4. Gravitational wave production

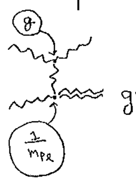
To have remnants from cosmological phase transitions, we need to consider probes which do not thermalize afterwards — otherwise all memory is lost by dissipation and random kicks (cf. p. 12).

To set the stage, let us recall that any electromagnetic plasma with charged constituents emits blackbody radiation, even if the plasma as a whole is neutral. The photons originate from microscopic collisions, and carry the energy

$$de_\gamma = 2\,\frac{d^3\vec{k}}{(2\pi)^3}\,k\left[\tfrac{1}{2}+n_B(k)\right].$$

Polarizations | phase space | energy | statistical weight (p. 1)

In the case of gravitons, they can also originate from collisions, but now the production rate is so small that they do not thermalize:



$$\Rightarrow\quad \frac{de_{GW}}{dt\,d\ln k} \;\; \underset{\left(\alpha=\frac{g^2}{4\pi}\right)}{\sim} \;\; \frac{\alpha}{m_{Pl}^2}\,k^4\,n_B(k)\,\phi(k).$$

need to extract the energy from thermal motion

Complicated but perhaps slowly varying function of dimension k^3

Given that $k^4 e^{-k/T}$ is maximized at $k\sim T$, the maximal rate is

$$\left.\frac{de_{GW}}{dt\,d\ln k}\right|_{max} \sim \frac{\alpha T^7}{m_{Pl}^2}.$$

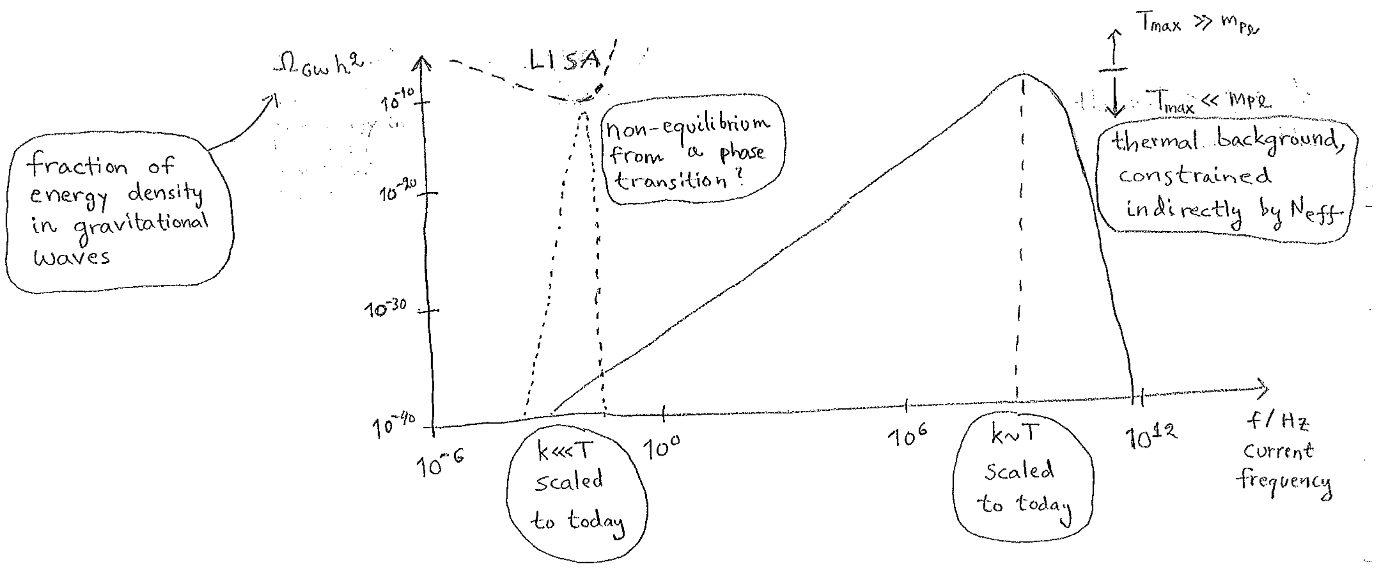
Normalizing by the energy density of radiation, we obtain a rate that can be compared with the Hubble rate:

$$\text{thermalization if}\quad \frac{1}{T^4}\left.\frac{de_{GW}}{dt\,d\ln k}\right|_{max} \gg \frac{T^2}{m_{Pl}} \;\Leftrightarrow\; \frac{\alpha T}{m_{Pl}} \gg 1$$

$$\Leftrightarrow\quad T \gg \frac{m_{Pl}}{\alpha}.$$

This is unlikely to be reached by typical reheating mechanisms.

