Phase transitions in the early universe

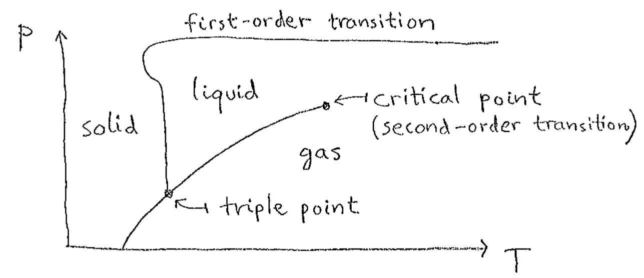
(M. Laine // GGI 2022)

1. Basics of phase transitions in field theory
2. Modern methods and applications
3. Nucleations and dynamics
4. Gravitational wave production
5. Baryogenesis and magnetic fields

1. Basics of phase transitions in field theory

What is a phase transition?

Phase diagram of water:



To get going, let us compute the partition function for a harmonic oscillator.

$$Z = \mathrm{Tr}\,(e^{-\beta \hat{H}}) \,, \quad \beta \equiv \frac{1}{T} \,, \quad \hat{H} = \text{Hamiltonian}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} = \underbrace{\hbar\omega}_{\equiv\,\varepsilon} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\Rightarrow \quad Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\varepsilon (n + \frac{1}{2})} = \frac{e^{-\beta\varepsilon/2}}{1 - e^{-\beta\varepsilon}} \,.$$

From the partition function we get the (canonical) free energy:

$$Z = e^{-\beta F} \quad \Leftrightarrow \quad F = -T \ln Z = \frac{\varepsilon}{2} + T \ln (1 - e^{-\beta\varepsilon}) \,.$$

We also need a "propagator" ($\hat{x} \leftrightarrow \hat{\phi}$):

$$\langle \hat{x}^2 \rangle = \frac{\mathrm{Tr}\,(\hat{x}^2 e^{-\beta\hat{H}})}{\mathrm{Tr}\,(e^{-\beta\hat{H}})} = (-T) \cdot \frac{2}{m} \frac{\partial}{\partial \omega^2} \ln \mathrm{Tr}\,(e^{-\beta \hat{H}})$$

$$= \frac{2\hbar^2}{m} \frac{\partial}{\partial \varepsilon^2} F = \frac{\hbar^2}{m} \frac{1}{\varepsilon} \frac{\partial F}{\partial \varepsilon}$$

$$= \frac{\hbar^2}{m} \frac{1}{\varepsilon} \left[\frac{1}{2} + T \, \frac{\beta e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}} \right]$$

$$= \frac{\hbar^2}{m} \cdot \frac{1}{\varepsilon} \cdot \Big[\underbrace{\frac{1}{2}}_{\substack{\text{"Vacuum contribution"} \\ \text{(independent of T)}}} + \underbrace{\frac{1}{e^{\beta\varepsilon} - 1}}_{\substack{\equiv\, n_B(\varepsilon) \\ \stackrel{!}{=}\ \text{Bose distribution}}} \Big] \,.$$

Scalar field theory is a collection of oscillators: $\varepsilon \to \varepsilon_k \equiv \sqrt{k^2+m^2}$.
Moreover that system is extensive, so it is useful to consider the free energy density rather than the free energy, $f \equiv F/V$.

$$Z_\phi = \prod_{\vec{k}} \exp\left\{-\frac{1}{T}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]\right\}$$

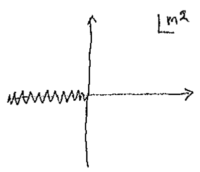
$$f_\phi = -\frac{T\ln Z_\phi}{V} = \frac{1}{V}\sum_{\vec{k}}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right] \underset{V\to\infty}{=} \int_{\vec{k}}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right].$$

The vacuum part $\int_{\vec{k}} \varepsilon_k/2$ looks divergent and requires renormalization.
The thermal part $f_T(m) \equiv T\int_{\vec{k}} \ln\left(1-e^{-\beta\varepsilon_k}\right)$ is exponentially convergent.
Let us look at it more precisely:

* low-T limit, $T \ll m$: $\varepsilon_k \geq m$, $\beta\varepsilon_k \geq \frac{m}{T} \gg 1 \Rightarrow e^{-\beta\varepsilon_k} \ll 1$

$$\Rightarrow f_T(m) \approx T\int_{\vec{k}}\left(-e^{-\beta\varepsilon_k}\right) \approx -\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}.$$

* high-T limit, $T \gg m$: try a Taylor expansion in m^2?
Actually, this is very non-trivial: there is a branch cut, but a generalized expansion exists:

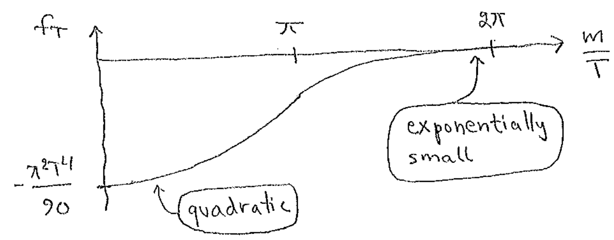


Some details on how to compute this kind of expansions are given on p.7.

$$\Rightarrow f_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} + O(m^4)$$

$m := \sqrt{m^2}$

* numerically:



We will need these results soon.

