativistic Fermions	Non-relativistic (Either)
n_i	$\frac{\zeta(3)}{\pi^2} g_i T^3$	$\left(\frac{3}{4}\right) \frac{\zeta(3)}{\pi^2} g_i T^3$	$g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$
ρ_i	$\frac{\pi^2}{30} g_i T^4$	$\left(\frac{7}{8}\right) \frac{\pi^2}{30} g_i T^4$	$m_i n_i$
p_i	$\frac{1}{3}\rho_i$	$\frac{1}{3}\rho_i$	$n_i T \ll \rho_i$

Number density, energy density, and pressure, for species in thermal equilibrium

total energy density in all relativistic species $\rho = \dfrac{\pi^2}{30} g_* T^4$

$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

entropy density in relativistic species $s = \dfrac{2\pi}{45} g_{*S} T^3$

$$g_* \approx g_{*S}$$

comoving entropy density is conserved $s \propto a^{-3}$

$$T \propto g_{*S}^{-1/3} a^{-1}$$

$$s = \frac{\rho + p}{T}$$

temperature will consistently decrease under adiabatic evolution in an expanding universe

Primordial nucleosynthesis

Observations of primordial nebulae reveal abundances of the light elements unexplained by stellar nucleosynthesis.

At temperatures below 1 MeV, the weak interactions are frozen out and neutrons and protons cease to interconvert. The equilibrium abundance of neutrons at this temperature is about 1/6 the abundance of protons (due to the slightly larger neutron mass). The neutrons have a finite lifetime ($\tau_n = 890$ sec) that is somewhat larger than the age of the universe at this epoch, $t(1\ \mathrm{MeV}) \approx 1$ sec, but they begin to gradually decay into protons and leptons. Soon thereafter, we reach a temperature somewhat below 100 keV, and Big-Bang nucleosynthesis (BBN) begins. At that point the neutron/proton ratio is approximately 1/7.

for every two neutrons and fourteen protons, we end up with one helium nucleus and twelve protons.

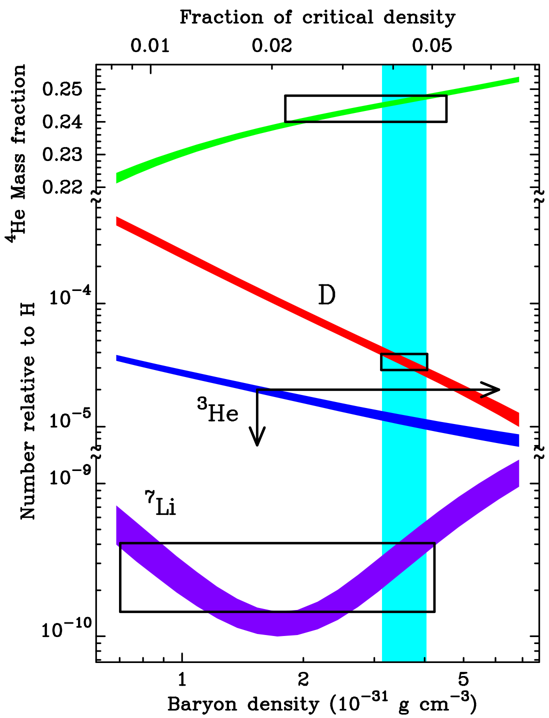
→ 25% of the baryons by mass are converted to helium.

trace amounts of deuterium (approximately 10^{-5} deuterons per proton), ^{3}He (also $\sim 10^{-5}$), and ^{7}Li ($\sim 10^{-10}$)

the relative abundances of the light elements depend essentially on just one parameter, the *baryon to entropy ratio*

$$\boxed{\eta \equiv \frac{n_B}{s} = \frac{n - n_{\bar{b}}}{s}}$$

$n_B = n_b - n_{\bar{b}}$ is the difference between the number of baryons and antibaryons per unit volume.



$$\eta = 6.1 \times 10^{-10} \begin{array}{l} +0.3\times 10^{-10} \\ -0.2\times 10^{-10} \end{array}$$

from precise measurements of the relative heights of the first two microwave background acoustic peaks

Baryogenesis

Outside of particle accelerators, antimatter can be seen in cosmic rays in the form of a few antiprotons, present at a level of around 10^{-4} in comparison with the number of protons this proportion is consistent with secondary antiproton production through $p + p \longrightarrow 3p + \bar{p}$, as the cosmic rays stream towards us.

