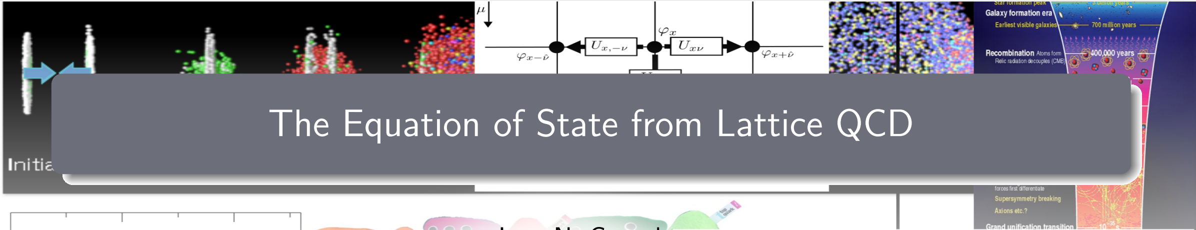
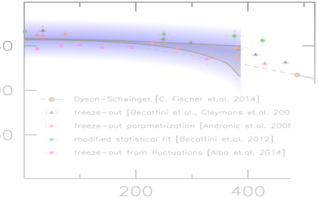
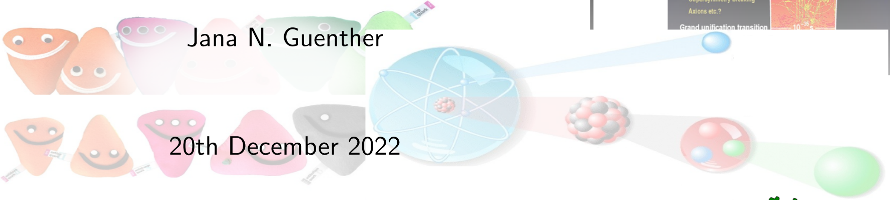


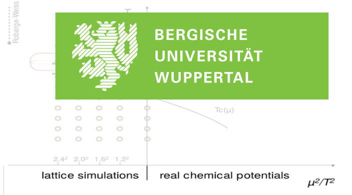
The Equation of State from Lattice QCD

Jana N. Guenther

20th December 2022



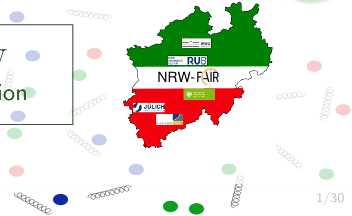




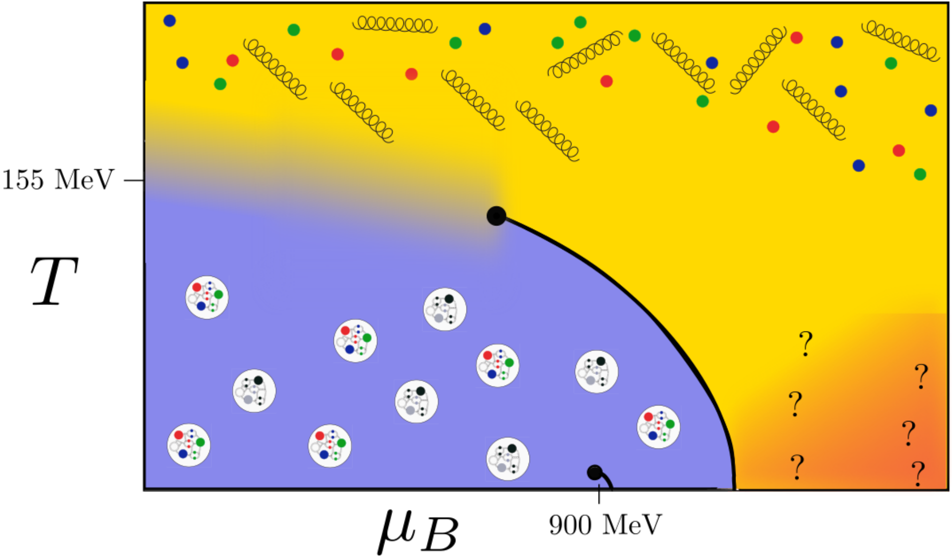
BERGISCHE UNIVERSITÄT WUPPERTAL

WB collaboration

BMW collaboration



The (T, μ_B)-phase diagram of QCD



1. Lattice QCD

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   - Rescaling and expansion - the analysis in [Borsanyi:2022qlh]
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The QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \overline{\psi} \left(\mathrm{i}\gamma_\mu D^\mu - m \right) \psi$$

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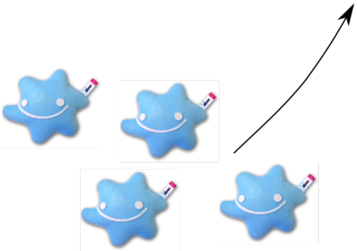

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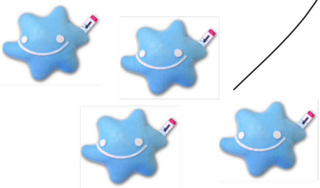

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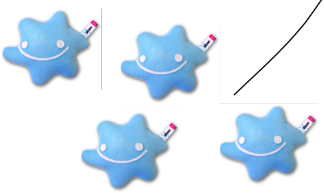


Lattice Simulation and Statistical mechanics

$$\int \mathcal{D}\phi(x) e^{\mathrm{i}\int \mathrm{d}^4 x \mathcal{L}}$$

Lattice Simulation and Statistical mechanics

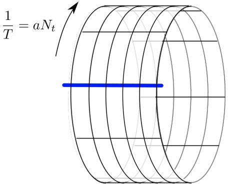
Zero-Temperature-LQCD:

$$\int \mathcal{D}\phi(x)e^{\mathrm{i}\int \mathrm{d}^4x\mathcal{L}} \xrightarrow{t\to \mathrm{i}\tau} \int \mathcal{D}\phi(x)e^{-S}$$

Infinite space time volume of a QFT in Euclidean space time

Lattice Simulation and Statistical mechanics

Zero-Temperature-LQCD:



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Lattice Simulation and Statistical mechanics

