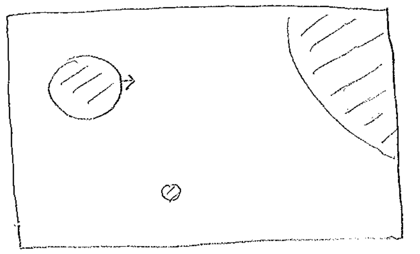
3. Nucleations and dynamics

Suppose that there is a strong first-order transition. How does it proceed in real time? Note that the latent heat associated with the transition needs to be put somewhere (e.g. universe expansion), so we cannot go instantaneously from one phase to another.

General process:

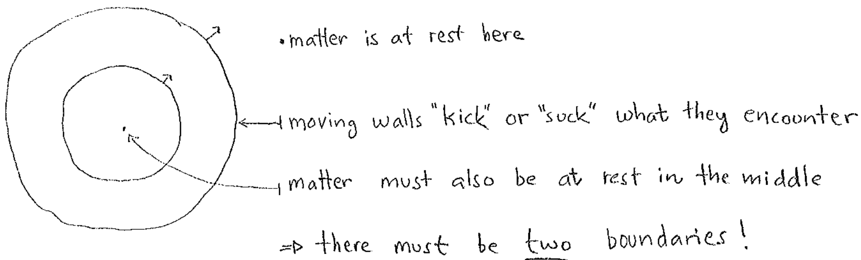
(i) nucleation of bubbles of the low-temperature phase



⟹ a classic & challenging theoretical problem (cf. p. 10–12).

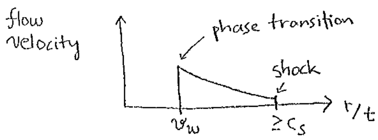
(ii) growth of the nucleated bubbles

There are non-trivial hydrodynamic boundary conditions to satisfy:

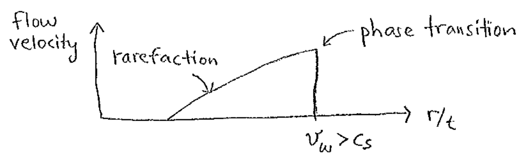


⟹ there must be two boundaries!

There are two classes of solutions, depending on whether the actual phase transition is the inner or the outer boundary.



deflagration



detonation

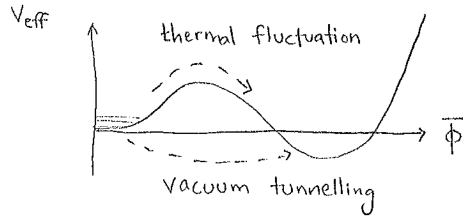
⟹ the determination of v_w is a classic & challenging theoretical problem.

(iii) bubble collisions, sound waves, turbulence, ...

Once the bubbles collide, the problem becomes very complicated, as there are no symmetries, and hydrodynamics is highly non-linear. These days such phenomena are studied with large-scale numerical simulations.

What is nucleation?

Intuitive picture :



But actually these are not separate phenomena, rather just different limits of a more general description.

In the path integral formalism, we say that the field space has several extrema : local minima of the action S_E, where the eigenvalues of $\delta^2 S_E / \delta\phi^2$ are all positive ; and unstable saddle points, where (at least) one eigenvalue is negative.

Specifically, consider a configuration $\phi = \hat{\phi}(\tau, \vec{x})$ with the properties that

* periodic in imaginary time: $\hat{\phi}(0,\vec{x}) = \hat{\phi}(\beta,\vec{x})$.
* asymptotically in false vacuum: $\lim_{|\vec{x}|\to\infty} \hat{\phi}(\tau,\vec{x}) = 0$.
* Solution of equations of motion: $\delta S_E / \delta\phi \,|_{\phi=\hat{\phi}} = 0$.

