Entropy of quantum states[1]

SM&FT 2022

Arturo Konderak

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Dipartimento Interateneo di Fisica "M. Merlin"
Istituto Nazionale di Fisica Nucleare *INFN*

[1] Paolo Facchi, Giovanni Gramegna, and Arturo Konderak. "Entropy of Quantum States". In: *Entropy* (2021)

Outline

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Entropy in quantum mechanics

- Von Neumann entropy[2]

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ρ density matrix

$$\rho = \sum_{j=1}^{N} \lambda_j |\psi_j\rangle\langle\psi_j| \text{ spectral decomposition}$$

$$\mathcal{S}_{\mathrm{VN}}(\rho) = -k_{\mathrm{B}}\mathrm{Tr}(\rho \log \rho) = -k_{\mathrm{B}} \sum_{j=1}^{N} \lambda_j \log \lambda_j$$

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- Shannon entropy of the Eigenvalues $\mathrm{H}(\lambda)$
- $\mathcal{S}_{\mathrm{VN}}(\rho) = 0$ if and only if $\rho = |\psi\rangle\langle\psi|$

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Motivation I: ambiguities

- If we only have access to some observables, we cannot characterize uniquely the density matrix

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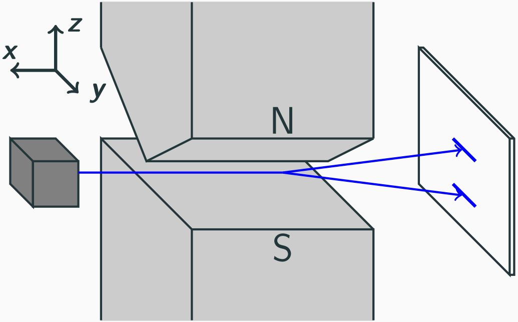
- Example: Stern-Gerlach experiment

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{C}$$



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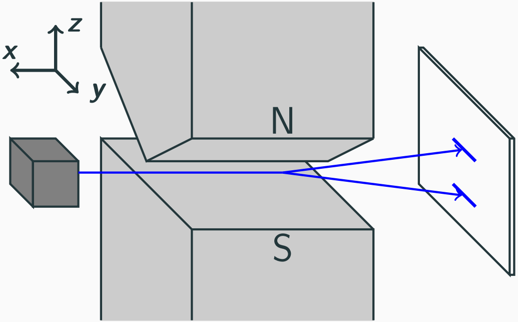

$$|\uparrow_x\rangle\langle\uparrow_x| \stackrel{?}{=} \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$$

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Motivation II: Black Hole entropy

- Hawking[4]

$$\mathcal{S}_{\rm BH} = \frac{k_{\rm B} A}{4\ell_p^2}$$

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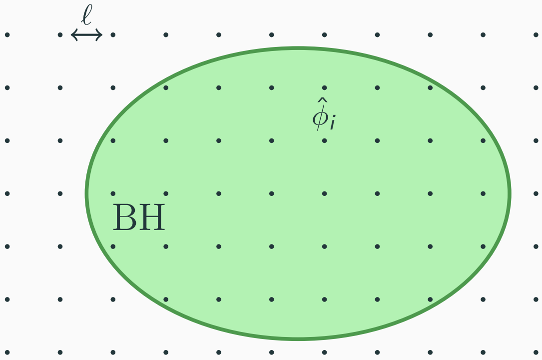
- On a lattice

$$\mathcal{S} \sim \frac{A}{\ell^2}$$



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Operators and states algebraic description

- Algebraic description: incomplete algebra of observables[6]

$$\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$$

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$$\omega : \mathfrak{A} \to \mathbb{C}$$

$$\omega(A^* A) \geqslant 0$$

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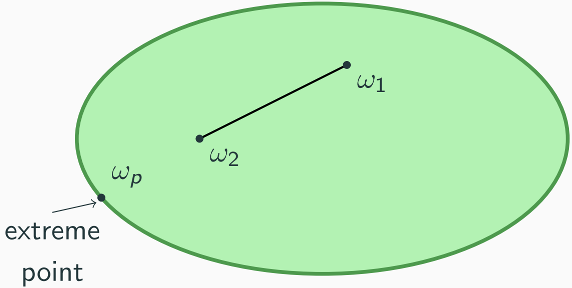
- Set of states is convex!