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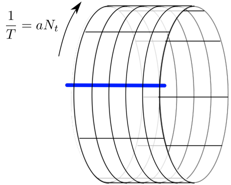
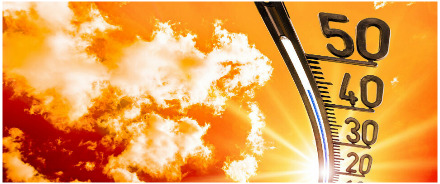


Finite-Temperature-LQCD:

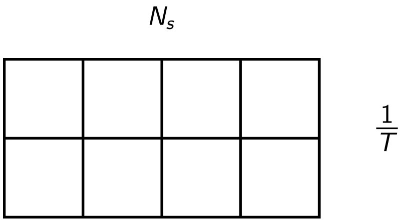


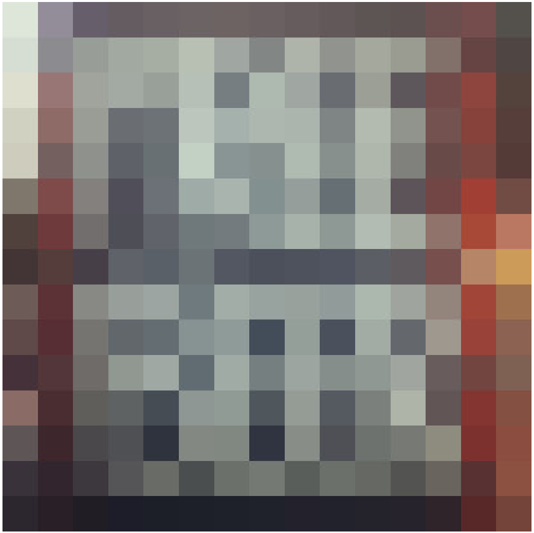
$$\int \mathcal{D}\phi(x) e^{\mathrm{i}\int \mathrm{d}^4 x \mathcal{L}} \xrightarrow{t \to \mathrm{i}\tau} \int \mathcal{D}\phi(x) e^{-S} \xrightarrow{\text{periodic boundary in a finite time}} \int \mathcal{D}\phi(x) e^{-\int_0^\beta \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L}} = Z$$

Infinite space time volume of a QFT in Euclidean space time

Partition function of a grand canonical ensemble at finite temperature

The continuum limit





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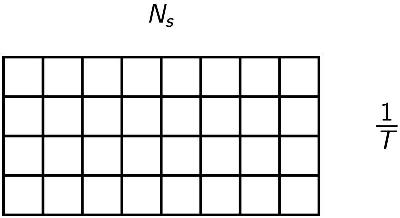

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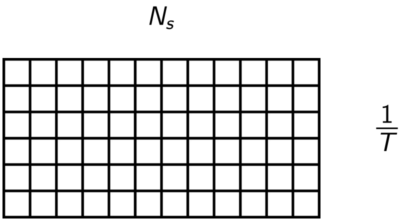

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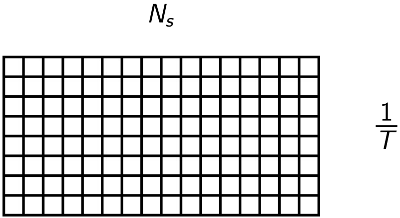
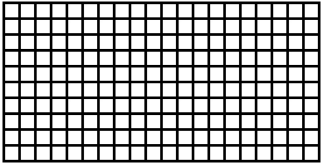


The continuum limit

N_s



$\frac{1}{T}$

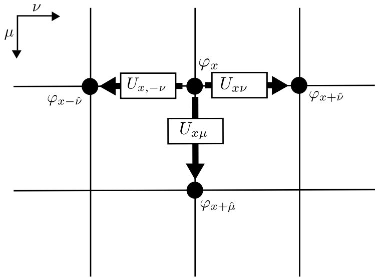
The continuum limit

N_s

$\frac{1}{T}$

How do we do that?

- We look at a 4d space-time lattice with size $N_s^3 \times N_t$
- Fermions live on the lattice sites gauge fields live on the links
- We use Euclidean space-time: $t \longrightarrow i\tau$

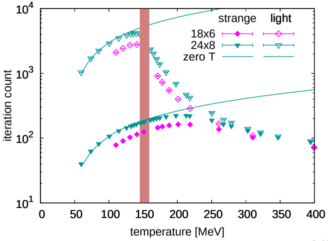


- We can do Monte-Carlo-Simulations (with importance sampling) to solve our integrals
- Everything is determined in terms of our lattice spacing a
- a has to be determined by comparison with physical observables (for example $a = \frac{(am_p)_{\text{lattice}}}{938 \text{ MeV}}$)
- We have to take the limits $a \longrightarrow 0$, $N_s \longrightarrow \infty$
- To do thermodynamics: $T = \frac{1}{aN_t}$

Why aren't we finished yet?

- Simulations take a lot of computer time
- Not everything can be calculated directly. For example:
  - Only thermal equilibrium
  - Only simulations at $\mu_B = 0 \Rightarrow \langle n_B \rangle = 0$ ↯ heavy ion collision experiments

1000 configurations on a $64^3 \times 16$ lattice cost about 1 million core hours



The sign problem

The QCD partition function:

$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-S_F(U,\psi,\bar{\psi})-\beta S_G(U)}$$
$$= \int \mathcal{D}U\ \det M(U) e^{-\beta S_G(U)}$$

- For Monte Carlo simulations $\det M(U) e^{-\beta S_G(U)}$ is interpreted as Boltzmann weight
- If there is particle-antiparticle-symmetry $\det M(U)$ is real
- If $\mu^2 > 0$ $\det M(U)$ is complex

Dealing with the sign problem

- (Sign) Reweighting techniques
- Canonical ensemble
- Lefshetz Thimble
- Density of state methods
- Dual variables
- Complex Langevin
- ...