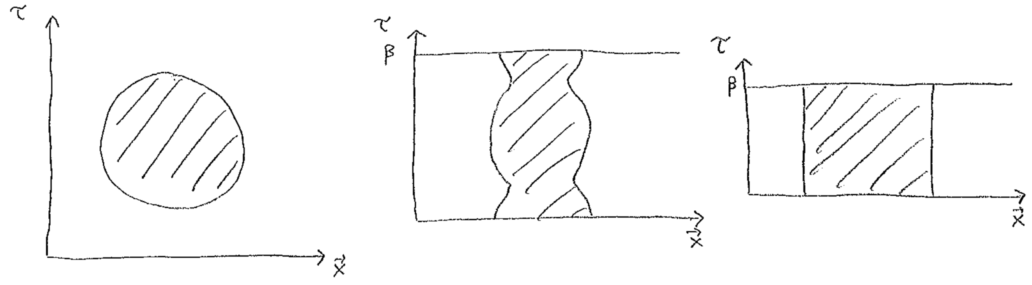
If we find a non-trivial such configuration, i.e. $\hat{\phi} \neq 0$, it is called a bounce.

Note that if a bounce exists, the fluctuation operator has zero modes:

$$\frac{\partial}{\partial x_i}\left\{ \frac{\delta S_E}{\delta\phi}\bigg|_{\phi=\hat{\phi}} = 0 \right\} \Rightarrow \frac{\delta^2 S_E}{\delta\phi^2}\bigg|_{\phi=\hat{\phi}} \partial_i \hat{\phi} = 0 .$$

Physically, the zero modes correspond to translations of the bubble: it can nucleate equally likely anywhere. (the bounce action is the same).

The bounce is maximally symmetric, within the given boundary conditions.



$T = 0$	$T \neq 0$	$T \gg 0$
$\Rightarrow$ 4d symmetry	$\Rightarrow$ (3+1)d symmetry	$\Rightarrow$ 3d symmetry
$\Rightarrow$ "instanton" for quantum tunnelling	$\Rightarrow$ "caloron" for thermally modified quantum tunnelling	$\Rightarrow$ $S_E[\hat{\phi}] = \beta \int d^3\vec{x}\, L_E^{(n=0)}$
		$\Rightarrow$ classical thermal tunnelling, e.g. "sphaleron"

Bounce in classical thermodynamics

Very good approximation in cosmology

Let us first recall some thermodynamics with $\mu = 0$; $e \equiv \frac{E}{V}$; $s \equiv \frac{S}{V}$, $f \equiv \frac{F}{V}$.

* $E = TS - pV \Rightarrow e = Ts - p$.
* $F = E - TS = -pV \Rightarrow f = -p$.
* $dF = -SdT - pdV \Rightarrow df\, V + f \cancel{dV} = -sV dT + f\cancel{dV} \Rightarrow s = -\frac{df}{dT} = \frac{dp}{dT}$.

Across a phase transition, f is continuous, but df/dT is not.

Then $\Delta e = T_c \Delta s - \Delta p = -T_c \underbrace{\Delta\left(\frac{df}{dT}\right)}_{\neq 0} + \underbrace{\Delta f}_{0} \neq 0$.

We call $L \equiv \Delta e$ the <u>latent heat</u>, where $\Delta e \equiv e|_{T>T_c} - e|_{T<T_c}$.

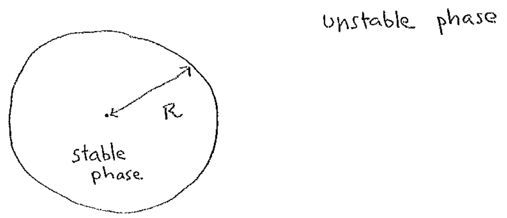
Let us now expand Δf to first order in a Taylor series in $T - T_c$:

$$\Delta f(T) = \underbrace{\Delta f(T_c)}_{0} + \underbrace{\frac{d}{dT}\Delta f\Big|_{T=T_c}}_{\boxed{\Delta\left(\frac{df}{dT}\right) = -\frac{\Delta e}{T_c} = -\frac{L}{T_c}}} (T - T_c) + \mathcal{O}(T-T_c)^2$$

$$= L\left(1 - \frac{T}{T_c}\right) + \mathcal{O}(T - T_c)^2 .$$

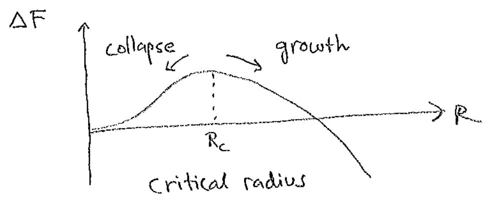
Physically: at $T<T_c$, the high-T phase has higher free energy ($\Delta f > 0$), so it is disfavoured.

Consider then a bubble of radius R at $T < T_c$:



Because of the volume, we <u>gain</u> the free energy $-\Delta f \cdot \frac{4}{3}\pi R^3$, however the surface <u>costs</u> $\sigma \cdot 4\pi R^2$, where σ = surface tension:

$$\Rightarrow S_E[\hat{\phi}] \approx \frac{\Delta F}{T} \approx \frac{1}{T}\left\{ -L\left(1 - \frac{T}{T_c}\right)\frac{4}{3}\pi R^3 + \sigma \cdot 4\pi R^2 \right\} .$$



Extremize with respect to $R \Rightarrow R_c = \frac{8\pi\sigma}{4\pi L\left(1-\frac{T}{T_c}\right)} = \frac{2\sigma}{L\left(1 - \frac{T}{T_c}\right)}$.

Insert back into $S_E[\hat{\phi}] \Rightarrow \Delta F = 4\pi\left(-\frac{8}{3} + 4\right)\frac{\sigma^3}{L^2\left(1-\frac{T}{T_c}\right)^2} = \frac{16\pi\sigma^3}{3L^2\left(1-\frac{T}{T_c}\right)^2}$.

Summary: $S_E[\hat{\phi}] \to \infty$ for $T \to T_c^-$

$\Rightarrow$ nucleation can only happen after some <u>supercooling</u>.

Exercise: Considering an equation of motion with spherical symmetry,

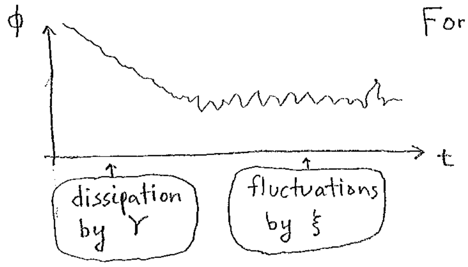
$$\frac{d^2\hat{\phi}}{dr^2} + \frac{2}{r}\frac{d\hat{\phi}}{dr} = V_{\text{eff}}'(\hat{\phi}) \; ; \quad \hat{\phi}'(0) = 0 \; ; \quad \hat{\phi}(\infty) = 0 ,$$

and a large radius ($T \to T_c^-$), show that in field theory,

$$\sigma \approx \int_0^{\bar{\phi}_{\text{broken}}} d\hat{\phi}\, \sqrt{2\left[V_{\text{eff}}(\hat{\phi}) - V_{\text{eff}}(\bar{\phi}_{\text{broken}})\right]} .$$

How to address nucleations and dynamics more systematically?

In principle: with "fluctuating hydrodynamics", having as degrees of freedom local temperature T, flow velocity u^μ, order parameter ϕ.