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Convex sets

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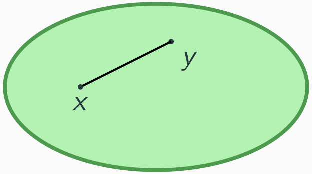

Convex sets

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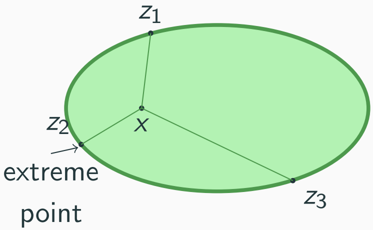
$$x = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3$$



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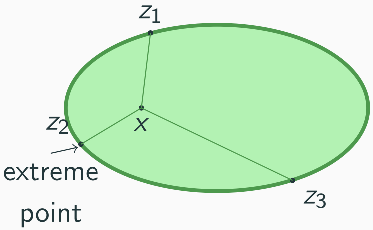

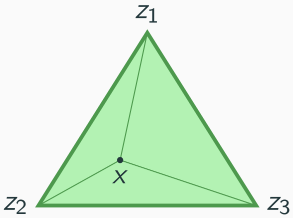

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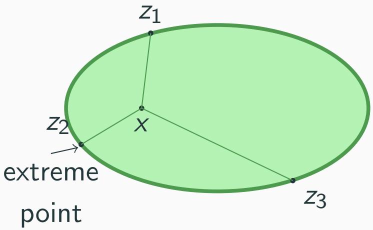
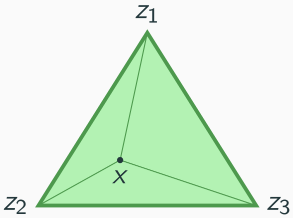
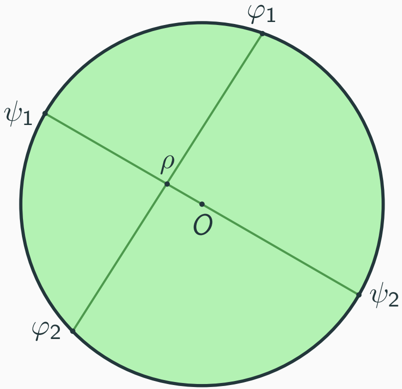
- QM: Not simplex states
  Example: the Bloch sphere





Schrödinger theorem

- Different decompositions for the same state (ambiguity in the preparation)



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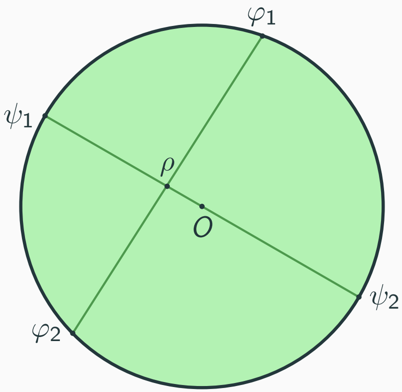

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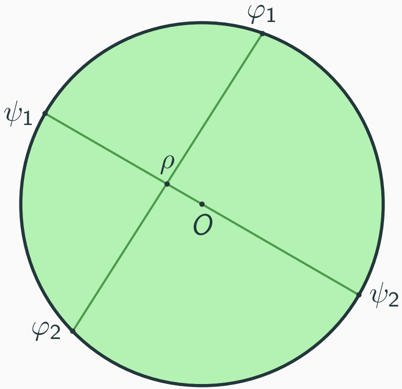
  $p_i = \sum_{j=1}^{N} |U_{ij}|^2 \lambda_j$ $\qquad U_{ij}$ unitary