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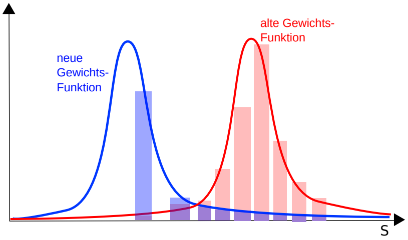

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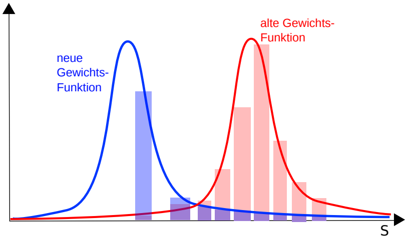
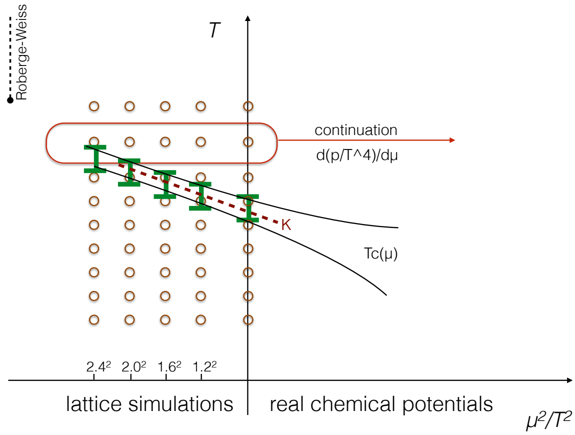


- (Taylor) expansion
- Imaginary μ

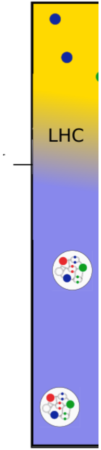
Analytic continuation from imaginary chemical potential



Common technique:

- [deForcrand:2002hgr]
- [Bonati:2015bha]
- [Cea:2015cya]
- [DElia:2016jqh]
- [Bonati:2018nut]
- [Borsanyi:2018grb]
- [Borsanyi:2020fev]
- [Bellwied:2021nrt]
- . . .

Expansion from $\mu = 0$



Taylor expansion

$$\frac{p}{T^4} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{j!k!}\chi_{jk}^{BS}\hat{\mu}_B^j\hat{\mu}_S^k$$

with $\hat{\mu} = \frac{\mu}{T}$

- rapid convergence in Stephan-Boltzmann ($T = \infty$) limit
- expansion coefficients are lattice observables

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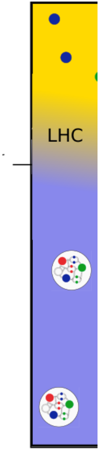

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Fugacity expansion/sector method

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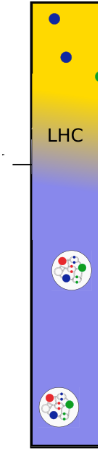

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- often the expansion is done for a specific choice of μ_S

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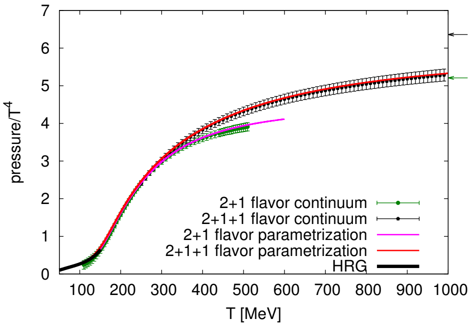
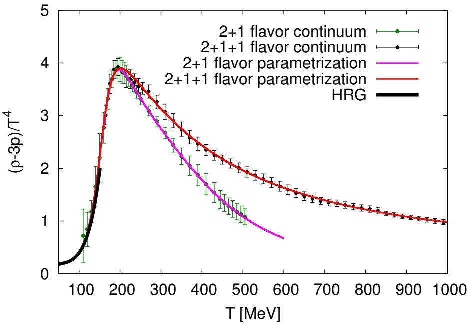
$\mu_B = 0$ and high T: Influence of the charm quark

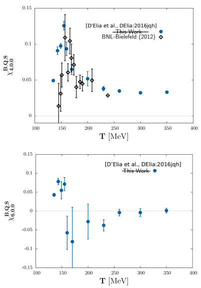




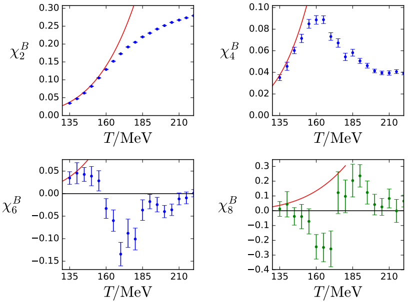
[Borsanyi:2016ksw]

χ^B

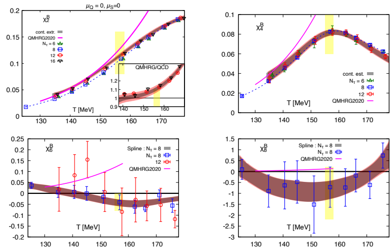
$$\chi_i^B = \frac{\partial^i (p/T^4)}{\partial \hat{\mu}_B^i}, \quad \frac{p}{T^4} = \chi_0^B + \frac{1}{2!}\chi_2^B \hat{\mu}_B^2 + \frac{1}{4!}\chi_4^B \hat{\mu}_B^4 + \frac{1}{6!}\chi_6^B \hat{\mu}_B^6 + \frac{1}{8!}\chi_8^B \hat{\mu}_B^8 + \dots, \quad \hat{\mu} = \frac{\mu}{T}$$



[DElia:2016jqh]

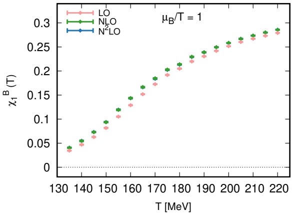


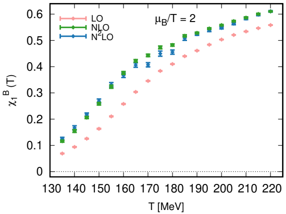
[Borsanyi:2018grb]

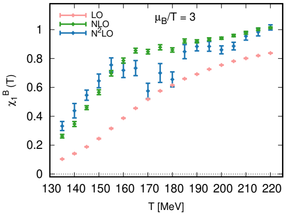


[Bollweg:2022rps]

Trouble with the equation of state







[Borsanyi:2021sxv], [Borsanyi:2018grb], $N_t = 12$

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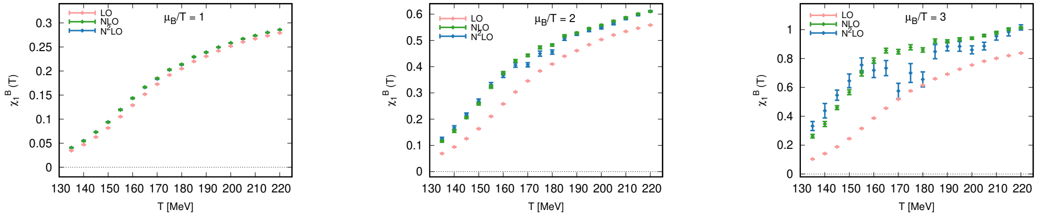

