Random Field Ising model, dimensional reduction and supersymmetry

Marco Picco*
Sorbonne Université and CNRS
Laboratoire de Physique Théorique et Hautes Energies

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*In collaboration with N. Fytas (Coventry Univ., UK), V. Martín-Mayor (Madrid Univ., Spain), G. Parisi (Roma Univ., Italy), and N. Sourlas (ENS Paris, France)

Introduction

The random–field Ising model (RFIM)

- RFIM is an old story starting in the 70's.
- It has been studied starting from the mean field theory, with perturbative renormalisation group, numerical simulations, etc.
- Generalization of the ferromagnetic Ising model, $J > 0$ and $S_x = \pm 1$:

$$\mathcal{H}^{(\mathrm{RFIM})} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x = E_J + E_{RF}$$

  with $\{h_x\}$ a random variable (quenched disorder), with zero mean and dispersion σ.
- E_J is just the ordinary Ising model. Without the random magnetic field, the model will be ordered at small temperature for $D > 1$.
- E_{RF} will try do destroy the ferromagnetic order. For a large enough σ, the spins will be aligned with the random field, such that $S_x = h_x/|h_x|$.

The random–field Ising model (RFIM)

- Ferromagnetic transition as we vary σ, from a ferromagnetic phase at small σ to a paramagnetic phase at large σ.

- Relevant dimensions : $3 \leq D \leq 6$

  $D_{ld} > 2$ Imry & Ma (1975) and $D = 6$ the upper critical dimension for RFIM.

- Mean Field Hamiltonian

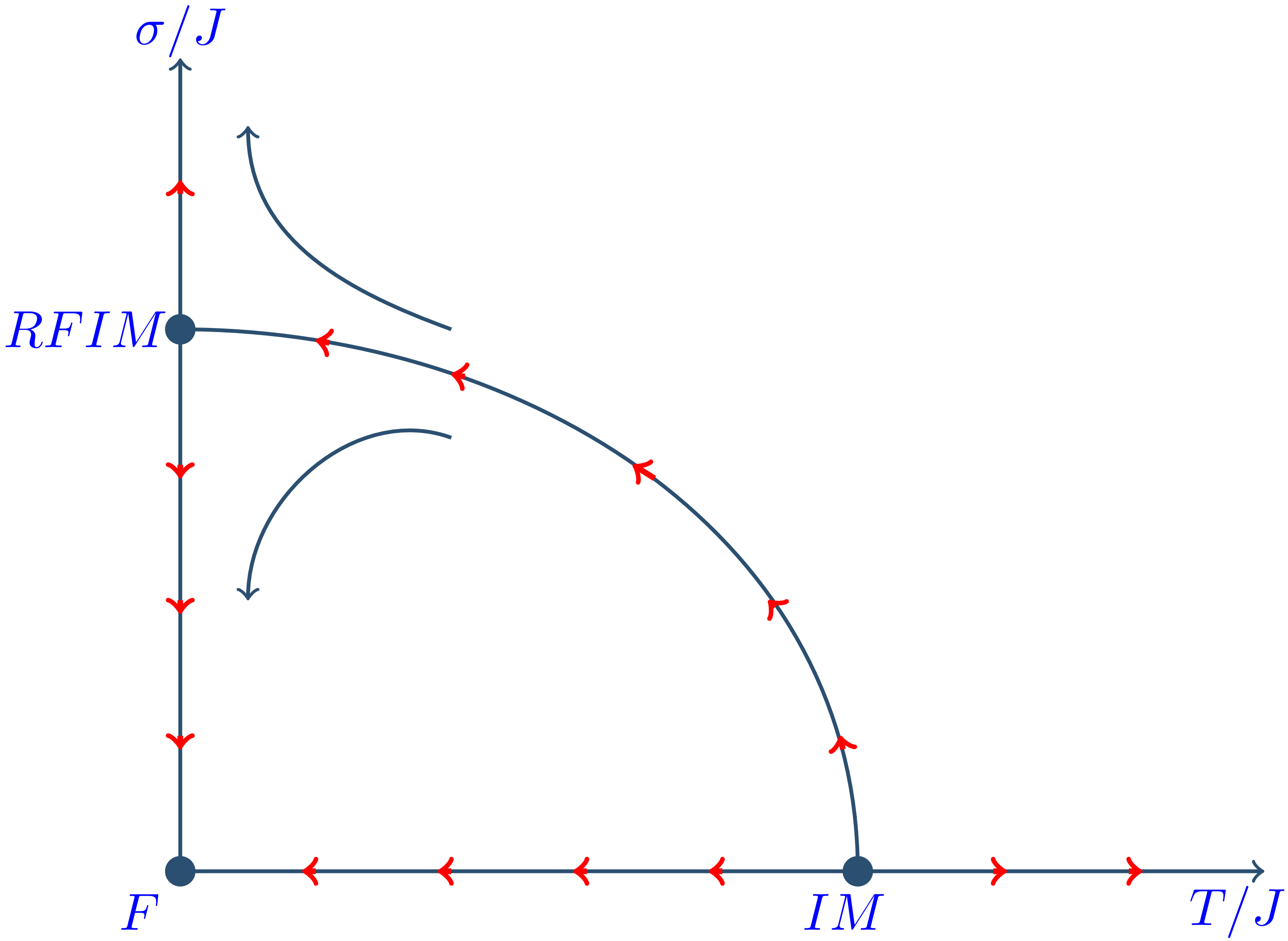
$$\mathcal{H}^{MF} = \int d^D r[(\nabla S(r))^2 + tS^2(r) + \lambda S^4(r) - H(r)S(r)]$$

