# Color-flavor reflection in the continuum limit of two-dimensional lattice gauge theories

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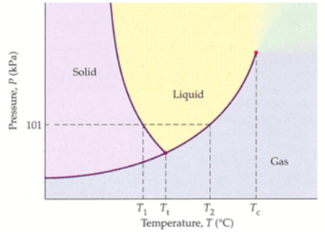
# Introduction and motivations

Critical phenomena have played a fundamental role in physics from the earliest time




Continuous transitions are characterized by critical exponents in 3D

$$\begin{aligned} \xi &\sim |T - T_c|^{-\nu} \\ \chi &\sim |T - T_c|^{-\gamma} \end{aligned}$$

Universality arguments are usually determined by very general properties of the system (Landau-Ginzburg-Wilson theory)

# Introduction and motivations

**Problem setting**

Do 2D lattice models with gauge invariance show universal properties?

Two-dimensional scalar systems on the lattice show universal behaviors (asymptotic freedom)

$$\hat{\xi} \underset{\beta \gg 1}{\sim} \beta^{p} e^{c\beta} , \quad \beta \equiv \frac{1}{T}$$

Example: 2D O($N_f$) vector models and NL$\sigma$Ms (N.B.: Unit-length fields!)

$$H = -\sum_{\boldsymbol{x},\mu,f} \phi_x^f \phi_{x+\mu}^f \quad \underset{\beta \to \infty\, ,\ \hat{\xi} \to \infty}{\longrightarrow} \quad \mathcal{S} = \frac{1}{2} \int \sum_{f=1}^{N_f} \partial_\mu \Phi^f(x) \partial_\mu \Phi^f(x) \, \mathrm{d}x$$

# Introduction and motivations

There exist 2D NL$\sigma$Ms with non-abelian gauge invariance described in terms of rank$-N_c$ projectors $P(x)$ (this is a $N_f \times N_f$ matrix)

$$\mathcal{S} = \frac{1}{2t} \int \mathrm{Tr}\, \partial_\mu P(x) \partial_\mu P(x) \, \mathrm{d}x$$

where $P^2(x) = P(x)$ and $\mathrm{Tr}\, P(x) = N_c$.

Non-abelian $\mathrm{SO}(N_c)$ gauge symmetry is manifest after having introduced the scalar fields $\varphi(x)$

$$P^{fg}(x) = \sum_{\alpha=1}^{N_c} \varphi^{\alpha f}(x) \varphi^{\alpha g}(x) \,, \quad \left( \varphi^{\alpha f}(x) \varphi^{\beta f}(x) = \delta^{\alpha\beta} \right)$$

# Introduction and motivations

The perturbative $\beta$-functions in $2+\epsilon$ dimensions suggest a hidden invariance [from Nucl. Phys. B **316**, 663 (1989)]

For the grassmannian manifold $O(N)/O(p)*O(N-p)$ we obtain

$$\begin{aligned}\beta(t) = \epsilon t - (N-2)t^2 - [2\boxed{p(N-p)} - N]t^3 - \left[\tfrac{3}{2}N\boxed{p(N-p)} - \tfrac{5}{4}N^2 + \boxed{p(N-p)} + \tfrac{1}{2}N\right]t^4 \\ - \left[\tfrac{1}{3}N^2\boxed{p(N-p)} + 5\boxed{p^2(N-p)^2} - \tfrac{5}{12}N^3 - \left(\tfrac{23}{6} + \tfrac{3}{2}\zeta(3)\right)N\boxed{p(N-p)}\right. \\ \left. + \left(-\tfrac{2}{3} + 3\zeta(3)\right)\boxed{p(N-p)} + \left(\tfrac{2}{6} + 3\zeta(3)\right)N^2 + \left(\tfrac{1}{3} - 12\zeta(3)\right)N + 12\zeta(3)\right]t^5 + O(t^6)\,.\end{aligned}$$

These $\sigma$-model QFTs are invariant under color-flavor reflection

$$\boxed{N_c \leftrightarrow N_f - N_c}$$

# Target and the lattice model

**Target**

If we introduced a (non-abelian) gauge symmetry into the lattice, would the continuum limit be described by a NL$\sigma$M QFT with gauge invariance?

We consider a scalar lattice model with $\mathrm{SO}(N_c)$ gauge symmetry and $\mathrm{O}(N_f)$ global invariance

$$\boxed{H = -\sum_{\boldsymbol{x},\mu} \mathrm{Tr}\, \phi_{\boldsymbol{x}}^t V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\mu} - \frac{\gamma}{N_c} \sum_{\boldsymbol{x}} \mathrm{Tr}\, \Pi_{\boldsymbol{x}} + V(\phi)}\,, \quad \mathrm{Tr}\, \phi_{\boldsymbol{x}} \phi_{\boldsymbol{x}} = 1$$

$$\Pi_{\boldsymbol{x}} = V_{\boldsymbol{x},1} V_{\boldsymbol{x}+1,2} V_{\boldsymbol{x}+2,1}^t V_{\boldsymbol{x},2}^t\,, \quad Z = \sum_{\{\phi, V\}} e^{-\beta H}$$