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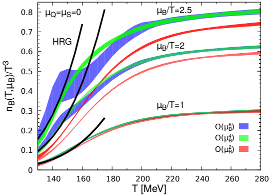

Taylor method

[Bazavov:2017dus]

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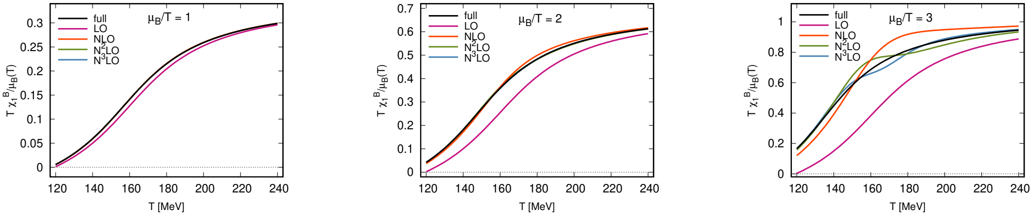

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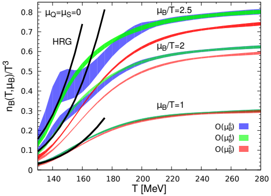

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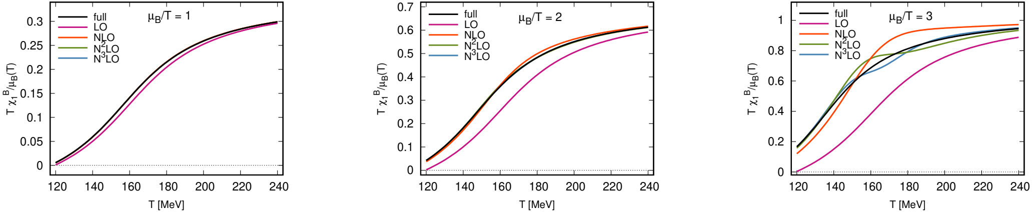
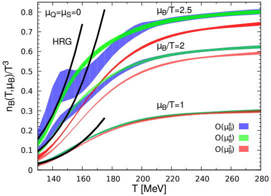


[Borsanyi:2021sxv]



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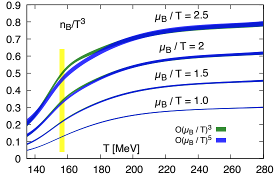

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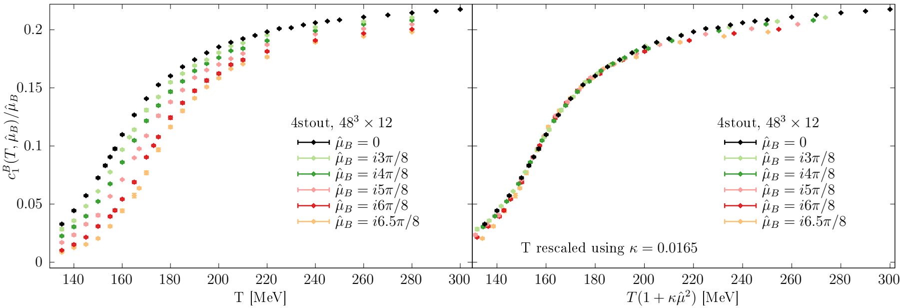
c_1^B

$$c_1^B(T,\hat{\mu}_B) = \chi_1^B - \frac{\chi_{11}^{BS}}{\chi_2^S}\chi_1^S = \chi_1^B$$



This rescaling will break down at large $T \longrightarrow$ rescaling with SBL

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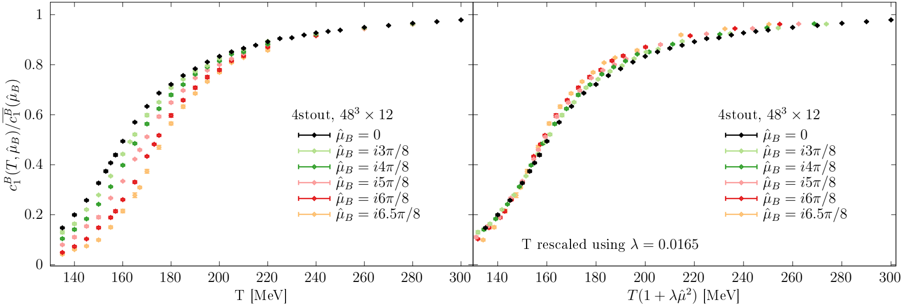

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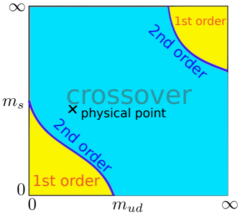
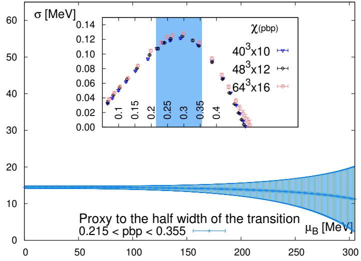
Why does the rescaling work?

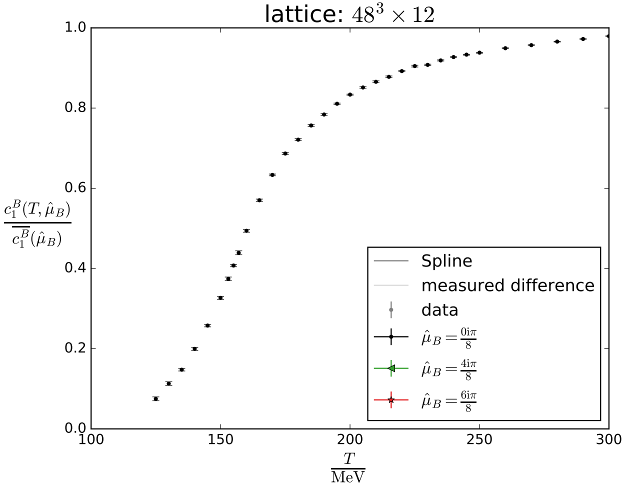
- It is an observation that it works
- It could be related to the critical scaling in the chiral limit
- If the universal contribution to EoS is large $\rightarrow$ single scaling variable
- If strength of transition is strongly Influenced by light quark masses $\rightarrow$ curves keep there shape
- Fits with the observation of constant width of the transition





