2. Modern methods and applications

Recap of lecture 1

$$V_{eff}^{(0)} + V_{eff}^{(1)} = -\frac{1}{2} m^2 \bar{\phi}^2 + \frac{1}{4}\lambda \bar{\phi}^4 + \int_{\vec{k}} \left[\frac{\varepsilon_k}{2} + T \ln\left(1 - e^{-\beta \varepsilon_k}\right)\right]_{\varepsilon_k = \sqrt{k^2 + m_{eff}^2}} .$$

Problems of this formula:

* recalling $m_{eff}^2 = -m^2 + 3\lambda \bar{\phi}^2$, ε_k can be complex for $\bar{\phi}^2 < \frac{m^2}{3\lambda}$, and $V_{eff}^{(1)}$ is ill-defined, at any temperature.
* as discussed on p. 4, even positive values of m_{eff}^2 are problematic, because $V_{eff}^{(0)} \sim V_{eff}^{(1)} \sim V_{eff}^{(n)}$ for $m_{eff} \lesssim \frac{\lambda T}{\pi}$.

Thermal field theory as an imaginary-time path integral

On p. 3 we already showed the recipe for an imaginary-time path integral, and let us now give some justifications.

Starting point: Feynman's path integral, generalized to field theory.

In these lectures we use the metric $\eta^{\mu\nu} = (+---)$.

$$\langle \phi_b | e^{-\frac{i}{\hbar}\hat{H}(t_b - t_a)} | \phi_a \rangle = \int_{\phi(t_a)=\phi_a}^{\phi(t_b)=\phi_b} \mathcal{D}\phi \; \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \int_V d^3\vec{x} \left[\frac{1}{2}\partial^\mu\phi \partial_\mu \phi - V(\phi)\right]\right\} .$$

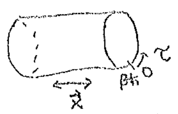
What interests us:

$$Z_\phi = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right) = \int_{-\infty}^{\infty} d\phi_a \, \langle \phi_a | e^{-\beta\hat{H}} | \phi_a \rangle .$$

The two can be related by a <u>Wick rotation</u>: $t_a \to 0$, $it_b \to \beta\hbar$, $it \to \tau$

$$\Rightarrow Z_\phi = \int_{\phi(\beta\hbar)=\phi(0)} \mathcal{D}\phi \; \exp\left\{ \frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int_V d^3\vec{x} \underbrace{\left[-\frac{1}{2}\partial_\tau\phi \partial_\tau\phi - \frac{1}{2}\partial_i\phi\partial_i\phi - V(\phi)\right]}_{\equiv -L_E} \right\}$$

Periodicity implies that fields live in a <u>compact time direction</u>:



In the <u>classical limit</u>: $\frac{1}{\hbar}\int_0^{\beta\hbar} \xrightarrow{\hbar \to 0} \beta$;

$$\exp\left\{ -\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int_V d^3\vec{x}\, L_E \right\} \longrightarrow \exp\left\{ -\beta \int_V d^3\vec{x}\, L_E\right\} .$$

„Boltzmann weight".

<u>Momentum space</u> (now with $\hbar \to 1$):

$$\phi(\tau, \vec{x}) = T \sum_{\omega_n} \int_{\vec{k}} \phi(K)\, e^{iK\cdot X} , \quad K\cdot X \equiv \omega_n \tau - \vec{k}\cdot\vec{x}$$

$$\Rightarrow \int_X \frac{1}{2}\left(\partial_\tau\phi\partial_\tau\phi + \partial_i\phi\partial_i\phi + m^2\phi^2\right) = \sum\!\!\!\!\!\!\int_K \frac{1}{2}\phi(-K)\left(\omega_n^2 + k^2 + m^2\right)\phi(K) .$$

<u>Propagator</u>:

$$\underbrace{\phi(K)\phi(Q)} = \frac{\delta(K+Q)}{\omega_n^2 + \varepsilon_k^2} , \quad \delta(K) = \int_X e^{iK\cdot X} = \beta\,\delta_{\omega_n,0}(2\pi)^d \delta^{(d)}(\vec{k}) ,$$

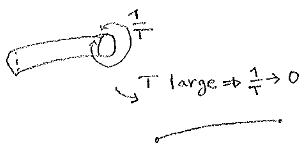
where d is space dimension.

Origin of infrared (IR) problem

The problems listed on p.5 are related to small values of $m_{\text{eff}}^2 \lesssim \left(\frac{\lambda T}{\pi}\right)^2$, and are therefore known as an IR problem.

For small λ, $\left(\frac{\lambda T}{\pi}\right)^2 \ll (2\pi T)^2$. In momentum space, this implies that $m_{\text{eff}}^2 \ll \omega_n^2$, whenever $n \neq 0$. Therefore IR problems (sensitivity to m_{eff}) can only concern the zero-mode $\omega_n = 0$; otherwise IR physics is "shielded" by $\omega_n \neq 0$.