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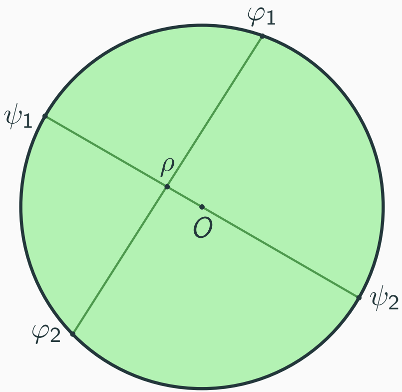
- The spectral decomposition is less random



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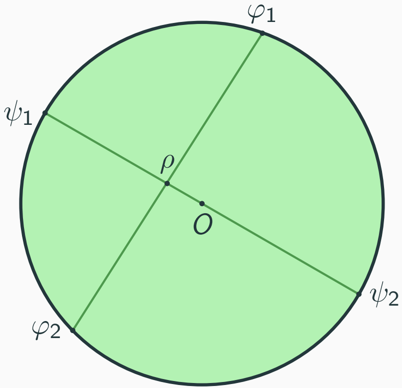

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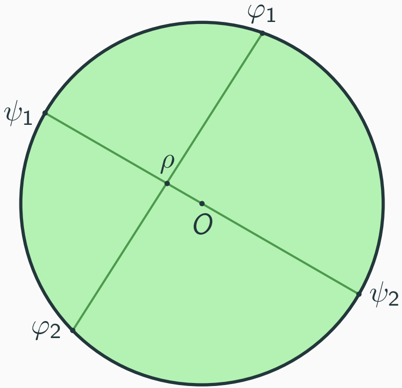

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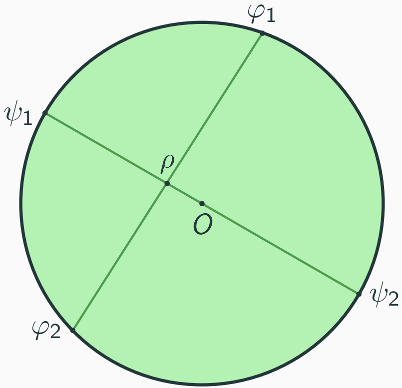

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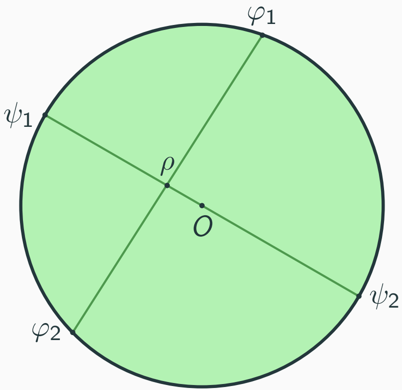
- The spectral decomposition has smallest entropy



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- This property can be used to define an entropy for convex sets

$$\mathcal{S}(\omega) = \inf \left\{ \mathrm{H}(\vec{p}) \,:\, \omega = \sum_{i} p_i \omega_i, \;\; \vec{p} \text{ probability vector}, \quad \omega_i \text{ pure states} \right\}$$

Finite-dimensional $*$-algebras

- **Structure theorem**: Every finite dimensional C^*-algebra is the sum of matrix algebras with multiplicities

$$\mathfrak{A} = (\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{I}_{m_1}) \oplus (\mathcal{B}(\mathcal{H}_2) \otimes \mathbb{I}_{m_2}) \oplus \cdots \oplus (\mathcal{B}(\mathcal{H}_N) \otimes \mathbb{I}_{m_N})$$

$$\mathfrak{A} \ni \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_1 \end{pmatrix} \oplus \begin{pmatrix} A_2 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} A_N & 0 & \cdots & 0 \\ 0 & A_N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{pmatrix}$$

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- A density matrix ρ represent a state ω in the sense $\mathrm{tr}(\rho A) = \omega(A)$ forall A in $\mathfrak{A}$
- **Result**: There is a unique density matrix $\rho_\omega \in \mathfrak{A}$ representing ω

$$\rho_\omega = p_1 \left(\rho_1 \otimes \frac{\mathbb{I}_{m_1}}{m_1} \right) \oplus p_2 \left(\rho_2 \otimes \frac{\mathbb{I}_{m_2}}{m_2} \right) \oplus \cdots \oplus p_N \left(\rho_N \otimes \frac{\mathbb{I}_{m_N}}{m_N} \right)$$