- Average of the random field done by introducing a replicated system :

$$\mathcal{H}^{MF} = \int d^D r[\sum_a ((\nabla S_a(r))^2 + tS_a^2(r) + \lambda S_a^4(r)) - \sigma \sum_{a,b} S_a(r)S_b(r)]$$

with $< H(r) >= 0$ and $< H(r)H(r') >= \sigma\delta(r - r')$

RG fixed point & phase diagram



Mean Field for the RFIM

- Propagator : $(k^2\delta_{a,b} - \sigma M_{a,b})^{-1} \rightarrow \frac{\delta_{a,b}}{k^2} - \frac{\sigma M_{a,b}}{k^2(k^2 - n\sigma)}$

- Then, two propagators :
  - a diagonal one corresponding to $G_{xy}^{(\mathrm{dis})} = \overline{\langle S_x S_y \rangle}$ and $\simeq 1/k^4$.
  - a non diagonal one corresponding to $G_{xy}^{(\mathrm{con})} = \overline{\langle S_x S_y \rangle} - \overline{\langle S_x \rangle \langle S_y \rangle}$ and $\simeq 1/k^2$.

- Bellow the upper critical dimension, each propagator will have an anomalous dimension.

- The RFIM bellow the upper critical dimension is characterized by three quantities, ν and the anomalous dimensions η and $\bar{\eta}$ for the two propagators.

RFIM & PRG

- The RFIM in $D < 6$ can be considered with the Perturbative Renormalization Group (PRG).

- The PRG can be carried out at all orders in $\epsilon = 6 - D$ and predicts for all critical exponents and at each order

$$\alpha^{RFIM,D} = \alpha^{IM,D-2} \rightarrow \text{Dimensional reduction}$$

(Aharony, Imry, and Ma, 1976 and Young, 1977).

- In particular, $\eta = \bar{\eta}$.

- Another prediction of the PRG is the universality: RFIM with different random fields distribution are in the same universality class.

Dimensional reduction versus sharp reality

- Parisi & Sourlas PRL **43**, 744 (1979): the dimensional reduction is explained by a hidden supersymmetry in the Random Field Ising model.

  Supersymmetry $\rightarrow$ Dimensional reduction.

- Failure: The 3D RFIM orders while the 1D IM does not!

- $4D$ and $5D$ RFIM ?

- Recent works suggested that dimensional reduction and supersymmetry is restored for $D \simeq 5$: Tissier, Tarjus (2011) with Functional renormalization group studies.
  Similar predictions by S. Hikami (2018) using bootstrap computations.

New results

Computational scheme

N.G. Fytas, V. Martín-Mayor, M. P., and N. Sourlas, PRL **116**, 227201 (2016), Phys. Rev. E 95, 042117 (2017), J Stat Phys (2018) 172: 665-672

- We consider a D dimensional hyper-cubic lattice with periodic boundary conditions and energy units $J = 1$.
- Gaussian distribution and Poissonian distribution : check for Universality.
- Optimization methods: Graph theoretical algorithms that calculate ground states of the model in polynomial time, avoiding equilibration problems: $L_{\max}^{D} = \{192^3, 64^4, 28^5\}$.
- Extensive averaging over **10 million samples**.
- Re-weighting extrapolation: From a single simulation we extrapolate the mean value of observables to nearby parameters of the disorder distribution.