→ no evidence for primordial antimatter in our galaxy.

if matter and antimatter galaxies were to coexist in clusters of galaxies, then we would expect there to be a detectable background of γ-radiation from nucleon-antinucleon annihilations within the clusters. This background is not observed

→ negligible antimatter on the scale of clusters

The baryon number density does not remain constant during the evolution of the universe, instead scaling like a^{-3}

define baryon asymmetry of the universe in terms of the quantity $\eta \equiv \dfrac{n_B}{s}$

range of η consistent with the deuterium and ^{3}He primordial abundances is

$$2.6 \times 10^{-10} < \eta < 6.2 \times 10^{-10}$$

as the universe cooled from early times to today, what processes were responsible for the generation of this very specific baryon asymmetry?

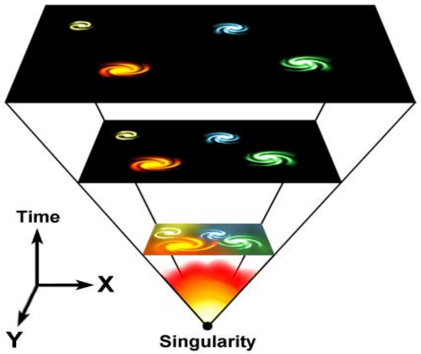
Sakharov : a small baryon asymmetry η may have been produced in the early universe if

- baryon number (B) violation,
- violation of C (charge conjugation symmetry) and CP (the composition of parity and C),
- departure from thermal equilibrium.

Successes of Hot big Bang Model

four pillars of the standard hot big bang model:

- expansion of the universe
- origin of the cosmic background radiation
- synthesis of light elements
- formation of galaxies and large scale structure



but …

There are some issues associated with the choice of initial conditions, to which the standard Hot Big bang cosmological model is not able to provide an answer.

It does not give a wrong answer, it is just unable to provide any answer.

Hence, in the context of the Hot Big bang cosmological model one must accept particular initial conditions.

Problems of the standard cosmological model:

- ***singularity problem***

$$\left.\begin{aligned} \rho &\sim a^{-3(1+w)} \\ a &\sim t^{\frac{2}{3(1+w)}} \end{aligned}\right\} \Longrightarrow \rho \to \infty \text{ as } t \to 0$$

& the corresponding solutions cannot be formally to the domain $t < 0$

- ***problem of large scale homogeneity and isotropy***

The universe is not completely homogeneous and isotropic even now, at least on a relatively small scale, so there is no reason to believe that it was initially homogeneous

The class of initial conditions for which the universe tends asymptotically (at large t) to a Friedmann universe is one of measure zero among all possible initial conditions

- ***flatness of space problem***

$$\Omega - 1 = \frac{K}{(aH)^2}$$

If Ω is unity at some time, it is always unity.

If $\Omega \neq 1$ at any time, it evolves in time.

Assuming that the spatial curvature is initially small

$$\begin{array}{llll} \text{mat. dom.} & a \propto t^{2/3},\ H \propto t^{-1} \Rightarrow \dfrac{1}{aH} \propto t^{1/3} & \Rightarrow |1-\Omega| \propto t^{2/3} \\ \text{rad. dom.} & a \propto t^{1/2},\ H \propto t^{-1} \Rightarrow \dfrac{1}{aH} \propto t^{1/2} & \Rightarrow |1-\Omega| \propto t\ . \end{array}$$

spatial curvature positive

$\Rightarrow$ expansion will stop and the universe will collapse.

spatial curvature is negative,

$\Rightarrow$ universe empty and cold

ΛCDM model $\quad |\Omega(t_0) - 1| < 10^{-2}$

$\Rightarrow$ $|\Omega(t_{\rm BBN}) - 1| \lesssim 10^{-17}$ strong tuning

- ***horizon problem or causality problem***

CMB data show that at $t \sim 10^5$ yr the universe was quite accurately homogeneous and isotropic on scales orders of magnitude greater than t (i.e., greater than the particle horizon), with temperatures T in different regions differing by less that $\mathrm{O}(10^{-4})\mathrm{T}$.

