Fluctuations of arctic curves and the Tracy-Widom distribution

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1. The six vertex model

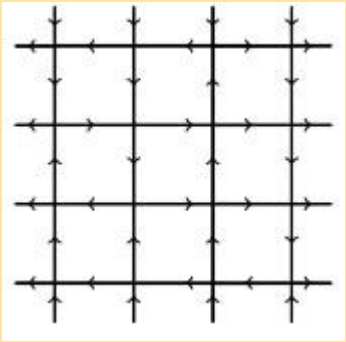
Basics

- Residual entropy of **ice** *[Pauling '35]*
- **Vertex weights**

$$W(a_i) = a,\ W(b_i) = b,\ W(c_i) = c$$

- Exact **free energy** on a square lattice with pbc *[Lieb '67]*
- Different kind of bcs (free and fixed)

Six possible vertex (v) configurations



$$Z = \sum_{\text{conf}} \prod_v W(v)$$

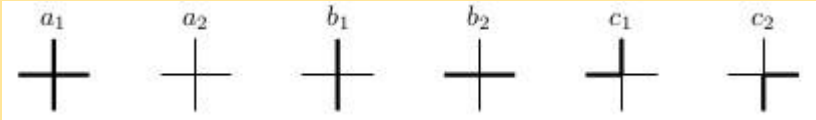
2. Domain wall boundary conditions

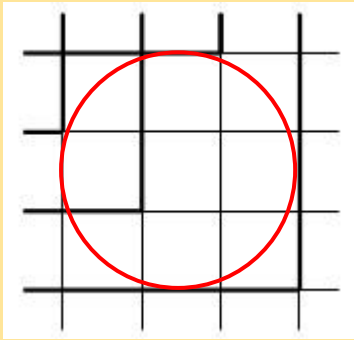
Domain wall bcs *[Korepin '82]*

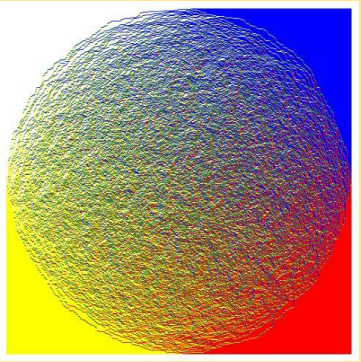
- Paths come in from the top and go out on the left
- Vertices fluctuate within **deterministic curves** *[Jokush, Propp and Shor '98]-[Colomo and Pronko 2010]* for $N \to \infty$
- Define for later purposes

$$\Delta = \frac{a^2+b^2-c^2}{2ab}$$

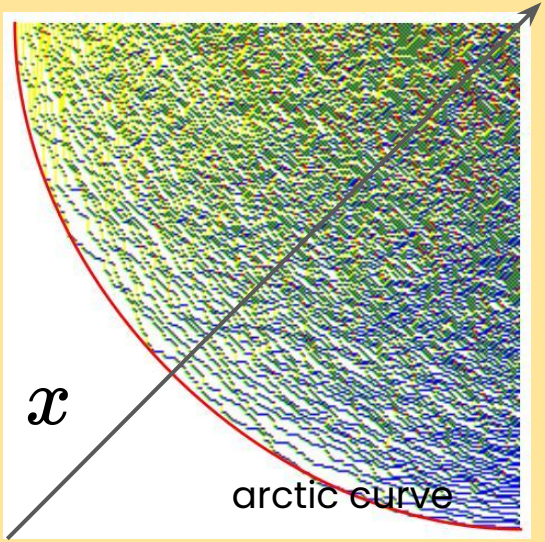
Non-intersecting path interpretation







3. Boundary fluctuations



Theorem *[Johansson 2005]*

Consider the first occurrence of a vertex different from the one on the frozen corner.

For $\Delta = 0$ *and* $N \to \infty$ *one has:*

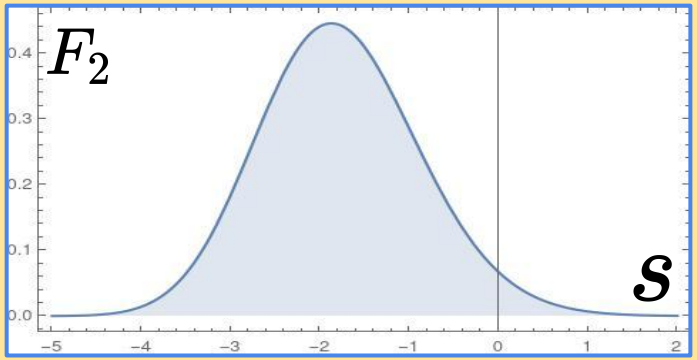
$$\text{Prob}\,(X) = F_2(-X) \;\text{ with }\; X = \frac{x - x_{ac}}{\beta N^{1/3}}$$

- **Purpose:** Test numerically universality of the **Tracy-Widom** distribution at $\Delta \neq 0$

4. Tracy-Widom distribution

- Fluctuations of the **largest eigenvalue** of an $N \times N$ Hermitian random matrix

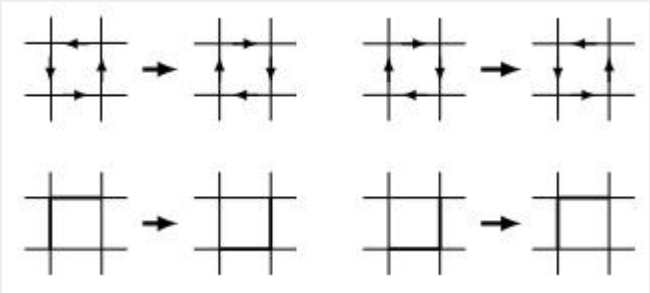
$$\text{Prob}\left(\frac{\lambda_{max} - \sqrt{2N}}{\beta N^{1/6}} > s\right) = F_2(s)$$