Observables

- Order-parameter density: $m = \frac{1}{L^D} \sum_x S_x$.

- Disconnected propagator : $\overline{\langle S_x S_y \rangle} \sim \frac{1}{r^{D-4+\bar{\eta}}} \to \chi_k^{(\mathrm{dis})} = L^D \overline{\langle |m_k|^2 \rangle}_k$.

- Connected propagator :
  $\overline{\frac{\partial \langle S_x \rangle}{\partial h_y}} \sim \frac{1}{r^{D-2+\eta}} \to \chi_k^{(\mathrm{con})} = \frac{1}{L^D} \sum_{x,y} \mathrm{e}^{\mathrm{i}k\cdot(x-y)} \frac{G_{xy}^{(\mathrm{con})} + G_{yx}^{(\mathrm{con})}}{2}$.

- Binder ratio and Correlation lengths (con and dis)

$$U_4 = \frac{\overline{\langle m^4 \rangle}}{\overline{\langle m^2 \rangle}^2} \; ; \; \xi^{\#} = \frac{1}{2\sin(\pi/L)} \sqrt{\frac{\chi^{\#}_{(0,\cdots)}}{\chi^{\#}_{(2\pi/L,0,\cdots)}} - 1} \, .$$

- Dimensionless quantities : $U_4(L,\sigma); \xi^{(\mathrm{dis})}(L,\sigma)/L$ and $\xi^{(\mathrm{con})}(L,\sigma)/L$.

Finite–size scaling

- Close to a critical point, a dimensionless quantity behaves as :

$$g(L,\sigma) = F_g(L^{1/\nu}(\sigma - \sigma_c)) + \mathcal{O}(L^{-\omega}) \cdots \quad (1)$$

with $F_g(x)$ some universal function and ω the leading irrelevant correction $\rightarrow \sigma_c, \omega, 1/\nu$

- We fit simultaneously several data sets: 2 field distributions and up to 3 crossing points: $\mathrm{Z}^{(\mathrm{x})}$, where Z $=$ G, or P and x $=$ (con), (dis), or U_4.

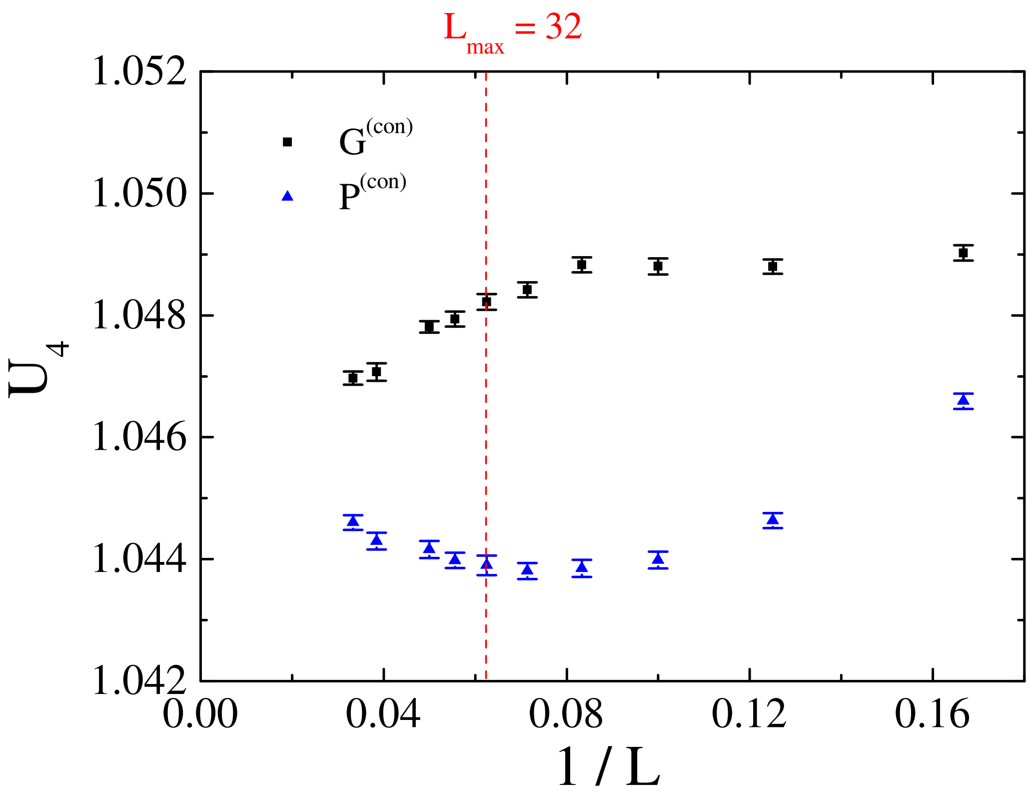
- Estimation of ω using joint fits for several magnitudes.