[Borsanyi:2020fev]

Measuring the shift

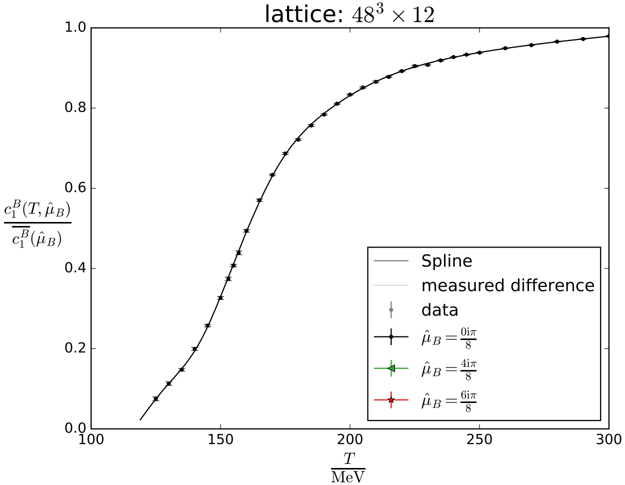


c_1^B: net baryon density

$\overline{c_1^B}$: SBL of net baryon density

$$\Pi(T, \hat{\mu}_B, N_\tau) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B}$$

Measuring the shift



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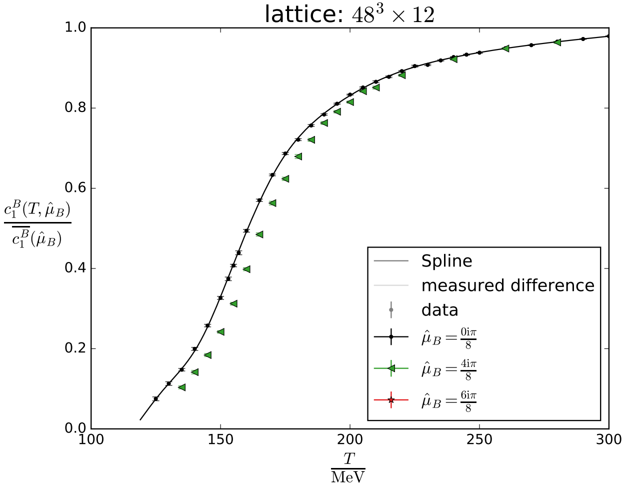

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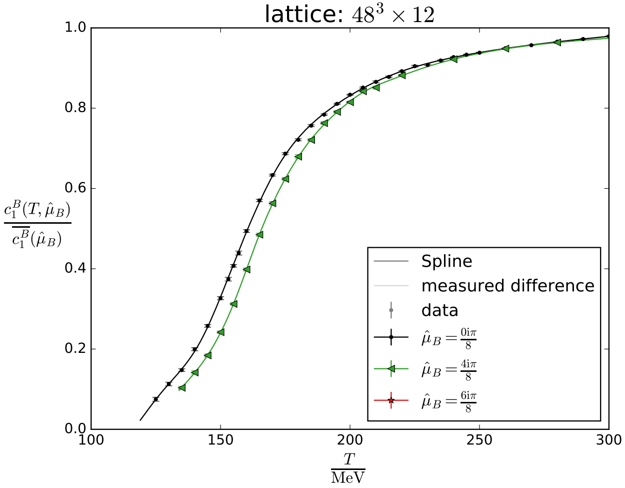
$$\Pi(T, \hat\mu_B, N_\tau) = \frac{T'(T, \hat\mu_B, N) - T}{T\hat\mu_B}$$

Measuring the shift



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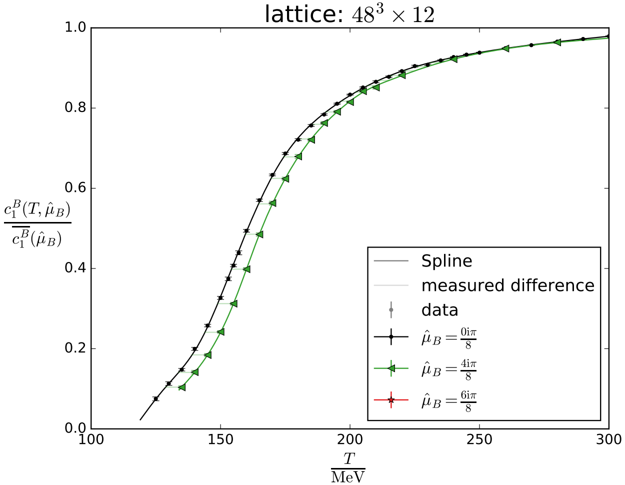

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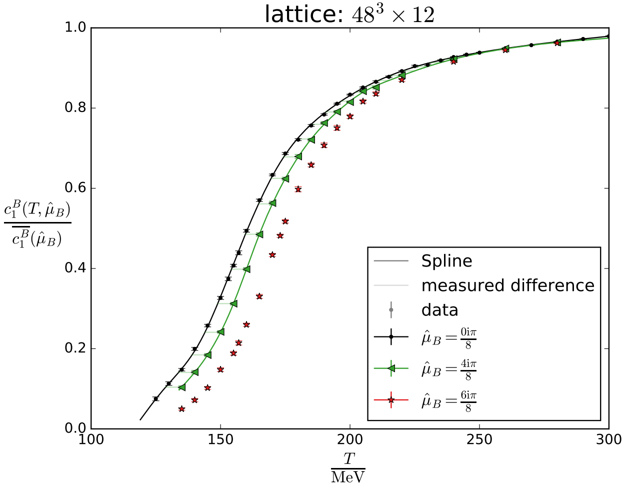