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- Equality when $m_i = 1$

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Entanglement between identical particles

- Are identical particles always entangled?

$$\psi = \psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1$$

- Entanglement characterized in terms of local algebras[8]
- Exponential property: Factorization of Fock space

$$\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$$

Bosons: Local algebras are the whole sets of bounded operators

Fermions: Local algebras has selections rule, angular momentum conserved

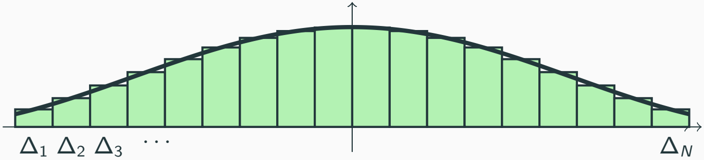
[8] Paolo Facchi and Arturo Konderak. "Entanglement between Identical particles". In: *In preparation* (2022)

Infinite dimensional algebras

- Infinite dimensional systems, continuous decomposition[9]:

$$\sum_k \lambda_k \omega_k \mapsto \int_K d\mu(\omega)\omega$$

- Riemann integral



- $\mathcal{S} = -k_{\mathrm{B}} \sum_k \Delta_k \log \Delta_k \to \infty$ for continuous distributions

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Conclusions

- ω represented by the density matrix

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- Von Neumann entropy for $m_i = 1$

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- Entropy

$$\mathcal{S}(\omega) = \inf \left\{ \mathrm{H}(\vec{p}) \, : \, \omega = \sum_i p_i \omega_i \right\} = \mathcal{S}_{\mathrm{VN}}(\rho_\omega) - \sum_{i=1}^{N} p_i \log m_i$$

- Von Neumann entropy for $m_i = 1$
- From thermodynamics

$$\mathcal{S}_{\mathrm{gas}} = S(\omega) + \sum_{i=1}^{N} p_i s_i$$

Thank you

for your attention!

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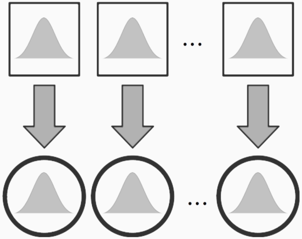

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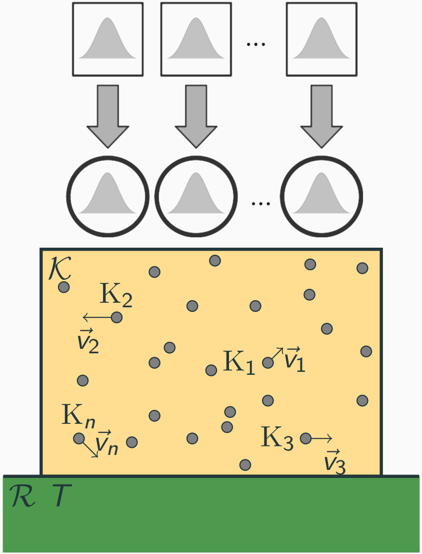

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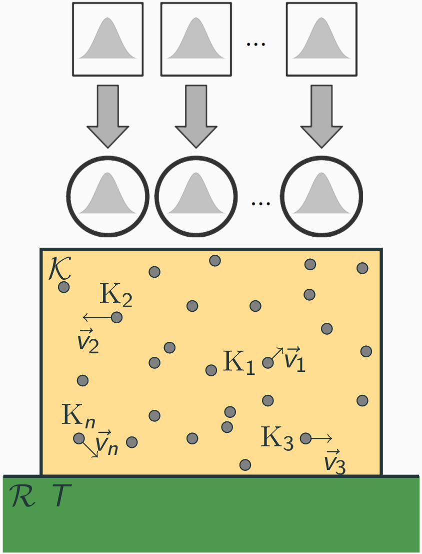