The probability that the T of these regions being correlated to that accuracy is at most 10^{-24}–10^{-30}

- ***structure formation problem***

what is the origin of initial inhomogeneities giving rise to the observed structure formation?

▪ baryon asymmetry problem

why the universe is made almost entirely of matter, with almost no antimatter?

why baryons are many orders of magnitude scarcer than photons with

$$\frac{n_{\mathrm{B}}}{n_{\gamma}} \sim 10^{-9}$$

$$\Omega_{0,\mathrm{mono}} \sim 10^{11} \left(\frac{T_{\mathrm{GUT}}}{10^{14}\ \mathrm{GeV}}\right)^{3} \left(\frac{m_{\mathrm{mono}}}{10^{16}\ \mathrm{GeV}}\right)$$

▪ unwanted relics problem

Primordial monopole problem:

Creation of superheavy t'Hooft-Plyakov magnetic monopoles, which should be copiously produced in all of GUTs when phase transitions at 10^{14}–10^{15} GeV

Monopole annihilation proceeds very slowly & present monopole density should be comparable to baryon density

⇨ catastrophic consequences:

$$M_{\mathrm{monopole}} \approx \mathcal{O}(m_{\mathrm{proton}})$$

giving an energy density in the universe about 15 orders of magnitude higher that the $\rho_c \sim 10^{29}\ \mathrm{g/cm^3}$

- ***vacuum energy problem***

a constant homogeneous scalar field over all space represents a restructuring of the vacuum --- space filled with a constant scalar field remains empty since the motion of objects passing through space is not disturbed

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, \mathrm{R} = 8\,\pi\,\mathrm{G}\,\mathrm{T}_{\mu\nu} = 8\,\pi\,\mathrm{G}\,(\tilde{\mathrm{T}}_{\mu\nu} + g_{\mu\nu}\,\mathrm{V}(\varphi))$$

but when the scalar field appears, there is a change in the vacuum energy density, which is described by $V(\phi)$; in GR this affects the properties of space-time

Data: $|\rho_{\mathrm{vac}}| = |\mathrm{V}(\varphi_0)| \lesssim 10^{-29}\ \mathrm{g/cm^3}$

Example $\mathrm{SU}(5)$

This value of $V(\phi)$ was attained after a series of phase transitions

- after 1st PT: $\mathrm{SU}(5) \to \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ the vacuum energy (the value of $V(\phi)$) decreased by $10^{80}\ \mathrm{g/cm^3}$
- after PT: $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1) \to \mathrm{SU}(3)\times\mathrm{U}(1)$ it was reduced by about $10^{25}\ \mathrm{g/cm^3}$
- after PT that formed baryons form quarks, the vacuum energy decreased by $10^{14}\ \mathrm{g/cm^3}$ and surprisingly enough after all of these enormous drops, it turned out to equal zero to an accuracy $\pm 10^{-29}\ \mathrm{g/cm^3}$

Cosmological Inflation

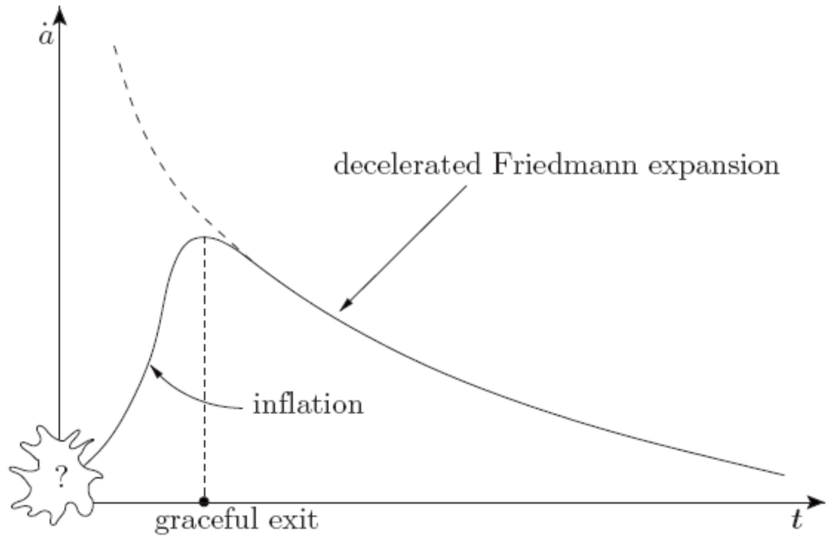
Inflation is a stage of accelerated expansion of the universe when gravity acts as a repulsive force.