Effective field theory approach



The general idea is to "turn the tables" and consider the "problematic" Matsubara zero modes as the key degree of freedom. Since zero modes do not depend on τ, the fields live in $d=3$ dimensions, and the procedure is called dimensional reduction (DR). The unproblematic modes $\omega_n \neq 0$ are "integrated out".

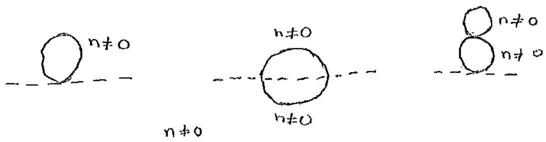
Key concepts:

* scale hierachy : $m_{\text{eff}} \ll 2\pi T = \min\{|\omega_n|, n \neq 0\}$.
* factorization: treat different scales with different methods.
* IR side: write down most general L_E for zero modes.
* UV side: determine parameters of L_E by "matching", so that the contributions from $\omega_n \neq 0$ are also included.

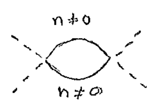
Implementation:

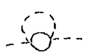
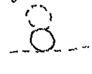
$$S_{\text{eff}} = \underbrace{\frac{1}{T}}_{\text{from } \int_0^\beta d\tau} \int_V d^3\vec{x} \left\{ \underbrace{\frac{1}{2}(\partial_i \phi_3)^2}_{\substack{\text{only spatial} \\ \text{derivatives,} \\ \text{for } \partial_\tau \phi_3 = 0}} + \frac{1}{2} \underbrace{m_3^2}_{m_3^2 \approx -m^2 + \frac{\lambda T^2}{4}} \phi_3^2 + \frac{1}{4}\lambda_3 \phi_3^4 + \underbrace{\ldots}_{\substack{\text{truncated} \\ \text{higher-dimensional} \\ \text{operators}}} \right\}.$$

* for normalization of ϕ_3 and for mass parameter m_3^2:

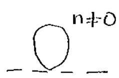


* for λ_3:



* tricky: is not  =  already accounted for by λ_3? Not completely, as there is a left-over yielding δm_3^2!
Mixed diagrams require always care, also  and .

Example: 1-loop contribution to m_3^2



$$\overbrace{\phi\phi \quad \frac{\lambda}{4}\phi}^{3}\underbrace{\phi\phi\phi}_{4} \longrightarrow \delta m_3^2 = 3\lambda \sum\!\!\!\!\!\!\int_{K,\,\omega_n\neq 0} \frac{1}{\omega_n^2+k^2+\underbrace{m^2}_{\text{can be omitted if } \ll (2\pi T)^2}}$$

Sums (p. 2): $T\sum_{\omega_n\neq 0}\frac{1}{\omega_n^2+k^2} = \frac{1}{k}\left[\frac{1}{2}+\frac{1}{e^{k/T}-1}\right]-\frac{T}{k^2}$.

Integral: $\int\frac{d^d\vec{k}}{(2\pi)^d}\,\frac{1}{k}\left[\frac{1}{2}+\frac{1}{e^{k/T}-1}-\frac{T}{k}\right]$

$$= \sum_{n=1}^{\infty}\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dk\,k^{d-2}\Big[e^{-\frac{kn}{T}} + \underbrace{\frac{1}{2}-\frac{T}{k}}_{\substack{\text{no scale}\,\to\,\text{vanish}\\ \text{in dimensional}\\ \text{regularization!}}}\Big]$$

$$\frac{d^d\vec{k}}{(2\pi)^d} = \frac{2\,dk\,k^{d-1}}{(4\pi)^{d/2}\Gamma(d/2)}$$

$$\frac{1}{e^{k/T}-1} = \frac{e^{-k/T}}{1-e^{-k/T}} = \sum_{n=1}^{\infty}e^{-\frac{kn}{T}}$$

$$= \sum_{n=1}^{\infty}\frac{2T^{d-1}}{(4\pi)^{d/2}\Gamma(d/2)\,n^{d-1}}\underbrace{\int_0^\infty dx\,x^{d-2}e^{-x}}_{\Gamma(d-1)}$$

$$= \frac{2T^{d-1}\Gamma(d-1)\zeta(d-1)}{(4\pi)^{d/2}\Gamma(d/2)}$$

$$\overset{d=3}{=} \frac{2T^2\zeta(2)}{4\pi\sqrt{4\pi}\,\Gamma(3/2)} = \frac{2T^2\frac{\pi^2}{6}}{4\pi\,2\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}} = \frac{T^2}{12} .$$

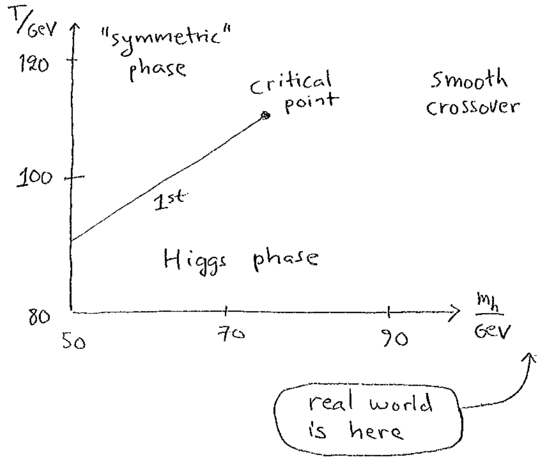
So $\delta m_3^2 = \frac{\lambda T^2}{4}$, as we saw on p. 4.