However, non-equilibrium processes, like phase transitions, could yield something.



Sketch: how to compute the gravitational wave production rate?

The energy density of gravitational radiation can famously not be localized, so the computation is delicate in general relativity. More simply, the result can be obtained by quantizing gravitons and considering a local Minkowskian frame.*

* See however comments on p.15.

Let us define the quantum-mechanical Hamiltonian

$$\hat{H} = \hat{H}_{\text{plasma}} + \hat{H}_{\text{gravitons}} + \hat{H}_{\text{int}} \;, \quad \hat{H}_{\text{int}} = \int_{\vec{x}} \varepsilon \, \underbrace{\hat{h}}_{\text{"graviton"}} \, \underbrace{\hat{J}}_{\text{operator containing plasma fields}} \;, \quad \varepsilon \ll 1 .$$

On-shell field operator: $\hat{h} = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\epsilon_k}} \left(\hat{a}_k e^{-iK\cdot X} + \hat{a}_k^{\dagger} e^{iK\cdot X} \right)$.

States: $|\vec{k}\rangle = \hat{a}_k^{\dagger}|0\rangle$; $|I\rangle \equiv \underbrace{|i\rangle}_{\text{plasma}} \otimes \underbrace{|0\rangle}_{\hat{h}}$; $|F\rangle \equiv \underbrace{|f\rangle}_{\text{plasma}} \otimes \underbrace{|\vec{k}\rangle}_{\hat{h}}$.

Transition matrix element: $T_{FI} = \langle F | \int_0^t dt' \, \hat{H}_{\text{int}}(t') | I \rangle$.

Phase space rate: $\frac{\dot{f}(\vec{k})}{(2\pi)^3} = \lim_{t,V\to\infty} \sum_{f,i} \frac{e^{-\beta E_i}}{Z} \frac{|T_{FI}|^2}{tV}$.

Inserting the on-shell field operator we find $\left(\langle \vec{k} | \hat{a}_q^{\dagger} | 0 \rangle = \delta^{(3)}(\vec{k}-\vec{q}) \right)$

$$T_{FI} = \varepsilon \int_{X'} \frac{e^{iK\cdot X'}}{\sqrt{(2\pi)^3 2\epsilon_k}} \langle f | \hat{J}(X') | i \rangle$$

$$\Rightarrow \quad |T_{FI}|^2 = \frac{\varepsilon^2}{(2\pi)^3 \, 2\epsilon_k} \int_{X',Y'} e^{iK\cdot(X'-Y')} \langle f | \hat{J}(X') | i \rangle \langle i | \hat{J}^{\dagger}(Y') | f \rangle .$$

Now sum over i with the Boltzmann weight $e^{-\beta E_i}$, and make use of translational invariance to cancel tV,** after setting $X' \to Y' + X$.

** against $\int_{Y'}$

$$\Rightarrow \quad \dot{f}(\vec{k}) = \frac{\varepsilon^2}{2\epsilon_k} \int_X e^{iK\cdot X} \underbrace{\langle \hat{J}^{\dagger}(0) \hat{J}(X) \rangle}_{\text{thermal average}} .$$

For gravitational waves:

* after canonical normalization of $\hat{h}$: $\varepsilon \to \frac{\sqrt{8\pi}}{m_{\text{Pl}}}$.

