Phase transitions in the early universe

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1. Electroweak phase transition beyond the Standard Model

Without supplemental particles the Standard Model (SM) has no first-order thermal phase transition. In fact it is a cross-over [1]. Beyond the SM extensions could, however, allow for a first-order phase transition. While supplementing the SM with additional scalars is one viable option to achieve this, also higher-dimensional operators can alter the phase transition. This exercise follows closely the calculations of [2].

Inspect the pure scalar sector of the Minkowskian SM Lagrangian

$$\mathcal{L}_{\rm M} = (D_\mu\phi)^\dagger(D_\mu\phi) - V(\phi) \;, \tag{1.1}$$

$$V(\phi) = \mu_h^2\phi^\dagger\phi + \lambda\big(\phi^\dagger\phi\big)^2 \;, \tag{1.2}$$

where D_μ is the covariant derivative acting on the SU(2) Higgs doublet ϕ. Using a constant background field $\bar{\phi}$, we parameterise

$$\phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(\bar{\phi} + h + iG^0) \end{pmatrix} \;, \tag{1.3}$$

where h is the physical Higgs field and G^+, G^0 are Goldstone bosons with $G^- = (G^+)^\dagger$.

Exercise 1.1. Construct the tree-level effective potential $V_{\rm eff}^{(0)}(\bar{\phi})$ by employing the parameterisation (1.3) in the Higgs potential (1.1). For this analysis assume vanishing $h, G^\pm, G^0 \to 0$. From the resulting potential, relate the $\overline{\rm MS}$ -renormalised parameters of the Lagrangian to physical observables using the vacuum expectation value of the Higgs $v = 246$ GeV and the physical Higgs mass $M_h = 125$ GeV.

Exercise 1.2. The effective potential receives loop corrections such that $V_{\rm eff}(\bar{\phi}) = V_{\rm eff}^{(0)}(\bar{\phi}) + V_{\rm eff}^{(1)}(\bar{\phi})$. The one-loop contribution to the effective potential then takes the finite-temperature form

$$V_{\rm eff}^{(1)} = \sum_i n_i \SumInt_P \ln(P^2 + m_i^2) \;, \tag{1.4}$$

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where $D = d + 1 = 4 - 2\epsilon$, the Euclidean four-momenta $P = (\omega_n, \mathbf{p})$, $\SumInt_P = T \sum_{\omega_n} \int_{\mathbf{p}}$, and $\int_{\mathbf{p}} = \left(\frac{\bar{\mu}^2 e^{\gamma}}{4\pi}\right)^{\epsilon} \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}$. The summation runs over all species $\{i\}$ that couple to ϕ and n_i is the number of degrees of freedom of the i-th field with mass m_i.

Show that the one-loop contribution to the effective potential (1.4) is of the form

$$V_{\text{eff}}^{(1)}(\bar{\phi}) = \sum_i n_i \Big[\int_P \ln\big(P^2 + m_i^2(\bar{\phi})\big) + J_{\text{B,F}}\Big(\frac{m_i^2(\bar{\phi})}{T^2}\Big) \Big] , \tag{1.5}$$

where $\int_P = \left(\frac{\bar{\mu}^2 e^{\gamma}}{4\pi}\right)^{\epsilon} \frac{\mathrm{d}^D P}{(2\pi)^D}$, $\bar{\mu}$ is the $\overline{\text{MS}}$ -renormalisation scale, and γ is the Euler-Mascheroni constant. The first term of eq. (1.5) is the zero-temperature Coleman-Weinberg potential [3] and temperature effects [4] are encoded in the thermal functions

$$J_{\text{B,F}}(m_i^2) = -T \int_{\mathbf{p}} \ln\big(1 \pm n_{\text{B,F}}(\varepsilon_p^i, T)\big) , \quad n_{\text{B,F}}(\varepsilon_p^i, T) = \frac{1}{e^{\varepsilon_p^i/T} \mp 1} , \quad \varepsilon_p^i = \sqrt{p^2 + m_i^2} . \tag{1.6}$$

Here, $n_{\text{B,F}}$ are the bosonic and fermionic distribution functions, respectively. Show that their expansion at high temperature i.e. $z \ll 1$ with $z^2 = m^2(\bar{\phi})/T^2$ follows

$$\mathcal{J}_{\text{B}}(z^2) = -\frac{\pi^2}{90} + \frac{1}{24} z^2 - \frac{1}{12\pi} (z^2)^{\frac{3}{2}} + \mathcal{O}(z^4) , \tag{1.7}$$

$$\mathcal{J}_{\text{F}}(z^2) = +\frac{7}{8}\frac{\pi^2}{90} - \frac{1}{48} z^2 + \mathcal{O}(z^4) . \tag{1.8}$$

where $\mathcal{J}_{\text{B,F}}(z^2) = J_{\text{B,F}}(z^2)/T^4$. Using the above expressions,

- derive the one-loop effective potential at leading order in the high-temperature expansion $z^2 = m^2(\bar{\phi})/T^2$.
- determine the functional form of the effective potential as a function of $\bar{\phi}$ through the corresponding leading-order terms at $\bar{\phi}^2$, $\bar{\phi}^4$ and $\bar{\phi}^6$.