- Individual extrapolation of all other observables fixing ω.

- Determination of the number of corrections to include is rather subtle and needs to be done case by case.

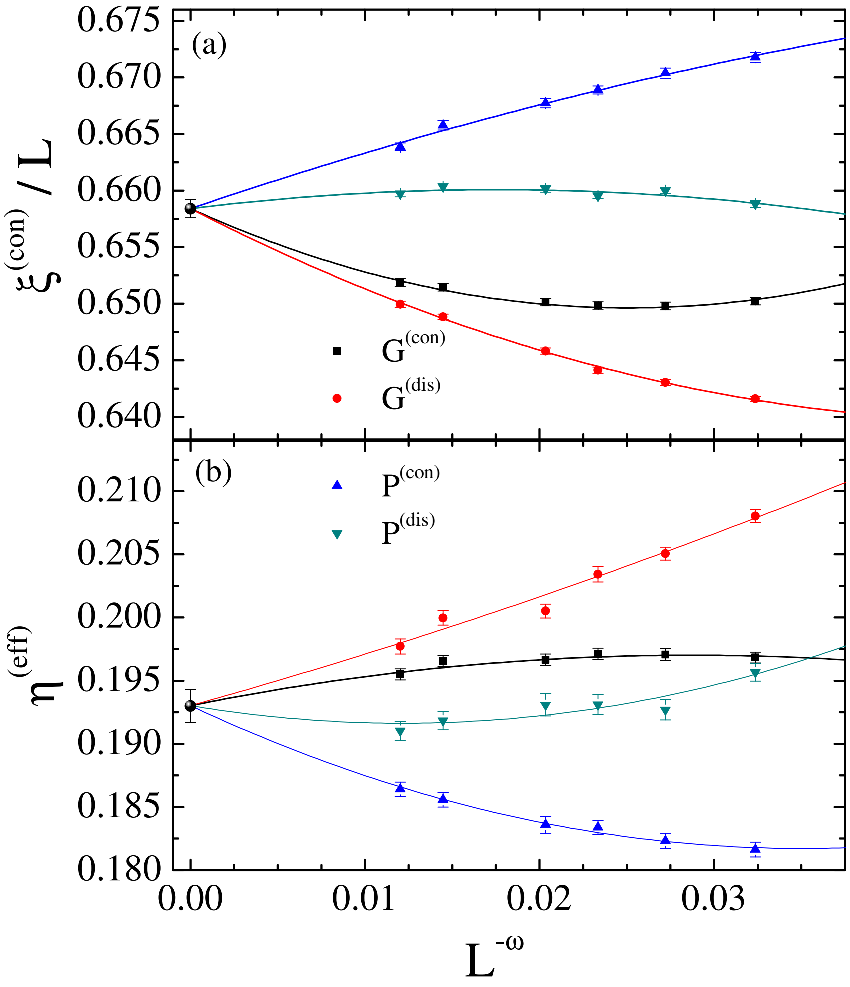
Non-monotonic behavior and universality

U_4 for the $4D$ RFIM.