# Finite Size Scaling (FSS) strategy

We used Monte Carlo simulations (metropolis+overrelaxation algorithms) to compute expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\{\phi, V\}} \mathcal{O} e^{-\beta H} \approx \frac{1}{N_{\text{meas}}} \sum_{k=1}^{N_{\text{meas}}} \mathcal{O}_k$$

To study the continuum limit of the lattice model, we have used FSS techniques (i.e. we keep $R_\xi \equiv \hat{\xi}/L$ fixed sending $L \to +\infty$)

$$\langle \mathcal{O} \rangle (R_\xi) \approx L^{-y_{\mathcal{O}}} \mathcal{F}(R_\xi) + ..$$

# Finite Size Scaling (FSS) strategy

RG invariant quantities are particularly useful quantities to test universal predictions ($y_{\mathcal{O}} = 0$)

$$U(R_\xi) \sim \mathcal{U}(R_\xi) + ..$$

Example: Binder cumulants (here $\mathcal{Q}_x^{fg} = \sum_{\alpha=1}^{N_c} \phi_{\boldsymbol{x}}^{\alpha f} \phi_{\boldsymbol{x}}^{\alpha g} - \delta^{fg}/N_f$)

$$U \equiv \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 \equiv \frac{1}{V^2} \sum_{\boldsymbol{x},\boldsymbol{y}} \mathrm{Tr}\, \mathcal{Q}_{\boldsymbol{x}} \mathcal{Q}_{\boldsymbol{y}}$$

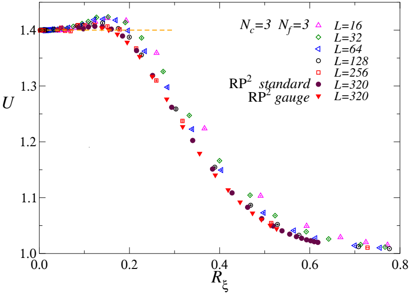
To verify whether two lattice systems exhibit the same critical behavior, we consider

$$\boxed{U(R_\xi) \underset{L \to +\infty}{\approx} \mathcal{U}(R_\xi)}$$

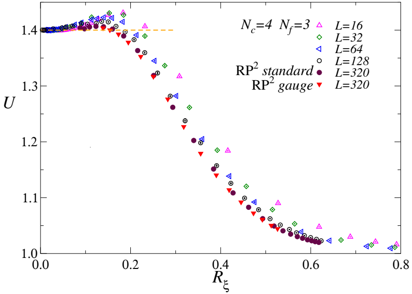
# Previous results [PRD **102**, 034512 (2020)]

$$H = -\sum_{\boldsymbol{x},\mu} \mathrm{Tr}\, \phi_{\boldsymbol{x}}^t V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\mu} \cancel{- \frac{\gamma}{N_c} \sum_{\boldsymbol{x}} \mathrm{Tr}\, \Pi_{\boldsymbol{x}}} + \cancel{V(\phi)}$$

$N_c{=}3,\ N_f{=}3$




$N_c{=}4,\ N_f{=}3$




$$\mathrm{Tr}\, B_{\boldsymbol{x}}^2 \to 1\,, \text{ where } B_{\boldsymbol{x}}^{fg} \equiv \sum_{\alpha=1}^{N_c} \phi_{\boldsymbol{x}}^{\alpha f} \phi_{\boldsymbol{x}}^{\alpha g}$$

# Results [PRE **105**, 054117 (2022)]

$$V(\phi_{\boldsymbol{x}}) = w \sum_{\boldsymbol{x}} \operatorname{Tr} B_{\boldsymbol{x}}^2 \,, \quad B_{\boldsymbol{x}}^{fg} \equiv \phi_{\boldsymbol{x}}^{\alpha f} \phi_{\boldsymbol{x}}^{\alpha g}$$

$$\boxed{\beta \to \infty} \longrightarrow \boxed{w \leq 0} \longrightarrow \boxed{\operatorname{Tr} B_{\boldsymbol{x}}^2 = 1} \longrightarrow \boxed{\mathbb{R}\mathrm{P}^{N_f - 1}}$$

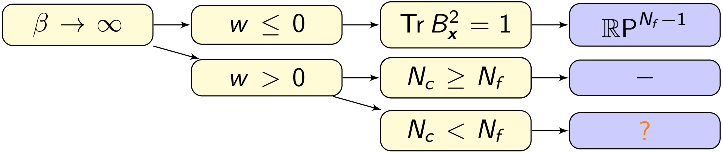
# Results [PRE **105**, 054117 (2022)]

$$V(\phi_{\boldsymbol{x}}) = w \sum_{\boldsymbol{x}} \mathrm{Tr}\, B_{\boldsymbol{x}}^2 \,, \quad B_{\boldsymbol{x}}^{fg} \equiv \phi_{\boldsymbol{x}}^{\alpha f} \phi_{\boldsymbol{x}}^{\alpha g}$$

$$\begin{tikzcd}
\beta \to \infty \arrow[r] \arrow[dr] & w \le 0 \arrow[r] & \mathrm{Tr}\, B_{\boldsymbol{x}}^2 = 1 \arrow[r] & \mathbb{R}\mathrm{P}^{N_f - 1} \\
 & w > 0 \arrow[r] & N_c \ge N_f \arrow[r] & -
\end{tikzcd}$$