$$\text{inflation} \quad \Leftrightarrow \quad \ddot{a} > 0$$

$$\text{inflation} \quad \Leftrightarrow \quad \frac{d}{dt}\left(\frac{1}{aH}\right) < 0$$

comoving Hubble length shrinks

$$\text{inflation} \quad \Leftrightarrow \quad \rho + 3p < 0$$



Lagrangean density $\mathcal{L}(\varphi, \partial^\mu \varphi)$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi)$$

the kinetic term of the field has the *kanonical* form and the field is minimally coupled

action

$$S = \int d^4x \sqrt{-g} \mathcal{L}$$

$V(\varphi)$ is the *potential* of the field

g is the determinant of the metric

Minimisation of the action leads to the *Euler–Lagrange equation*

$$\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial \varphi} - \partial_\mu \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial[\partial_\mu \varphi]} = 0$$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi)$$

$$\Rightarrow \quad \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial \varphi} - \partial_\mu \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial[\partial_\mu \varphi]} = 0$$

flat spacetime (Minkowski space)

$$g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

$$\ddot{\varphi} - \nabla^2 \varphi + V' = 0$$

spatially flat FRW metric

$$g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}, a^{-2}, a^{-2})$$

$$\ddot{\varphi} + 3H\dot{\varphi} - a^{-2}\nabla^2\varphi + V' = 0$$

$$\nabla^2 \varphi \equiv \delta^{ij} \partial_i \partial_j$$

Consider modeling matter in the early universe by a real scalar field ϕ, with potential $V(\phi)$

$$T_{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial_\nu\varphi + g_{\mu\nu}\mathcal{L}$$

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) + V(\phi)\right]$$

homogeneous case, $\partial_i\varphi = 0$ for the background evolution

all quantities depend only on cosmological time t and set $k = 0$.

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

The equation of motion for the scalar field is $$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0$$

Friedmann equation $$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$ $$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V\right]$$

slow-roll approximation

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$ ➡ if $\dot{\phi}^2 \ll V(\phi)$ then the potential energy of the scalar field is the dominant contribution to both the energy density and the pressure.

➡ equation of state is $p \simeq -\rho$.

approximately that of a cosmological constant

➡ accelerating expansion

slow-roll approximation

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0$$

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\dot{\phi} \simeq -\frac{V'(\phi)}{3H}\,,$$

$$H^2 \simeq \frac{8\pi G}{3}V(\phi)$$

prime denotes a derivative with respect to ϕ.

slow-roll approximation $\quad \begin{array}{rcl} |\epsilon| & \ll & 1 \\ |\eta| & \ll & 1 \end{array}$

slow-roll parameters

$$\epsilon \equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2$$

$$\eta \equiv M_p^2 \frac{V''}{V} \ .$$

the slow roll conditions yield inflation

$$\ddot{a}/a > 0$$

$$H = \frac{\dot{a}}{a} \quad \Rightarrow \quad \dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \quad \Rightarrow \quad \frac{\ddot{a}}{a} = \dot{H} + H^2 \quad \Longrightarrow \quad \text{inflation} \Leftrightarrow -\frac{\dot{H}}{H^2} < 1$$