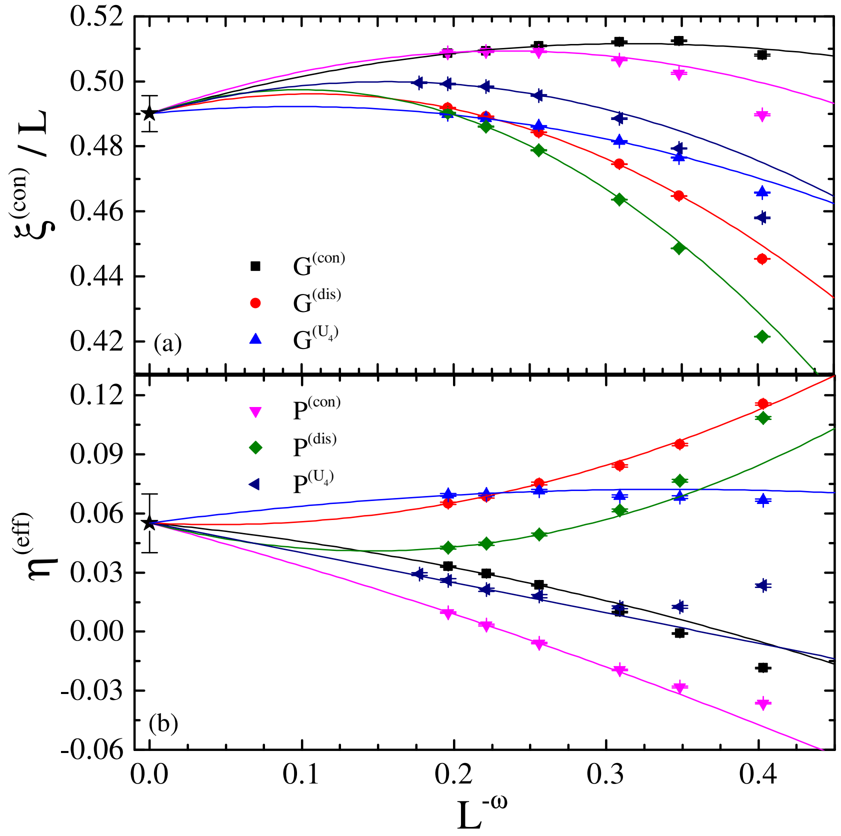
Higher-order corrections are necessary: $g_L = g^* + a_1 L^{-\omega} + a_2 L^{-2\omega} + \cdots$

Results for the 4D RFIM



$\omega = 1.30(9)$; $\xi^{(\text{con})}/L = 0.6584(8)$; $\eta = 0.1930(13) \neq 0.25 = \eta^{(\text{2DIM})}$

Results for the 5D RFIM



$$\omega = 0.66(15) \sim 0.82966(9) = \omega^{(\text{3D IM})}$$
$$\eta = 0.055(15) \sim 0.036298(2) = \eta^{(\text{3D IM})}$$

Summary of results for $3 \leq D < 6$

	3D RFIM	4D RFIM	5D RFIM	2D IM	3D IM	MF
ν	1.38(10)	0.8718(58)	0.626(15)	1	0.629971 (4)	1/2
η	0.5153(9)	0.1930(13)	0.055(15)	0.25	0.036298(2)	0
$\bar{\eta}$	1.028(2)	0.3538(35)	0.052(30)	0.25	0.036298(2)	0
$\Delta_{\eta,\bar{\eta}} = 2\eta - \bar{\eta}$	0.0026(9)	0.0322(23)	0.058(7)	0.25	0.036298(2)	0
β	0.019(4)	0.154(2)	0.329(12)	0.125	0.326419(3)	1/2
γ	2.05(15)	1.575(11)	1.217(31)	1.875	1.237075(10)	1
θ	1.487(1)	1.839(3)	2.00(2)	2	2	2
α	-0.16(35)	0.12(1)	-	-	-	-
α (from hyperscaling)	-0.09(15)	0.12(1)	0.12(5)	0	0.110087 (12)	0
$\alpha + 2\beta + \gamma$	2.00(31)	2.00(3)	2.00(11)	2	2.000000 (28)	2
$\sigma_c(G)$	2.27205(18)	4.17749(6)	6.02395(7)	-	-	-
$\sigma_c(P)$	1.7583(2)	3.62052(11)	5.59038(16)	-	-	-
U_4	1.0011(18)	1.04471(46)	1.103(16)			
$\xi^{(\mathrm{con})}/L$	1.90(12)	0.6584(8)	0.4901(55)			
$\xi^{(\mathrm{dis})}/L$	8.4(8)	2.4276(70)	1.787(8)			
ω	0.52(11)	1.30 (9)	0.66(+15/-13)		0.82966(9)	0

* In $D = 4$, RFIM different from IM in $D = 2$ and $2\eta - \bar{\eta} \neq 0$

* Within our numerical resolution: 5D RFIM $\rightarrow$ 3D IM

Supersymmetry ?

N.G. Fytas, V. Martín-Mayor, G. Parisi, M. P., and N. Sourlas, PRL **122**, 240603 (2019).