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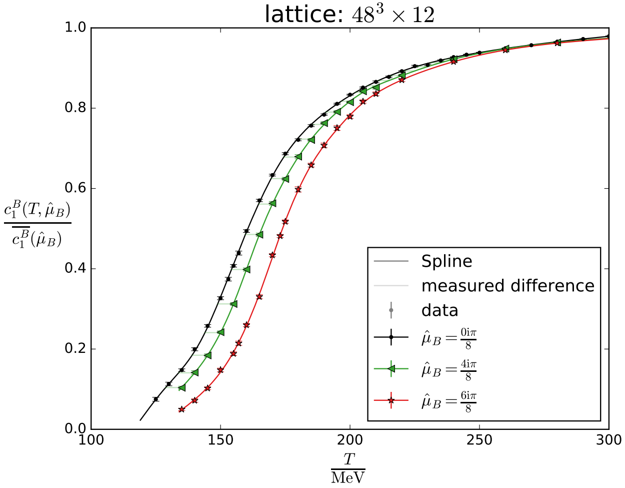

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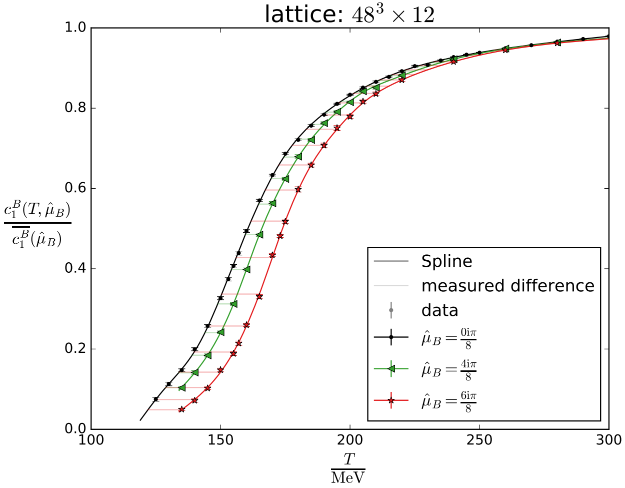

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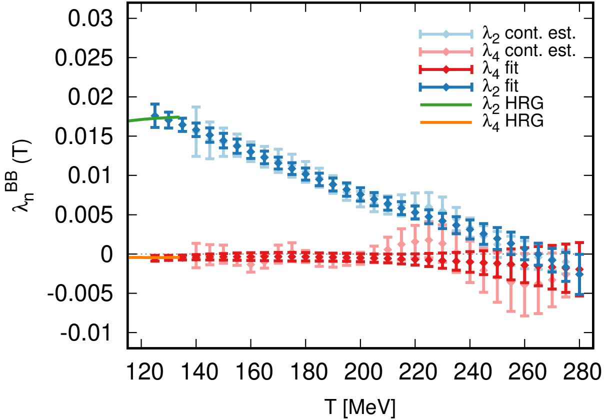
The expansion coefficients

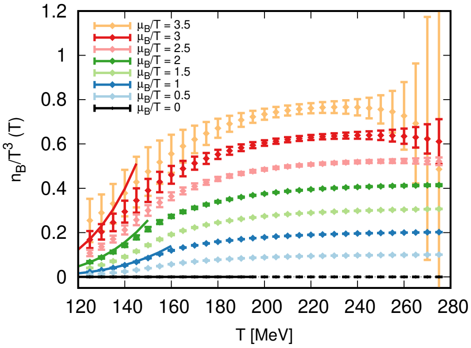


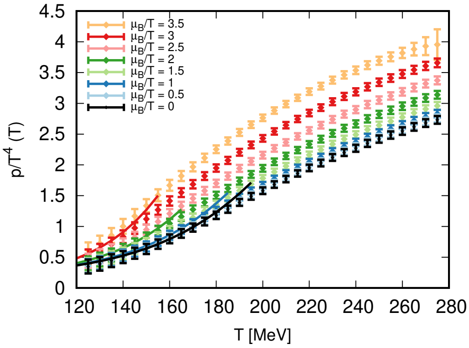
$$\Pi(T,\hat{\mu}_B,N_\tau) = \frac{T'(T,\hat{\mu}_B,N)-T}{T\hat{\mu}_B}$$

$$\Pi(T,\hat{\mu}_B,N_\tau) = \lambda_2^A + \lambda_4^A\hat{\mu}_B^2 + \lambda_6^A\hat{\mu}_B^4 + \frac{1}{N_\tau^2}\left(\alpha^A + \beta^A\hat{\mu}_B^2 + \gamma^A\hat{\mu}_B^4\right)$$

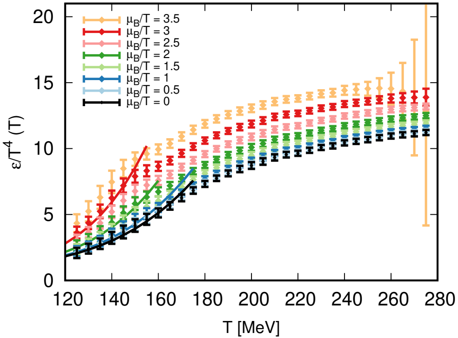
We make a fit to calculate derivatives and constrain it with the HRG.

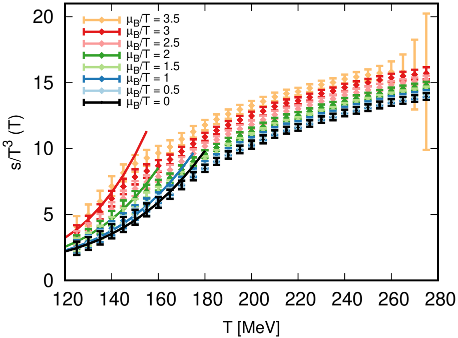
Results at $n_S = 0$ and $\mu_Q = 0$



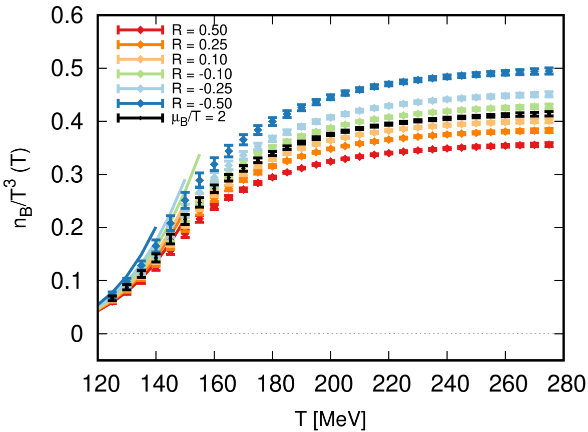


Results at $n_S = 0$ and $\mu_Q = 0$ II





Strange Baryon density

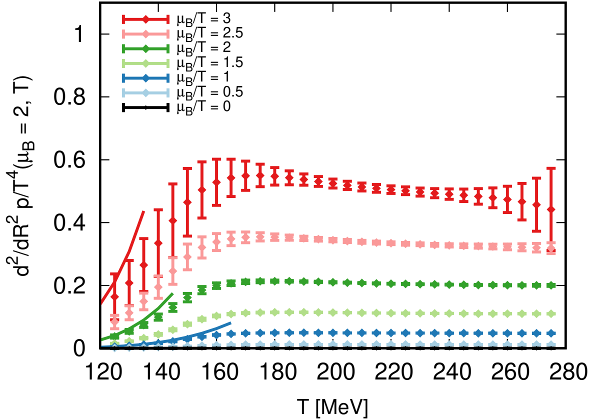


Expanding the baryon density:

$$\frac{\chi_1^B(T,\hat{\mu}_B,R)}{\chi_1^B(T,\hat{\mu}_B,R=0)} \approx 1 + R\frac{\chi_{11}^{BS}(T,\hat{\mu}_B,R=0)}{\chi_2^S(T,\hat{\mu}_B,R=0)}$$

where all quantities on the right hand side are along the strangeness neutral line.

Strange Pressure



At the strangeness neutral line the $\mathcal{O}(R)$ correction of the pressure vanishes. The leading order correction gives:

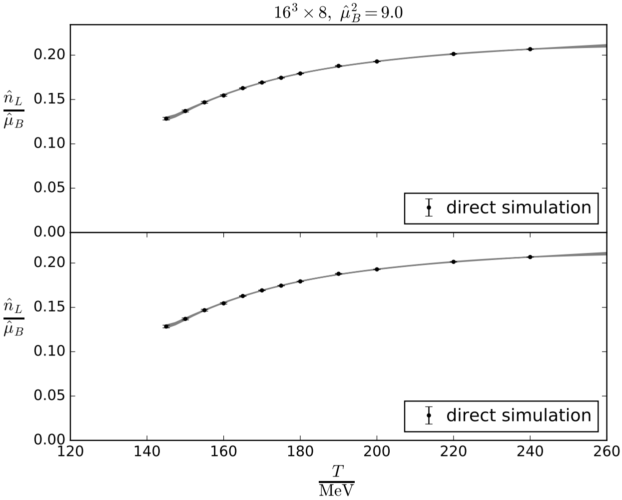
$$\hat{p}(T, \hat{\mu}_B, R) \approx \hat{p}(T, \hat{\mu}_B, R) + \frac{1}{2}\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}(T, \hat{\mu}_B)\, R^2,$$

where

$$\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}(T, \hat{\mu}_B) = \frac{\left(\chi_1^B(T, \hat{\mu}_B)\right)^2}{\chi_2^S(T, \hat{\mu}_B)}.$$

1. Lattice QCD
2. Equation of state
   - Rescaling and expansion - the analysis in [Borsanyi:2022qlh]
   - Results at $n_S = 0$ and $\mu_Q = 0$
   - Beyond strangeness neutrality
   - Cross-Check

Does it work? - Check in a small volume [Borsanyi:2022soo]



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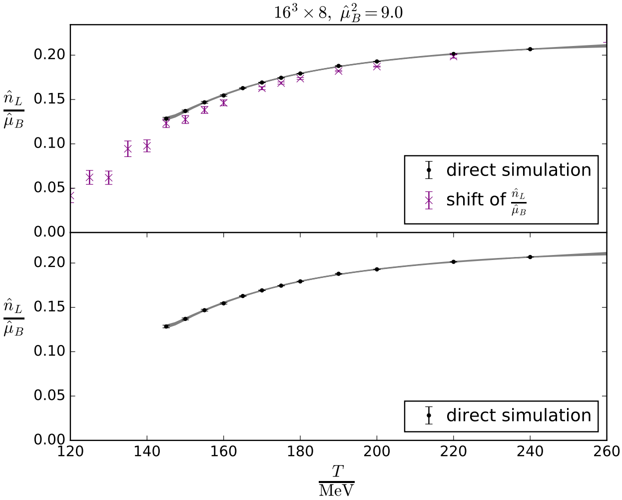

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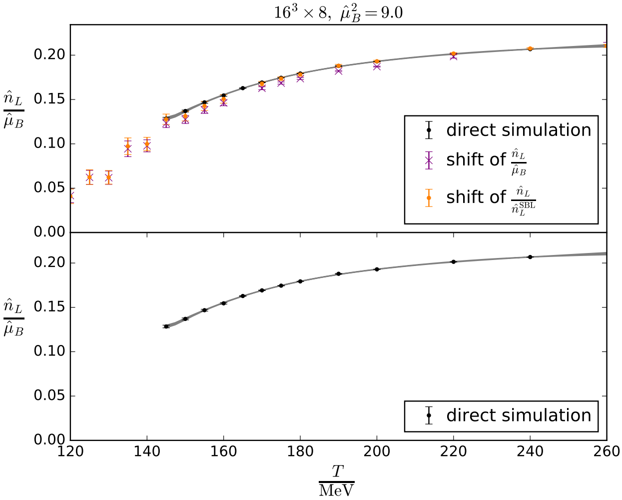

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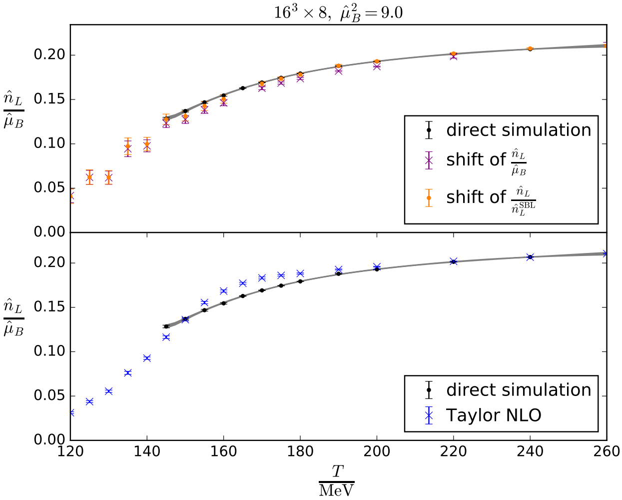