* $\hat{J} \to \sum_{\mu\nu} \underbrace{\epsilon^{TT(\lambda)}_{\mu\nu,\vec{k}}}_{\text{physical polarizations}} \underbrace{\hat{T}^{\mu\nu}}_{\text{energy-momentum tensor}}$

* $d\dot{e}_{GW} = \dot{f} \, \epsilon_k \frac{d^3\vec{k}}{(2\pi)^3}$

$$\Rightarrow \quad \frac{de_{GW}}{dt \, d^3\vec{k}} = \underbrace{\frac{1}{(2\pi)^3} \cdot \frac{8\pi}{m_{\text{Pl}}^2} \cdot \frac{\epsilon_k}{2\epsilon_k}}_{\frac{1}{2\pi^2 m_{\text{Pl}}^2}} \cdot \sum_{\lambda} \epsilon^{TT(\lambda)*}_{\mu\nu,\vec{k}} \, \epsilon^{TT(\lambda)}_{\rho\sigma,\vec{k}} \underbrace{\int_X e^{iK\cdot X} \langle \hat{T}^{\mu\nu}(0) \hat{T}^{\rho\sigma}(X) \rangle}_{\equiv \, G_<^{\mu\nu;\rho\sigma}(K) \text{ "Wightman function"}} .$$

Towards a practical measurement

(i) the polarization sum $\mathbb{L}_{\mu\nu;\rho\sigma} \equiv \sum_\lambda \epsilon^{TT(\lambda)*}_{\mu\nu,\vec{k}}\, \epsilon^{TT(\lambda)}_{\rho\sigma,\vec{k}}$ has the properties of being transverse ($K^\mu \mathbb{L}_{\mu\nu;\rho\sigma} = K^\nu \mathbb{L}_{\mu\nu;\rho\sigma} = \ldots = 0$), traceless ($\eta^{\mu\nu}\mathbb{L}_{\mu\nu;\rho\sigma} = \eta^{\rho\sigma}\mathbb{L}_{\mu\nu;\rho\sigma} = 0$), and normalized to two physical degrees of freedom ($\eta^{\mu\rho}\eta^{\nu\sigma}\mathbb{L}_{\mu\nu;\rho\sigma} = 2$). Exercise:

$$\mathbb{L}_{\mu\nu;\rho\sigma} = \frac{\mathbb{K}^T_{\mu\rho}\mathbb{K}^T_{\nu\sigma} + \mathbb{K}^T_{\mu\sigma}\mathbb{K}^T_{\nu\rho} - \mathbb{K}^T_{\mu\nu}\mathbb{K}^T_{\rho\sigma}}{2}\,, \quad \mathbb{K}^T_{\mu\nu} \equiv \eta_{\mu i}\eta_{\nu j}\Big(\delta_{ij} - \frac{k_i k_j}{k^2}\Big).$$

We did operate in a local Minkowskian frame, however. More generally, we would need to do a computation similar to the one on p. 12, implying (in the classical limit)

$$\langle h(X) h(Y)\rangle = \epsilon^2 \int_{Z,W} G_R(X-Z)\, G_R(Y-W)\, \langle J(Z) J(W)\rangle\,.$$

Green's function in curved background

(ii) the representation as a Wightman function is quite general, and can also be used for gravitational waves originating from quantum-mechanical processes, like particle collisions. In this case it can be represented as

$$G^{\mu\nu;\rho\sigma}_{<}(K) = 2 n_B(k)\, \mathrm{Im}\, G^{\mu\nu;\rho\sigma}_R(K)\,,$$

where G_R is a retarded Green's function, and the imaginary part is a reflection of the optical theorem.

(iii) in the case of phase transitions, however, we are interested in momenta $k \lll T$, where the fields contain many quanta and are classical. Then operator ordering plays no role, and we express $T^{\mu\nu}$ in terms of classical fields (T, u^μ, ϕ).

(iv) Concretely,

$$T^{\mu\nu} = T^{\mu\nu}_{\text{plasma}} + T^{\mu\nu}_{\phi}\,;$$

$$T^{\mu\nu}_{\text{plasma}} = T^{\mu\nu}_{\text{ideal}} + \big(\text{gradient corrections, proportional to viscosities}\big)\,;$$