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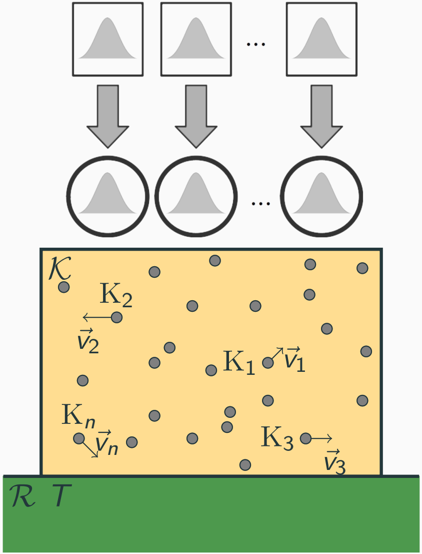

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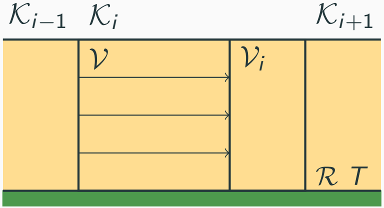

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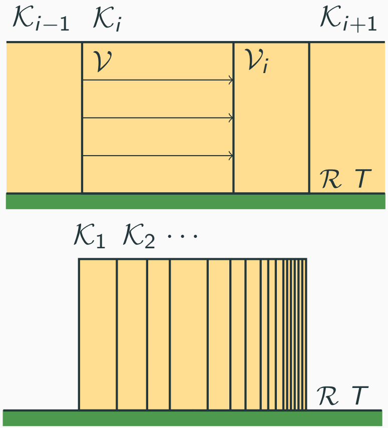

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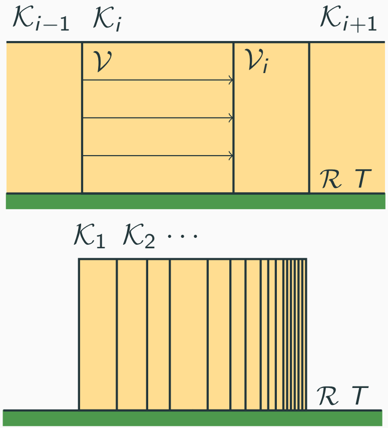

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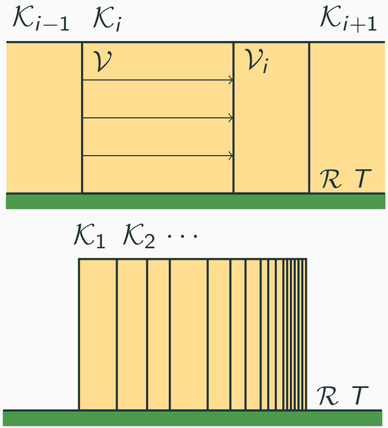

Entropy Backup

- Same phase: Zeno effect

$$\phi \to \psi$$

$$\psi_\nu = \cos \frac{\pi\nu}{2k}\phi + \sin \frac{\pi\nu}{2k}\psi$$

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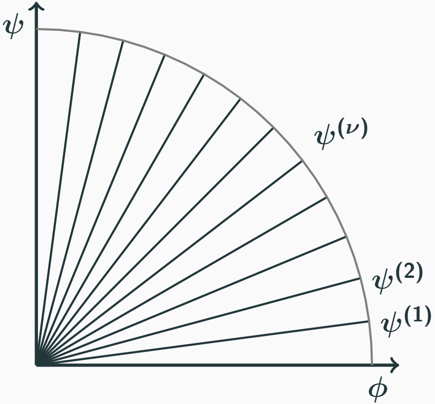