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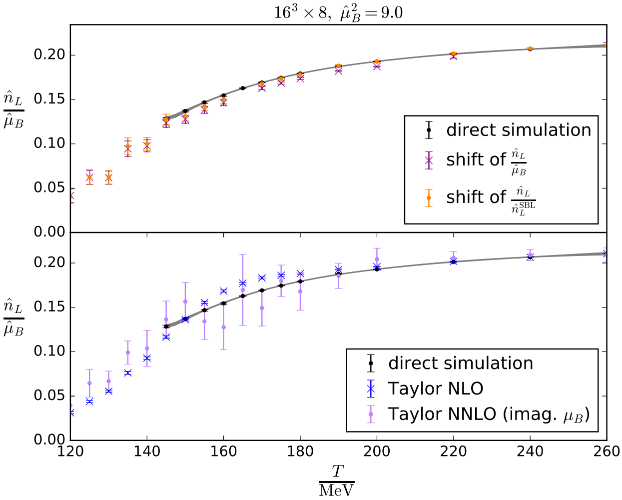

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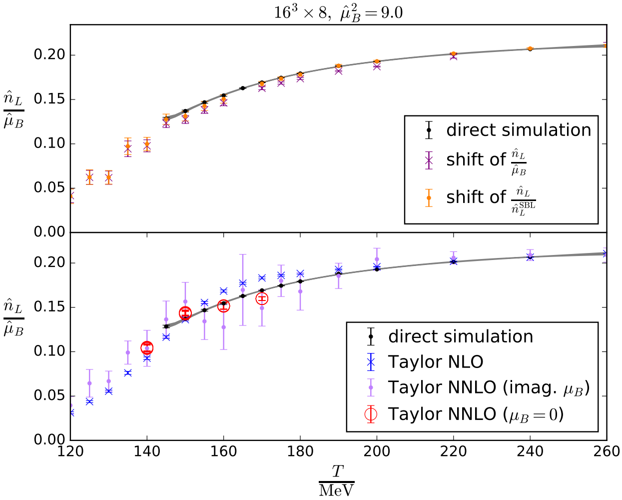

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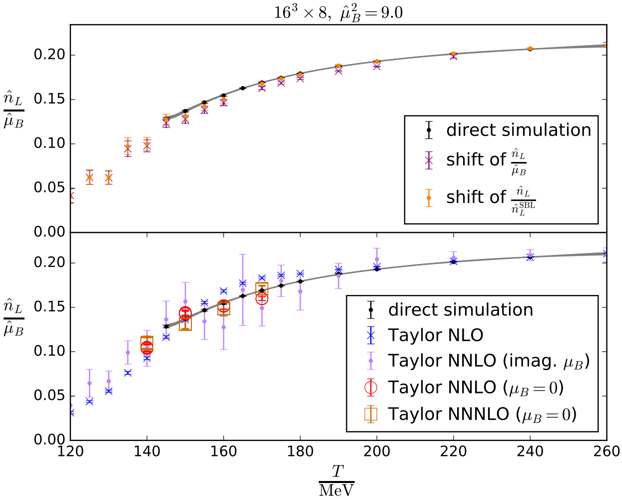
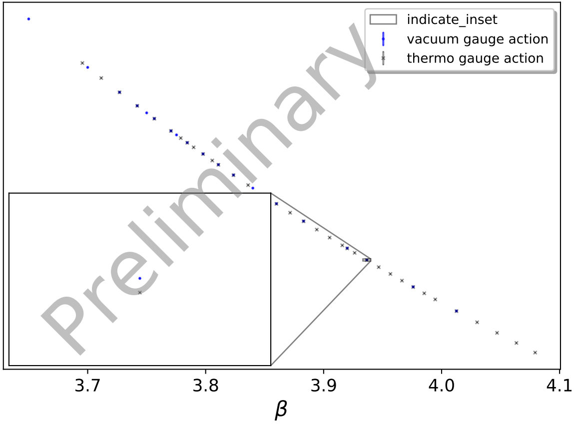


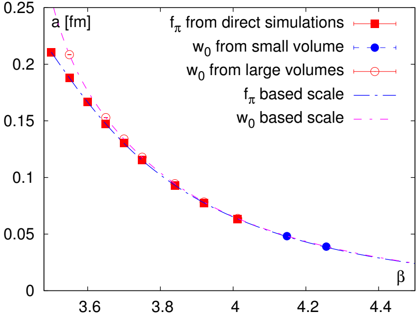
Outlook

- Update of the Equation of State at $\mu = 0$
- Addition of a magnetic field to the Equation of state at $\mu \neq 0$
- Further investigation of strangeness effects



Cross-Check

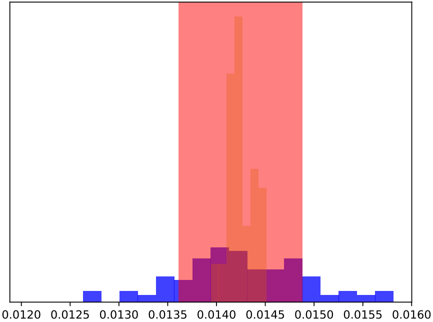
Lattice Setup



- Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions
- 2+1+1 flavour, on LCP with pion and kaon mass
- Simulation at $\langle n_S \rangle = 0$
- Continuum estimate from lattice sizes: $32^3 \times 8$, $40^3 \times 10$, $48^3 \times 12$ and $64^3 \times 16$
- $\frac{\mu_B}{T} = \mathrm{i}\frac{j\pi}{8}$ with $j = 0, 3, 4, 5, (5.5), 6$ and 6.5
- Two methods of scale setting: f_π and w_0, $Lm_\pi > 4$

Systematic Errors

- 3 different sets of spline node points at μ_B=0
- 2 different sets of spline node points at finite imaginary μ_B
- w_0 or f_π based scale setting
- 2 different chemical potential ranges in the global fit: $\hat{\mu}_B \leq 5.5$ or $\hat{\mu}_B \leq 6.5$
- 2 functions for the chemical potential dependence of the global fit: linear or parabola
- including the coarsest lattice, $N_\tau = 8$, or not, in the continuum extrapolation.



histogram for λ_2 at $T = 155$ MeV

In total we perform 96 Fits. We weight every result with a $Q > 0.01$ uniformly