- So far, we have checked about dimensional reduction which seems to exists between $D = 5$ RFIM and $D = 3$ IM.
- What about supersymmetry predicted by Parisi and Sourlas (1979) ? Dimensional reduction is a consequence of supersymmetry, not the other way around !!!
- Dimensional reduction was measured on the exponents. We can also consider $U_4, \xi/L$. These quantities are associated to boundary conditions, which are not the same in $3D$ and $5D$.
- We consider measurements in $5D$ with the geometry :

$$L_x = L_y = L_z = L \; ; \; L_t = L_u = RL \; ; \; R \geq 1 \tag{2}$$

and look for the limit $R \to \infty$

Supersymmetry ?

- The correction limit is to take $R \to \infty$ **before** the thermodynamic limit, $L \to \infty$.

- This corresponds to restoring a partial supersymmetry $O(2,2)$ in place of the original supersymmetry $O(D,2)$, which was broken by the boundary conditions.

- We consider the disconnected correlation function

$$G^{(\mathrm{dis})}_{(x_1,u);(x_2,u)} = \overline{\langle S_{x_1,u} S_{x_2,u} \rangle} \; , \tag{3}$$

with x_1 or x_2 the 3 dimensional part and u the 2 dimensional part.

- Supersymmetry prediction

$$G^{(\mathrm{dis})}_{(x_1,u);(x_2,u)} = \mathcal{Z} G^{\mathrm{3d\ Ising}}_{x_1;x_2} \tag{4}$$

with $\mathcal{Z}$ a position-independent normalization constant.

Supersymmetry ?

- In practice, we first define a Fourier transform as :

$$\chi_k^{(\mathrm{dis})} = \frac{1}{L^{D-2}} \sum_{x_1,x_2} \mathrm{e}^{ik\cdot(x_1-x_2)}\, \overline{G^{(\mathrm{dis})}_{(x_1,u_1);(x_2,u_2)}} \tag{5}$$

- Compute a correlation length ($\mathcal{Z}$ disappeared !!!).

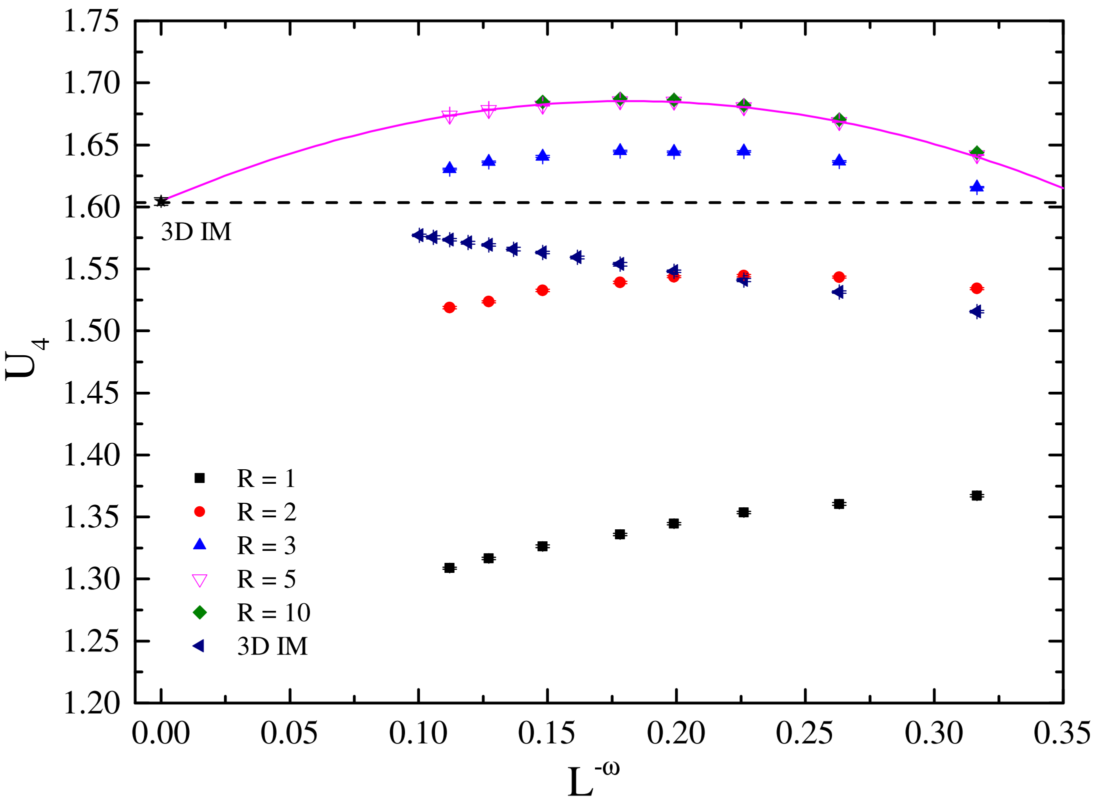
$$\xi^{(\mathrm{dis})} = \frac{1}{2\sin(\pi/L)} \sqrt{\frac{\chi^{(\mathrm{dis})}_{(0,0,0)}}{\chi^{(\mathrm{dis})}_{(2\pi/L,0,0)}} - 1}\ . \tag{6}$$