Towards Matsubara / imaginary-time formalism

The following representation can be proven* for the "propagator":

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\varepsilon_k/T}-1}\right] \overset{!}{=} T\sum_{\omega_n} \frac{1}{\omega_n^2+\varepsilon_k^2}, \qquad \omega_n \equiv 2\pi T n, \quad n \in \mathbb{Z}.$$

The ω_n are called Matsubara frequencies and $\omega_n = 0$ is "Matsubara zero mode".
Particularly interesting is the behaviour for $\varepsilon_k \ll \pi T$.

LHS: $\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{\varepsilon_k/T + \varepsilon_k^2/2T^2 + \ldots}\right] = \frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{T}{\varepsilon_k}\left(1 - \frac{\varepsilon_k}{2T} + \ldots\right)\right] = \frac{T}{\varepsilon_k^2} + O\left(\frac{1}{T}\right).$

RHS: from $\omega_n = 0$: $\frac{T}{\varepsilon_k^2}$ $\Rightarrow$ much simpler to extract the limit!

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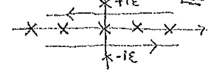
* Exercise: verify the above sum by considering the contour integral

$$\int_{\rightleftarrows} \frac{d\omega}{2\pi i}\, \frac{1}{\omega^2+\varepsilon^2}\, i\, n_B(i\omega)$$



and closing the contour in two possible ways.

Recipe of imaginary-time formalism (more on proof on p.5)

The Matsubara formalism corresponds to an "imaginary-time"/"Euclidean" path integral:

$$\mathcal{Z}_\phi = \int_{\text{b.c.}} \mathcal{D}\phi \, e^{-S_E},$$
$$S_E \equiv \int_0^\beta d\tau \int_V d^3\vec{x}\, L_E,$$
$$L_E \equiv -\mathcal{L}_M(t \to -i\tau).$$

Here boundary conditions (b.c.) are periodic for bosons (antiperiodic for fermions).
Check: $\phi(\beta,\vec{x}) = \phi(0,\vec{x}) \Rightarrow e^{i\omega_n\beta} = 1 \Rightarrow \omega_n = 2\pi T n$ OK!
~~Apply~~ this to a real scalar field with $V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$.

$$\mathcal{L}_M = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$
$$\Rightarrow L_E = \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi).$$

Effective potential

We return to a finite volume for a moment. Let $\bar{\phi}$ be the mode with $\omega_n=0, \vec{k}=\vec{0}$, and ϕ' the modes with $K=(\omega_n,\vec{k}) \neq 0$. Note that $\int_0^\beta d\tau \int_V d^3\vec{x}\, \phi' \overset{!}{=} 0$.

$$\Rightarrow \mathcal{Z}_\phi = \int_{-\infty}^{\infty} d\bar{\phi} \int \mathcal{D}\phi' \, e^{-S_E[\phi=\bar{\phi}+\phi']}$$
$$\equiv \int_{-\infty}^{\infty} d\bar{\phi}\, e^{-\frac{V}{T}V_{\text{eff}}(\bar{\phi})} \approx \int_{-\infty}^{\infty} d\bar{\phi}\, e^{-\frac{V}{T}\left[V_{\text{eff}}(\bar{\phi}_{\min}) + \frac{1}{2}V''_{\text{eff}}(\bar{\phi}_{\min})(\bar{\phi}-\bar{\phi}_{\min})^2 + \ldots\right]}$$
$$\approx e^{-\frac{V}{T}V_{\text{eff}}(\bar{\phi}_{\min})}\sqrt{\frac{c}{V}}$$
$$\Rightarrow f_\phi = V_{\text{eff}}(\bar{\phi}_{\min}) + \underbrace{\mathcal{O}\left(\frac{\ln V}{V}\right)}_{\boxed{\text{vanishes in thermodynamic limit}}}.$$