For ϕ, this amounts to a Langevin equation:

$$\partial^\mu\partial_\mu\phi + \Upsilon\, u^\mu\partial_\mu\phi + V'_{\rm eff}(\phi) = \xi \quad ; \quad \langle \xi(x)\xi(y)\rangle = \Omega\,\delta^{(4)}(x-y).$$

friction that transfers energy from ϕ to medium: this determines v_w (p.9)

random kicks that transfer energy from medium to ϕ; this leads to nucleations

Going to the medium rest frame, $u^\mu \to (1,\vec{0})$, and setting $V'_{\rm eff} \approx m^2\phi$, this can be solved with Green's functions:

$$\phi(x) = \int_Y G_R(X-Y)\,\xi(Y),$$

$$G_R(x) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{\vec{k}} e^{-i\omega t + i\vec{k}\cdot\vec{x}}\,\frac{1}{-\omega^2 + k^2 - i\Upsilon\omega + m^2}.$$

In order to be convinced that this makes sense, let us compute the equal-time correlator of ϕ:

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle &= \int_{Z,W} G_R(X-Z)\,G_R(Y-W)\,\langle\xi(Z)\xi(W)\rangle \\
&= \Omega\int_Z G_R(X-Z)\,G_R(Y-Z) \\
&= \Omega\int_{K,Q}\underbrace{\int_Z e^{-iK\cdot(X-Z)-iQ\cdot(Y-Z)}}_{\delta(K+Q)\,e^{-iK\cdot(X-Y)}}\,G_R(K)\,G_R(Q) \\
&\overset{x^0=y^0}{=} \Omega\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\frac{1}{-\omega^2 - i\Upsilon\omega + \epsilon_k^2}\cdot\frac{1}{-\omega^2 + i\Upsilon\omega + \epsilon_k^2}
\end{aligned}$$

The integral can be carried out with the residue theorem:

$$\omega^2 + i\Upsilon\omega - \epsilon_k^2 = \Big(\omega + \tfrac{i\Upsilon}{2} - \sqrt{\epsilon_k^2 - \Upsilon^2/4}\Big)\Big(\omega + \tfrac{i\Upsilon}{2} + \sqrt{\epsilon_k^2 - \Upsilon^2/4}\Big) \Rightarrow \text{poles in lower half-plane}$$

$$\omega^2 - i\Upsilon\omega - \epsilon_k^2 = \Big(\omega - \tfrac{i\Upsilon}{2} - \sqrt{\;\;}\Big)\Big(\omega - \tfrac{i\Upsilon}{2} + \sqrt{\;\;}\Big) \Rightarrow \text{poles in upper half-plane}$$

If we close the contour in the upper half-plane, this leads to

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle &\overset{x^0=y^0}{=} \Omega\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\frac{2\pi i}{2\pi}\Bigg\{\underbrace{\frac{1}{i\Upsilon(i\Upsilon+2\sqrt{\;\;})\,2\sqrt{\;\;}}}_{\text{pole at } \omega=\frac{i\Upsilon}{2}+\sqrt{\;\;}} + \underbrace{\frac{1}{(i\Upsilon-2\sqrt{\;\;})\,i\Upsilon\,(-2\sqrt{\;\;})}}_{\text{pole at } \omega=\frac{i\Upsilon}{2}-\sqrt{\;\;}}\Bigg\} \\
&= \frac{\Omega}{\Upsilon}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\Bigg\{\frac{\cancel{i\Upsilon}-2\sqrt{\;\;}}{\cancel{-\Upsilon^2}-4(\epsilon_k^2-\cancel{\frac{\Upsilon^2}{4}})} - \frac{\cancel{i\Upsilon}+2\sqrt{\;\;}}{\cancel{-\Upsilon^2}-4(\epsilon_k^2-\cancel{\frac{\Upsilon^2}{4}})}\Bigg\}\frac{1}{2\sqrt{\;\;}} \\
&= \frac{\Omega}{2\Upsilon}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\cdot\frac{1}{\epsilon_k^2}.
\end{aligned}$$

This agrees with the $\omega_n = 0$ contribution on p.5 if Ω and Υ satisfy the fluctuation-dissipation relation $\Omega = 2\Upsilon T$. So Langevin generalizes DR!