- Similar argument also for the Binder ratio :

$$U_4(L) = \frac{\overline{\langle m_u^4 \rangle}}{\overline{\langle m_u^2 \rangle}^2}\ . \tag{7}$$

Check of Supersymmetry

$U_4(L, R)$ vs. $L^{-\omega}$ for various R values, as computed in the $D = 5$ RFIM.



Similar results for other quantities. Strong support for restoration of supersymmetry for $D = 5$ (and not for $D = 4$).

Back in $4D$

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- It was argued by Brézin and De Dominicis (1998) that there exists additional interactions which become relevant as one decreases the dimension bellow the upper critical dimension $D = 6$.

- Recently, Kaviraj, Rychkov, Trevisani (2020, 2021) considered again the problem of the existence of relevant operators starting from the upper critical dimension.

- Claim : there exists two operators which become relevant around $d_c = 4.5$:
  "We thus predict that for $d < d_c$ the Parisi-Sourlas fixed point is destabilized, and the RFIM transition is described by another, non-supersymmetric, fixed point ..."

- In our results, we claimed to observe universality for the $4D$ RFIM and the absence of dimensional reduction. This was for Random fields with a Gaussian or Poissonian distribution by adjusting a single parameter σ, the variance of the distribution.

Back in $4D$

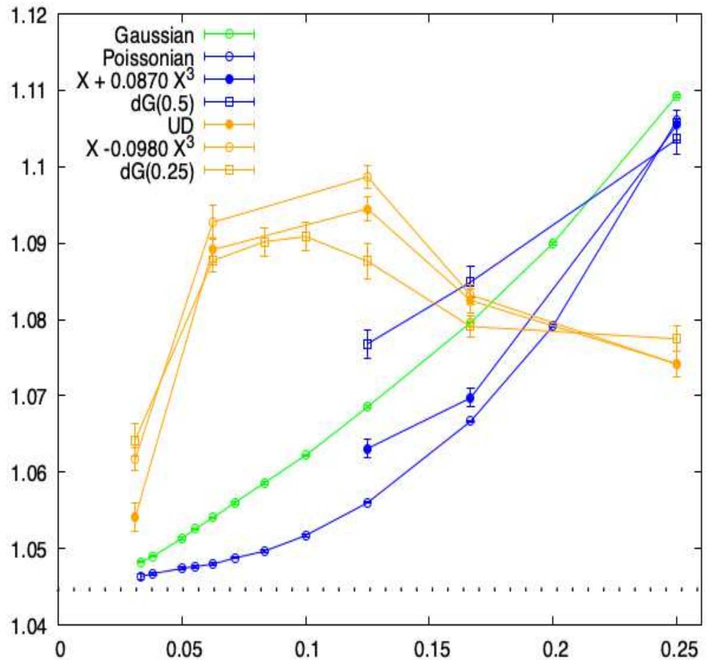
- Can we redo it while adjusting the parameters corresponding to these two operators, thus with three parameters ? And what are these operators on the lattice ?
- Brézin and De Dominicis (1998) argued that one of the relevant operators couples to the fourth cumulant of the distribution of the random fields (the kurtosis).
- Goal : finding the distribution for Random Fields such that the coupling to the two operators is fine-tuned to zero.
- Just changing the kurtosis (K_4) gives no new result if $K_4 > 3$ while a small change for $K_4 < 3$ but we flow back to the Gaussian case. ($K_4 = 3$ for the Gaussian and 6 for Poissonian).
- K_4 corresponds to one operator. One expects that the second operator couples with the sixth cumulant, K_6.