If the slow-roll approximation is valid,

$$H^2 = \frac{V}{3M_{\rm Pl}^2} \quad \Rightarrow \quad 2H\dot{H} = \frac{V'\dot{\varphi}}{3M_{\rm Pl}^2} \quad \Rightarrow \quad H^2\dot{H} = \frac{V'H\dot{\varphi}}{6M_{\rm Pl}^2} \overset{3H\dot{\varphi}=-V'}{=} -\frac{V'^2}{18M_{\rm Pl}^2}$$

$$\Rightarrow \quad -\frac{\dot{H}}{H^2} = \frac{V'^2}{18M_{\rm Pl}^2}\frac{9M_{\rm Pl}^4}{V^2} = \frac{1}{2}M_{\rm Pl}^2\left(\frac{V'}{V}\right)^2 = \varepsilon \ll 1 \ .$$

$\Longrightarrow$ if the slow-roll approximation is valid, inflation is guaranteed.

number of *e-folds* $$N(t) \equiv \ln\left(\frac{a(t_{\text{end}})}{a(t)}\right)$$

$$\frac{da}{a} = d\ln a = H\,dt = H\,\frac{d\varphi}{\dot\varphi}$$

$$N(\varphi) = \ln\frac{a(t_{\text{end}})}{a(t)} = \int_t^{t_{\text{end}}} H(t)dt = \int_{\varphi}^{\varphi_{\text{end}}} \frac{H}{\dot\varphi}d\varphi \overset{\text{slow roll}}{\approx} \boxed{\frac{1}{M_{\text{Pl}}^2}\int_{\varphi_{\text{end}}}^{\varphi}\frac{V}{V'}d\varphi}\,,$$

the end of inflation ϕ_{end}, defined by $\epsilon(\phi_{\text{end}}) = 1$

The conditions $\varepsilon \ll 1$ and $|\eta| \ll 1$ are necessary, but not sufficient for the slow-roll approximation only constrain the form of the potential, and identify from the potential a *slow-roll section*, where the slow-roll approximation *may* be valid.

$\ddot\varphi + 3H\dot\varphi = -V'$ is second order $\Rightarrow$ it accepts arbitrary φ and $\dot\varphi$ as initial conditions

$\Rightarrow$ $\dot\varphi^2 \ll V$, $|\ddot\varphi| \ll 3H|\dot\varphi|$ may not hold initially, even if φ is in the slow-roll section.

Example:

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 \qquad \Rightarrow \qquad V'(\varphi) = m^2 \varphi\,, \quad V''(\varphi) = m^2$$

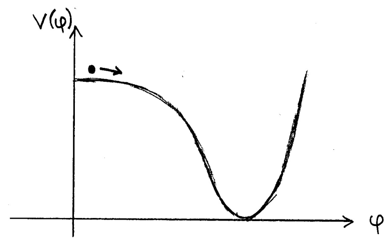
$$\left.\begin{aligned} \varepsilon(\varphi) &= \frac{1}{2} M_{\rm Pl}^2 \left(\frac{2}{\varphi}\right)^2 \\ \eta(\varphi) &= M_{\rm Pl}^2 \frac{2}{\varphi^2} \end{aligned}\right\} \quad \Rightarrow \quad \varepsilon = \eta = 2\left(\frac{M_{\rm Pl}}{\varphi}\right)^2$$

$$\varepsilon, \eta \ll 1 \quad \Rightarrow \quad \varphi^2 \gg 2 M_{\rm Pl}^2$$

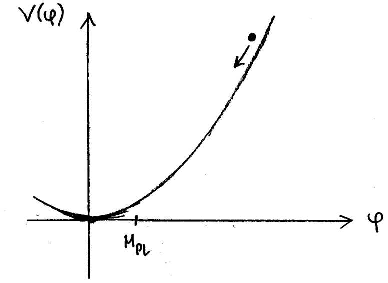
Models of inflation

A scalar field model of inflation consists of the potential for the inflation and its couplings to other fields. In most models, couplings to other fields don't matter during inflation, and only the inflaton is dynamically important. However, these couplings usually come into play when inflation ends. Inflation can end because the slow-roll approximation is no longer valid, as the field has rolled down the potential. In this case inflation ends when either $\varepsilon(\varphi)$ or $|\eta(\varphi)|$ becomes of order unity. Another possibility is that inflation ends while the inflaton undergoes slow-roll, because other fields coupled to the inflaton become dynamically important and terminate inflation. An example of this is *hybrid inflation*, where there is an extra scalar field in addition to the inflaton.