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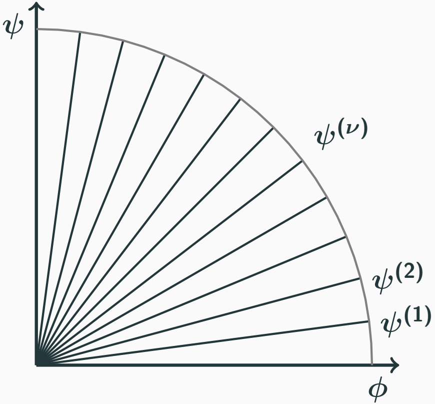

Schematic proof of structure theorem Backup

- **Structure theorem**: Every finite dimensional C^*-algebra is the sum of matrix algebras with multiplicities

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$$\mathcal{H}_i^{(m_i)} = \underbrace{\mathcal{H}_i \oplus \mathcal{H}_i \oplus \cdots \oplus \mathcal{H}_i}_{m_i}, \qquad \pi_i^{(m_i)} = \underbrace{\pi_i \oplus \pi_i \oplus \cdots \oplus \pi_i}_{m_i}.$$

$$\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_{m_i} \in \mathcal{H}_i^{(m_i)} \longleftrightarrow \xi_1 \otimes e_1 + \xi_2 \otimes e_2 + \ldots \xi_{m_i} \otimes e_{m_i} \in \mathcal{H}_i \otimes \mathbb{C}^{m_i},$$

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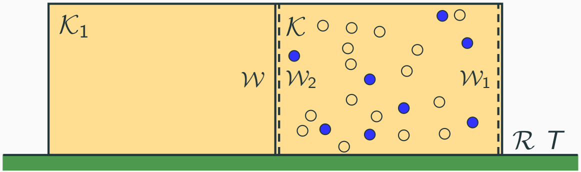

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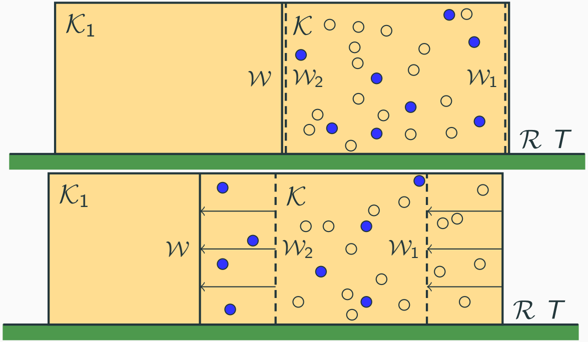

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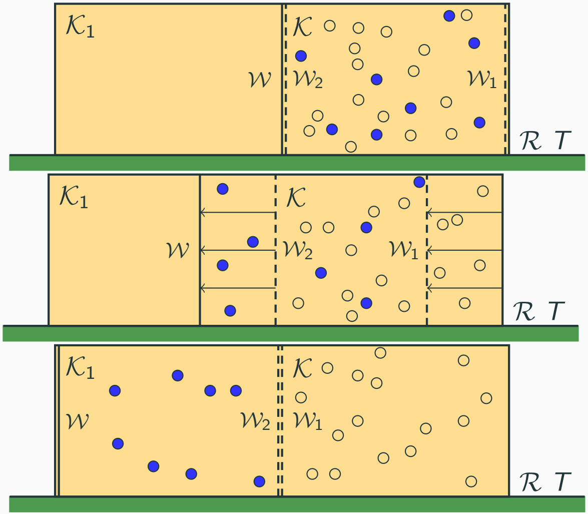
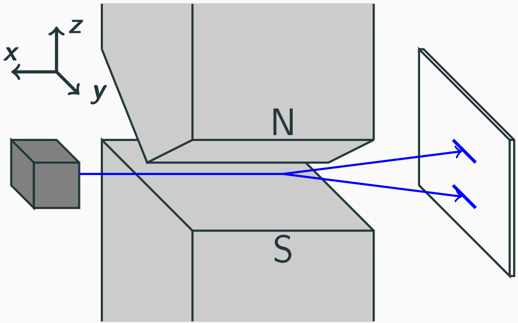


Stern-Gerlach experiment II Backup

- Example: Stern-Gerlach experiment
  algebra
  $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{C}$$