Back in $4D$

- Changing together K_4 and K_6, we observe a new behaviour for small values of these parameters.
- Good new : probably a new universality class with small K_4 and K_6.
- Good new : K_4 and K_6 control the cross over. Other cumulants are not relevant.
- Bad new : $(K_4, K_6) \simeq (1, 1)$ which is the bimodal distribution and is a mess. In particular, degeneracy of the ground states.
- Ongoing work : determine precisely the value of K_4, K_6 for the unstable fixed point. Check dimensional reduction and SUSY.

Back in $4D$

U_4 vs $1/L$:



Back in $4D$

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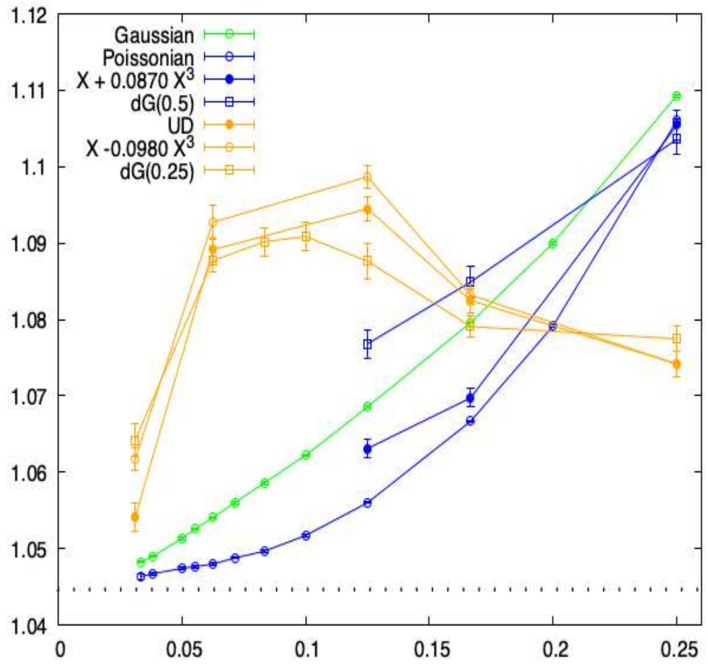

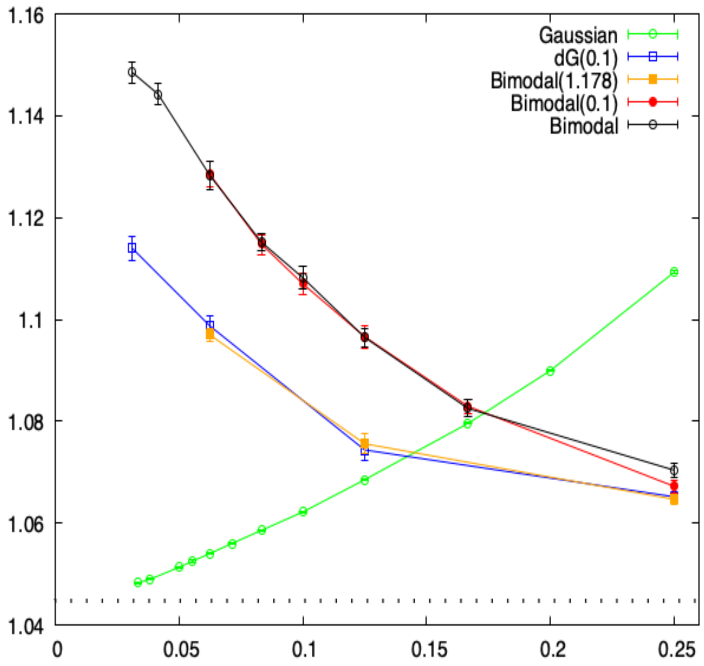

Conclusions

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1. We have developed powerful numerical and finite-size scaling tools for the study of the RFIM (hopefully useful for other disordered systems).
2. We have shown universality in the RFIM.
3. We provided high-accuracy estimates for various universal ratios and the whole set of critical exponents and relevant dimensions $D = \{4, 5\}$.

   Our estimates for the critical exponents indicate that dimensional reduction seems to be at play at, or close to, $D = 5$.
4. All the predictions of supersymmetry are satisfied between the $D = 5$ RFIM and the $D = 3$ Ising model with a good precision.
5. $D = 4$ is still (or again) a work in progress.