Inflation models can be divided into two classes:



1. Small field inflation: $\Delta\varphi < M_{\rm Pl}$ in the slow-roll section.



2. Large field inflation: $\Delta\varphi > M_{\rm Pl}$ in the slow-roll section.

Here $\Delta\varphi$ change in φ during (the observationally relevant part of) inflation.

Example: Consider a simple potential of the form $V(\varphi) = A\varphi^n$. This is a large field model, since $V'/V = n/\varphi \quad \Rightarrow \quad \varepsilon \ll 1$ requires $\varphi^2 \gg \frac{1}{2}n^2 M_{\rm Pl}^2$.

power-law inflation

$$V(\varphi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\varphi}{M_{\rm Pl}}\right) , \qquad p > 1$$

V_0 and p are constants.

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'$$

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V\right]$$

exact solution

$$a(t) = a_0 t^p$$

$$\varphi(t) = \sqrt{2p} M_{\rm Pl} \ln\left(\sqrt{\frac{V_0}{p(3p-1)}}\frac{t}{M_{\rm Pl}}\right)$$

slow-roll parameters $\qquad \varepsilon = \frac{1}{2}\eta = \frac{1}{p}$ $\qquad$ independent of φ

In this model inflation never ends unless other physics intervenes.

Reheating

As inflation ends, this energy is transferred in the reheating process to a thermal bath of particles produced in the reheating.

After inflation, the inflaton field φ begins to oscillate at the bottom of the potential $V(\varphi)$

The inflaton field is still homogeneous, $\varphi(t, \vec{x}) = \varphi(t)$, so it oscillates in the same phase everywhere

coherent oscillation

The oscillation period soon becomes much shorter than the expansion time scale H^{-1}

When the inflaton field is oscillating around the minimum of the potential, the energy stored in the inflaton field is transferred into particles, both by decay into quanta of the inflaton field, which subsequently decay, and direct decay into other fields via coupling between them and the inflaton.

If the decay is slow $\quad \dot{\rho}_\varphi + 3H\rho_\varphi = -\Gamma_\varphi \rho_\varphi \quad \Gamma_\varphi = 1/\tau_\varphi$, the *decay width*

energy transfer to other particles

inflaton decay time τ_φ

Thermalisation

The particles produced from the inflaton will interact, create other particles through particle reactions, and the resulting soup will eventually reach thermal equilibrium with some temperature $T_{\rm reh}$. This *reheating temperature* is determined by the energy density $\rho_{\rm reh}$ at the end of the reheating epoch:

$$\boxed{\rho_{\rm reh} = \frac{\pi^2}{30} g_*(T_{\rm reh}) T_{\rm reh}^4}$$

Necessarily $\rho_{\rm reh} < \rho_{\rm end}$ (end = end of inflation). If reheating takes a long time, we may have $\rho_{\rm reh} \ll \rho_{\rm end}$. it is possible that some particles (such as gravitinos) never reach thermal equilibrium, since their interactions are too weak.

In any case, as long as the momenta of the particles are much higher than their masses, the energy density of the universe behaves like radiation, regardless of the momentum space distribution.

So the background expansion rate is the same. After thermalisation of at least the baryons, photons and neutrinos is complete, the standard Hot Big Bang era begins.

Comments:

Inflation does not replace the Big Bang model, but it enriches it; it was proposed in order to address shortcoming of the BB model. Inflation does not resolve all problems of the BB model which are related with the initial conditions.

Despite decades of studies and countless inflationary models, inflation remains a paradigm in search of a theory.

Inflation gave as a bonus adiabatic scalar perturbations that can explain the origin of initial perturbations leading to scalar formation from gravitational instability.

The inflationary mechanism may be the appropriate one, but its origin may not be a scalar field but a modification of gravity having each origin in the quantum gravity era (like the Starobinsky model)

Standard inflation requires particular conditions as the outcome of a quantum era setting the beginning of the semiclassical era.

There are other models suggested in the literature that can do the job of inflation: bounce cosmologies