Exercise 1.3. By adding the sextic interaction $|\phi|^6$ of dimension six, the Higgs potential (1.1) in the symmetric phase is augmented by the operator

$$\mathcal{O}_6 = M^{-2} \big(\phi^\dagger \phi\big)^3 , \tag{1.9}$$

this is the minimal SM effective theory (SMEFT). In relation to the SM in Exercise 1.1 and Exercise 1.2

- visualise the difference between the tree-level effective potentials of the pure SM and the SMEFT.
- include the M-dependence in μ_h^2 and λ. For the additional parameter M there will be now a barrier in the effective potential already at tree-level. Determine how the global minimum depends on M.
- derive the one-loop effective potential at leading order in the high-T expansion.

2. Dimensionally reduced EFT of the SM

Construct the corresponding dimensionally reduced EFT starting from the pure scalar SM Lagrangian (1.1); see e.g. [2,6].

Exercise 2.1. In this scenario only two effective parameters need to be matched, namely λ_3 and $\mu^2_{h,3}$. In the symmetric phase, first draw the corresponding Feynman diagrams for the 2-point and 4-point scalar correlator in the pure scalar sector both in the 4d and 3d theory. Then using Feynman rules relate the diagrams to an integral expression. Since both the effective theory and the full theory are matched at the low energy scale, expand in soft momenta and masses $p, m_i \ll 2\pi T$.

Exercise 2.2. By following the derivation in the lecture, show that a general sum-integral can be expressed in $d = 3 - 2\epsilon$ as

$$Z_\alpha \equiv \SumInt_P{}' \frac{1}{[P^2]^\alpha} = \left(\frac{\bar{\mu}^2 e^\gamma}{4\pi}\right)^\epsilon 2T \frac{[2\pi T]^{d-2\alpha}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \zeta_{2\alpha - d} \;, \tag{2.1}$$

with $P = (\omega_n, \mathbf{p})$ and by using the d-dimensional vacuum integral

$$I_\alpha(m^2) \equiv \int_{\mathbf{p}} \frac{1}{[p^2 + m^2]^\alpha} = \left(\frac{\bar{\mu}^2 e^\gamma}{4\pi}\right)^\epsilon \frac{[m^2]^{\frac{d}{2} - \alpha}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \;, \tag{2.2}$$

with the sum representation of the Riemann zeta function $\zeta_s = \sum_{n=1}^\infty n^{-s}$.

Exercise 2.3. Schematically the matching can be illustrated for the quartic terms of the effective action

$$\Big(\lambda + \Gamma_{\rm 4d}\Big)\varphi^4_{\rm 4d} = T\Big(\lambda_3 + \Gamma_{\rm 3d}\Big)\varphi^4_{\rm 3d} \;, \tag{2.3}$$

where $\varphi_{\rm 3d} = \varphi_{\rm 4d}(1 + \mathcal{O}(\lambda))$ and loop corrections are collected in Γ. Argue that the loop corrections in the EFT, $\Gamma_{\rm 3d}$, vanish in the matching and extract the corresponding matching coefficients.

Exercise 2.4. With the 3d effective theory constructed above, compute the 3d effective potential from

$$V_{3d}(\bar{\phi}) \equiv \sum_i n_i \, J_{\rm 3d}(m_i^2) \;, \tag{2.4}$$

$$\begin{aligned} J_{\rm 3d}(m^2) = \int_{\mathbf{p}} \ln(p^2 + m^2) &= -\frac{1}{2}\left(\frac{\bar{\mu}^2 e^\gamma}{4\pi}\right)^\epsilon \frac{[m^2]^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(-\frac{d}{2})}{\Gamma(1)} \\ &= -\frac{(m^2)^{\frac{3}{2}}}{12\pi} + \mathcal{O}(\epsilon) \;. \end{aligned} \tag{2.5}$$

How does the resulting expression differ from the effective potential from Exercise 1.

3. Surface tension of a bubble

In the thin-wall limit, the difference in free energy if a bubble exists or not is given by

$$\Delta F = \sigma A_b - \Delta p V_b \, , \tag{3.1}$$

where σ is the *surface tension* and $\Delta p = -\Delta V_{\text{eff}}$ is the *pressure difference*. A_b is the surface area and V_b the volume of the bubble. Below we will derive this correspondence.

Assume the limit where the bubble has a thin wall that separates the new stable true vacuum on the inside from the false vacuum in the exterior. The radius of the bubble is R and $r = |\mathbf{x}|$ will be the radial coordinate of the bubble, such that

$$\begin{aligned} &\phi(r) = \bar{\phi}_{\text{broken}} \, , && V_{\text{eff}}(\phi) = V_{\text{true}} \, , && \text{for } r \ll R \, , \\ &\phi(r) = 0 \, , && V_{\text{eff}}(\phi) = V_{\text{false}} \, , && \text{for } r \gg R \, . \end{aligned} \tag{3.2}$$

Exercise 3.1. Derive an expression for $\Delta F = F_{\text{bub}} - F_{\text{nobub}}$. By ignoring the wall curvature of the bubble, rewrite the free energy difference in spherical coordinates.

Exercise 3.2. In the thin wall limit, the wall of the bubble is assumed to be small compared to its radius R. In other words, one is close to the limit of $T_{\text{c}} \to T_{\text{c}}^{-}$. For a wall of extent 2δ, show that the resulting expression is of the form (3.1) and identify the term $\Delta V_{\text{eff}} V_b$ and the remaining integral as σA_b.

Exercise 3.3. Use the Euler-Lagrange equations to derive the corresponding equation of motion (e.o.m.) for ϕ in d-dimensions and extract from it an expression for $\frac{\mathrm{d}\phi}{\mathrm{d}r}$. What is the correct sign of the derivative in the regime $[R-\delta, R+\delta]$? Using the found derivative, compute the remaining integral from Exercise 3.2 and identify from it the surface tension.

References

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