- Vicious Random walkers, random permutations, KPZ equation (***Universality***, *see [Deift 2006], also in Quanta magazine 2014*)

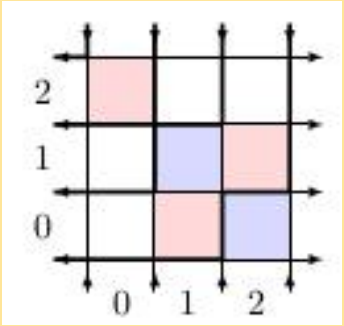
5. Monte Carlo algorithm

- **Local** Glauber dynamics



- Flip a vertex with probability

$$P(v) = \frac{\prod_{v' \in \text{plaquette}(v)} W(v')}{R}$$



- **Algorithm** proposed by *[Allison, Reshetikhin 2006]*
- Rejections for $\Delta < -1$

- Density profiles, several bcs. *[Lyberg, Korepin and V. 2016, 2018].*
- **GPU** implementation *[Keating, Sridhar 2018]*

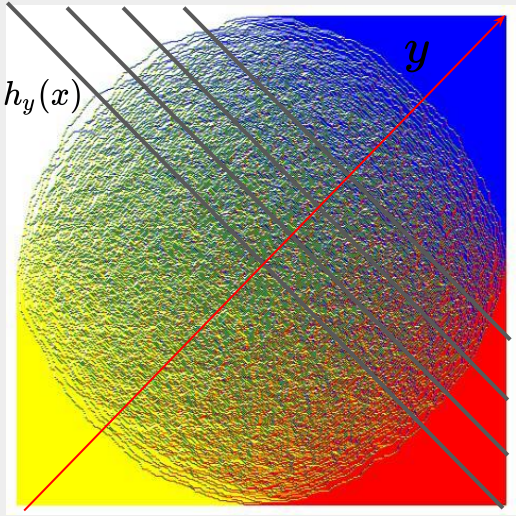
6. Method and Results [with I. Lyberg, V. Korepin, in prep.]

- Construct a normalized histogram $h_y(x)$ for each $y = 0, \ldots, N_\ell$
- Linear change of variable (ensure same variance and mean)

$$f_y(x) = \alpha_y h_y(\alpha_y x + \gamma_y); \quad \alpha_y = \frac{\sigma_y}{\sigma_0}; \quad \gamma_y = \frac{\mu_y}{\alpha_y}$$

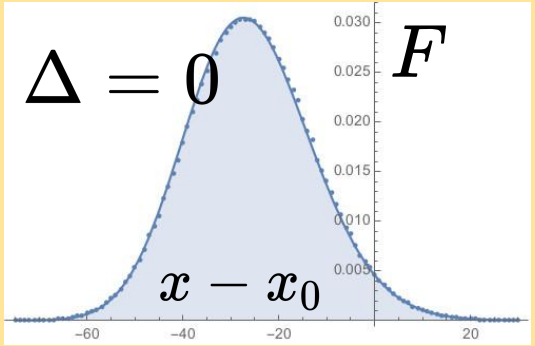
- Define the average

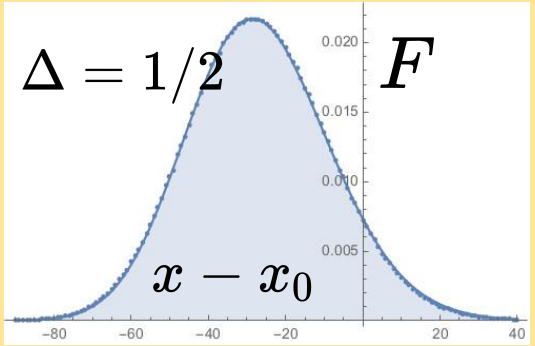
$$F(x) = \frac{1}{N_\ell + 1} \sum_y f_y(x)$$

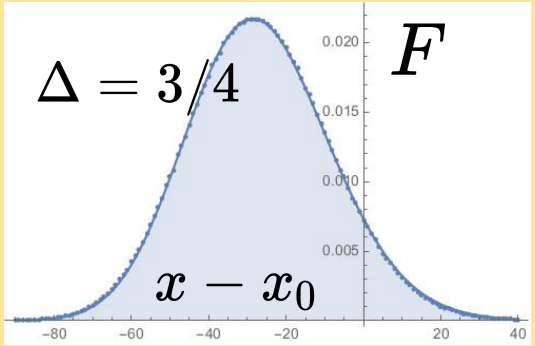


Claim: $F(x) \overset{N \gg 1}{\to} \frac{1}{\beta_N^{1/3}} F_2\left(\frac{x - x_0}{\beta_N^{1/3}}\right)$ and moreover $\beta_N = bN + O(1)$

7. Method and Results [with I. Lyberg, V. Korepin in prep.]







$N\backslash\Delta$	0	$1/2$	$3/4$
64	1.81(1)	2.85(5)	4.50(5)
128	3.33(2)	5.33(3)	8.65(7)
256	6.33(4)	10.11(4)	16.8(6)

- Fitted values of β_N

8. Conclusions, recap

- In the six vertex model with DWbc the case $\Delta = 0$ maps to free fermions. Obtaining results away from this point is hard.

- Provided numerical evidence of the existence of Tracy-Widom scaling for fluctuations of the arctic curves for $\Delta \neq 0$ (Universality)

- Analytical approaches (?)