Insert $\phi = \bar{\phi} + \phi'$ in L_E:

$$\frac{1}{2}(\partial_\mu\phi)^2 \to \frac{1}{2}(\partial_\mu\phi')^2$$
$$-\frac{1}{2}m^2\phi^2 \to -\frac{1}{2}m^2\bar{\phi}^2 - m^2\bar{\phi}\phi' - \frac{1}{2}m^2\phi'^2$$
$$\frac{1}{4}\lambda\phi^4 \to \underbrace{\frac{1}{4}\lambda\bar{\phi}^4}_{\boxed{\text{independent of } \tau,\vec{x}}} + \underbrace{\lambda\bar{\phi}^3\phi'}_{\boxed{\text{vanishes because } \int d\tau\int d^3\vec{x}\,\phi'=0}} + \underbrace{\frac{3}{2}\lambda\bar{\phi}^2\phi'^2}_{\boxed{\text{like free theory but with } m^2_{\text{eff}} \equiv -m^2+3\lambda\bar{\phi}^2}} + \underbrace{\lambda\bar{\phi}\phi'^3 + \frac{1}{4}\lambda\phi'^4}_{\boxed{\text{interactions}}}$$

So, in summary:

- $*$ tree-level potential: $V^{(0)}_{\text{eff}}(\bar{\phi}) = -\frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4$
- $*$ 1-loop potential: $V^{(1)}_{\text{eff}}(\bar{\phi}) = f_\phi(m_{\text{eff}}) = \int_{\vec{k}}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]_{\varepsilon_k=\sqrt{k^2+m^2_{\text{eff}}}}$
- $*$ higher order corrections, with powers of λ.

What can be said about the properties of the transition?

Let us insert the high-T expansion from p. 2 and see what kind of an effect $V_{\text{eff}}^{(1)}$ has:

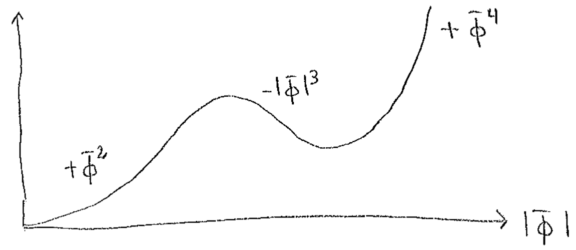
* leading term $-\frac{\pi^2 T^4}{90}$ $\Rightarrow$ independent of $\bar\phi$ $\Rightarrow$ no effect.
* NLO term $\frac{m_{\text{eff}}^2 T^2}{24} = \frac{-m^2 + 3\lambda\bar\phi^2}{24} T^2$ $\Rightarrow$ thermal mass correction!

$$V_{\text{eff}}^{(0)} + V_{\text{eff}}^{(1)} \approx \left[\bar\phi\text{-indep.}\right] + \frac{1}{2}\left(-m^2 + \frac{\lambda T^2}{4}\right)\bar\phi^2 + \frac{1}{4}\lambda\bar\phi^4$$

Thus there is a phase transition: symmetry gets restored at $T_c \approx \frac{2m}{\sqrt{\lambda}}$.

* NNLO term $-\frac{m_{\text{eff}}^3 T}{12\pi}$; let us for simplicity set $m^2 = 0$

$$\Rightarrow \quad V_{\text{eff}}^{(0)} + V_{\text{eff}}^{(1)} \approx \left[\bar\phi\text{-indep.}\right] + \frac{\lambda}{8} T^2 \bar\phi^2 - \frac{T}{12\pi}(3\lambda)^{3/2}|\bar\phi|^3 + \frac{1}{4}\lambda\bar\phi^4$$



This looks like a discontinuous, i.e. first-order transition.

How reliable are such predictions?

Whenever we do something perturbatively, we should estimate afterwards whether there is a small expansion parameter, i.e. whether the series converges (asymptotically).

Since the change from a continuous to a discontinuous transition is due to a 1-loop effect, it is questionable whether there is convergence in the present case.

Schematically:

$$\text{expansion parameter} \sim \frac{\overbrace{T\lambda^{3/2}|\bar\phi|^3}^{\text{from } V_{\text{eff}}^{(1)}}}{\underbrace{\pi\lambda\bar\phi^4}_{\text{from } V_{\text{eff}}^{(0)}}} \sim \frac{T\lambda}{\pi\lambda^{1/2}|\bar\phi|} \sim \frac{T\lambda}{\pi m_{\text{eff}}} \; ;$$

no convergence if $m_{\text{eff}} \lesssim \frac{\lambda T}{\pi}$!

Physically, these effects originate from the Bose-enhanced $\frac{T}{\epsilon_k^2}$ that we saw on p. 2: there are very many quanta at small energies, which combine to give a large contribution.

Upshot (obtained with techniques discussed in 2nd lecture):

in O(N) scalar field theory, the transition is of 2nd, not 1st order.