Extension of the effective theory to the Standard Model*

*Here the temporal gauge fields A_0^a, B_0 have been omitted; we return to the justification on p. 18.

$$L_{eff} = \underbrace{\frac{1}{4}G^a_{ij}G^a_{ij}}_{SU(2)} + \underbrace{\frac{1}{4}F_{ij}F_{ij}}_{U_Y(1)} + (D_i\phi_3)^\dagger(D_i\phi_3) + \underbrace{m_3^2}_{-\frac{m_h^2}{2}+\mathcal{O}(T^2)}\phi_3^\dagger\phi_3 + \underbrace{\lambda_3}_{\frac{g_w^2}{8}\frac{m_h^2}{m_w^2}+\dots}(\phi_3^\dagger\phi_3)^2 + \dots$$

Upshot:
- UV side: parameters determined up to 2-loop level
- IR side: L_{eff} studied with lattice simulations
- phase diagram:



Example of a strong first-order phase transition

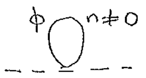
Consider a theory with two real scalar fields:

$$\mathcal{L}_M = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\partial^\mu\chi\partial_\mu\chi - V(\phi,\chi)\,; \quad V(\phi,\chi) = -\frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4} - \frac{M^2\chi^2}{2} + \frac{\kappa\chi^4}{4} + \frac{\gamma\phi^2\chi^2}{2}\,;$$

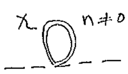
$$L_E = -\mathcal{L}_M(t\to -i\tau)\,;$$

$$L_{eff} = \frac{1}{2}\partial_i\phi_3\partial_i\phi_3 + \frac{1}{2}\partial_i\chi_3\partial_i\chi_3 + V_3(\phi_3,\chi_3)\,;\; V_3(\phi_3,\chi_3) = \frac{m_3^2\phi_3^2}{2} + \frac{\lambda_3\phi_3^4}{4} + \frac{M_3^2\chi_3^2}{2} + \frac{\kappa_3\chi_3^4}{4} + \frac{\gamma_3\phi_3^2\chi_3^2}{2}.$$

Thermal masses can be computed like on p. 7:



$$\phi\phi \quad \frac{\lambda}{4}\phi\phi\phi\phi \quad \Rightarrow \quad m_3^2 \supset 3\lambda\cdot\frac{T^2}{12}$$



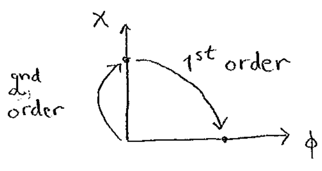
$$\phi\phi \quad \frac{\gamma}{2}\phi\phi\chi\chi \quad \Rightarrow \quad m_3^2 \supset \gamma\,\frac{T^2}{12}$$

And similarly for χ $\Rightarrow$
$$\begin{cases} m_3^2 \approx -m^2 + \dfrac{(3\lambda+\gamma)T^2}{12}, \\ M_3^2 \approx -M^2 + \dfrac{(3\kappa+\gamma)T^2}{12}. \end{cases}$$

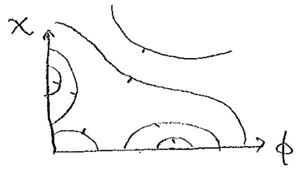
Therefore symmetries tend to get restored at
$$\begin{cases} T_\phi^2 \approx \dfrac{12\,m^2}{3\lambda+\gamma}, \\ T_\chi^2 \approx \dfrac{12\,M^2}{3\kappa+\gamma}. \end{cases}$$

Idea: two-step transition

* Suppose that symmetry first gets broken in χ-direction ($T_\chi > T_\phi$)
* however at low T we return to ϕ_{min} ($V_{eff}(\phi_{min},0) < V_{eff}(0,\chi_{min})$)
* then there could be a strong "tree-level" transition, proceeding through a saddle point:

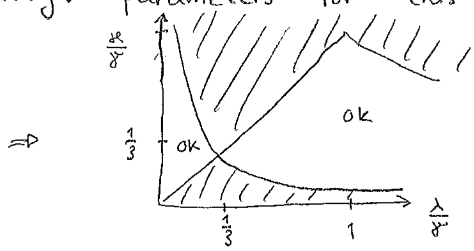


height contours:



Are there enough parameters for this? Exercise!

$\Rightarrow$



Summary: first order transitions are possible in extensions of the Standard Model.