# Results [PRE **105**, 054117 (2022)]

$$V(\phi_{\boldsymbol{x}}) = w \sum_{\boldsymbol{x}} \operatorname{Tr} B_{\boldsymbol{x}}^2 \,, \quad B_{\boldsymbol{x}}^{fg} \equiv \phi_{\boldsymbol{x}}^{\alpha f} \phi_{\boldsymbol{x}}^{\alpha g}$$

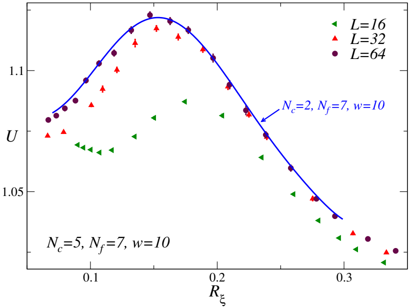


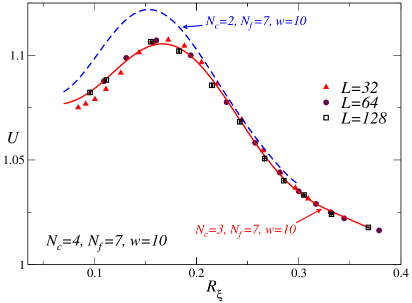
# Results [PRE **105**, 054117 (2022)]

$$H = -\sum_{\boldsymbol{x},\mu} \mathrm{Tr}\, \phi_{\boldsymbol{x}}^t V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\mu} \cancel{- \frac{\gamma}{N_c} \sum_{\boldsymbol{x}} \mathrm{Tr}\, \Pi_{\boldsymbol{x}}} + w \sum_{\boldsymbol{x}} \mathrm{Tr}\, B_x^2$$

$N_c=5,\ N_f=7$, w=10

$N_c=4,\ N_f=7$, w=10



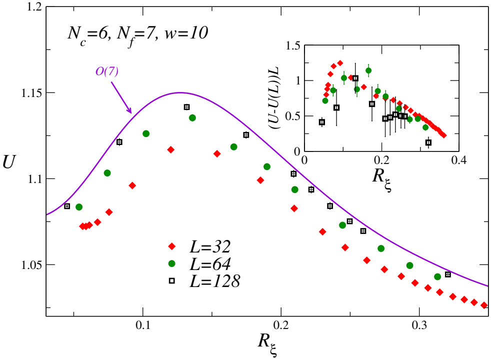




$$U(R_\xi) \approx \mathcal{U}(R_\xi)$$

# Results [PRE **105**, 054117 (2022)]

$$H = -\sum_{\boldsymbol{x},\mu} \mathrm{Tr}\, \phi^t_{\boldsymbol{x}} V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\mu} \cancel{- \frac{\gamma}{N_c} \sum_{\boldsymbol{x}} \mathrm{Tr}\, \Pi_{\boldsymbol{x}}} + w \sum_{\boldsymbol{x}} \mathrm{Tr}\, B_x^2$$

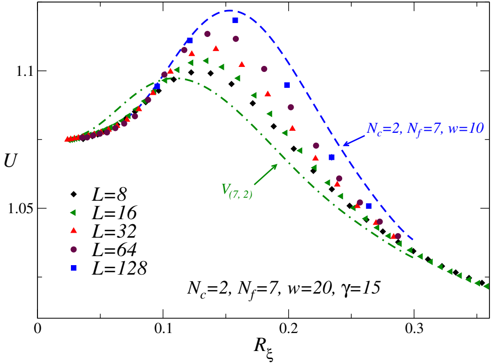
$N_c$=6, $N_f$=7, w=10



# Results [PRE **105**, 054117 (2022)]

$$H = -\sum_{\boldsymbol{x},\mu} \mathrm{Tr}\, \phi_{\boldsymbol{x}}^t V_{\boldsymbol{x},\mu} \phi_{\boldsymbol{x}+\mu} - \frac{\gamma}{N_c} \sum_{\boldsymbol{x}} \mathrm{Tr}\, \Pi_{\boldsymbol{x}} + w \sum_{\boldsymbol{x}} \mathrm{Tr}\, B_x^2$$

$N_c$=2, $N_f$=7, w=20, $\gamma = 15$



# Conclusions

We have considered the continuum limit of two-dimensional lattice models in the presence of $\mathrm{SO}(N_c)$ local symmetry and $\mathrm{O}(N_f)$ global invariance.

By tuning the quartic coupling associated with the interaction term $V(\phi) = w \sum_{\boldsymbol{x}} \mathrm{Tr}\, B_{\boldsymbol{x}}^2$, we have shown that

- if $w \leq 0$ we observe the same critical behavior as the $\mathbb{R}\mathrm{P}^{N_f - 1}$ model with the same global symmetry
- if $w > 0$ the lattice model shows an emergent color-flavor reflection invariance

Thank you for your attention!

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based on PRE **105**, 054117 (2022)