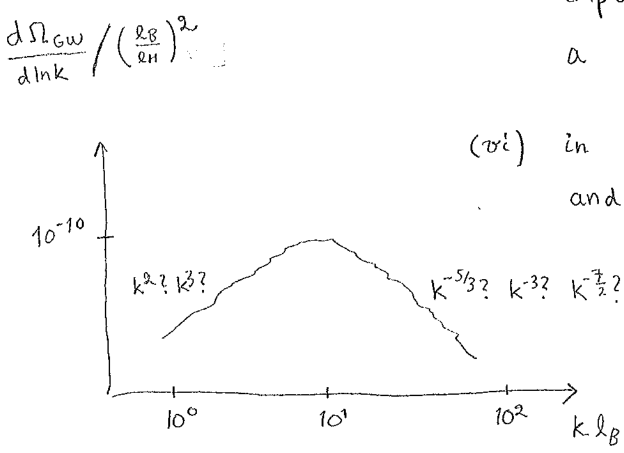
$$T^{\mu\nu}_{\text{ideal}} = (e+p)\, u^\mu u^\nu - p\, \eta^{\mu\nu}\,; \quad p = p_r - V_{\text{eff}}(\phi, T)\,;$$

$$T^{\mu\nu}_{\phi} = \partial^\mu\phi\, \partial^\nu\phi - \eta^{\mu\nu}\, \frac{\partial^\alpha\phi\, \partial_\alpha\phi}{2}\,.$$

(v) we may recall that the TT projection to tensor modes implies that symmetries need to be broken, in order for a propagating mode to be generated: quadrupoles must be present.

(vi) in practice, large-scale numerical simulations are used, and some semi-analytic models are being constructed.

defined on p.16



Something about scales

Even though we cannot compute the gravitational wave production rate analytically, we can estimate some of its basic properties.

* for reference: $T = 2.7\,K$; $1\,eV = 11600\,K$; $\hbar = 6.6\times10^{-16}\,eV\,s$

$$\Rightarrow f = \frac{\omega}{2\pi} = \frac{\hbar\omega}{2\pi\hbar} \underset{\hbar\omega\sim T}{=} \frac{2.7\,K}{2\pi\,\hbar} = \frac{2.7}{11600}\cdot\frac{1}{2\pi\times6.6\times10^{-16}\,s}$$

$$\sim 10^{11}\,\frac{1}{s} \sim 100\,\text{GHz}.$$

* the wavelength of the gravitational waves produced is at most the horizon length of the electroweak epoch, since the physics of bubble nucleation and growth is causal statistical physics.

At $T \sim T_{ew} \sim 100\,GeV$: $\ell_H \sim t_{ew} \sim \frac{m_{Pl}}{T^2_{ew}}$.

horizon length or Hubble radius: $\ell_H \equiv H^{-1}$

Scale till today : $\ell_H\big|_{now} = \frac{a(T_{now})}{a(T_{ew})}\cdot\ell_H\big|_{ew} \sim \frac{T_{ew}}{T_{now}}\cdot\frac{m_{Pl}}{T^2_{ew}}$

$$\sim \frac{10^{19}\,GeV}{10^{-3}\,eV\cdot100\,GeV} \sim 10^{20}\,\frac{1}{eV} \sim \frac{10^{29}}{GeV}$$

$$\underset{GeV\cdot fm\sim5}{\sim} 10^{13}\,m \underset{c=3\times10^8\,\frac{m}{s}}{\sim} 10^4\,s$$

$$\Rightarrow f_H\big|_{ew} \sim 10^{-4}\,Hz \quad\Rightarrow f_{causal}\big|_{ew} > 10^{-4}\,Hz.$$

* LISA armlength $3\times10^9\,m$ $\Rightarrow$ $f\big|_{LISA} \sim (10^{-2}\ldots10^{-1})\,Hz$.

* how much energy can be put into gravitational waves? The production rate must be quadratic in suppressing factors. Normalizing to the energy density of radiation, one factor is L/e, because at most L^{**} can be converted to e_{GW}. Another is ℓ_B/ℓ_H, where ℓ_B is the bubble distance scale, because this sets the pattern for breaking symmetries. Further suppressions are by velocities, efficiency, etc..

** here L is the latent heat from p. 11

$$\Rightarrow \Omega_{GW}h^2 \lesssim 10^{-5}\left(\frac{L}{e}\right)^2\left(\frac{\ell_B}{\ell_H}\right)^2 .$$

* LISA sensitivity could be down to $\Omega_{GW}h^2 \sim 10^{-12}$.

Summary : If one degree of freedom undergoes a transition, e.g. the Higgs field, and $\ell_B/\ell_H < 10^{-2}$, as nucleation computations often indicate, then we get

$$\sim 10^{-5}\,(10^{-2})^2(10^{-2})^2 \sim 10^{-13} .$$

But perhaps something more remarkable happens !?