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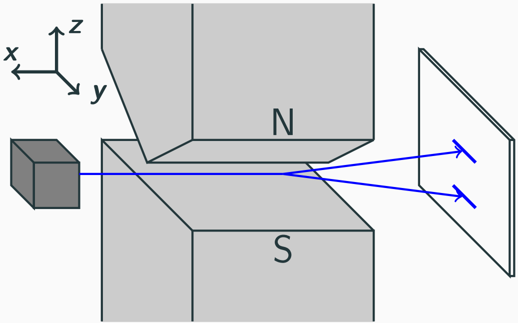

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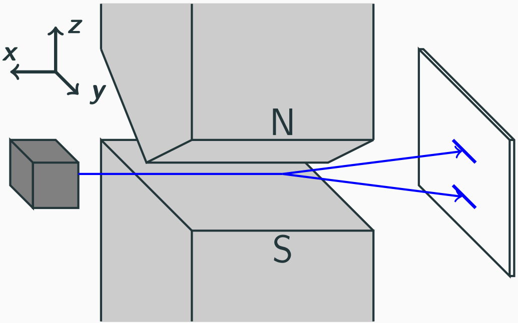
- The representative state is

$$\rho_\omega = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



$$|\uparrow_x\rangle\langle\uparrow_x| \stackrel{?}{=} \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$$

- Quantum mechanics:
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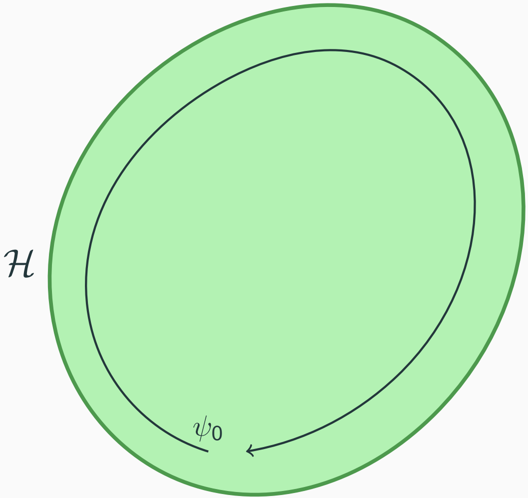

Thermodynamics, pure states Backup

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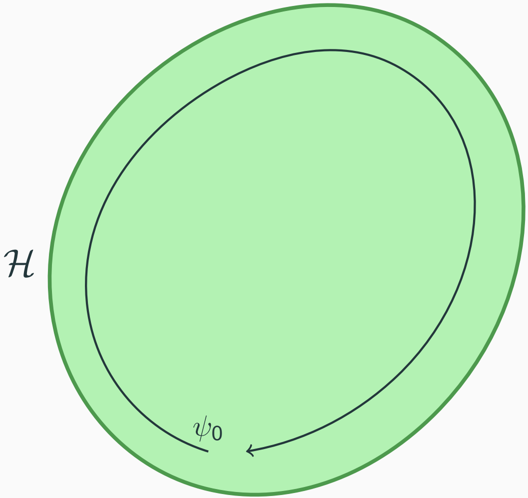

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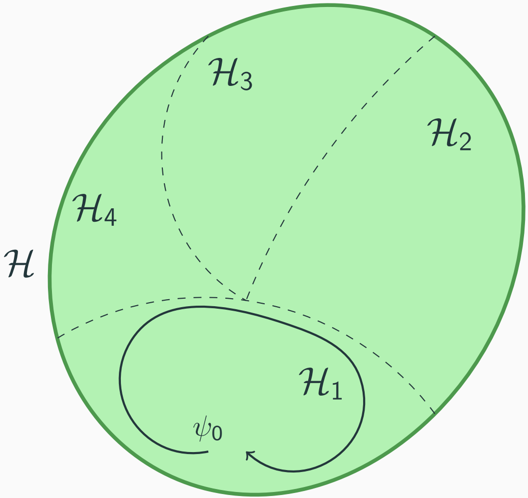

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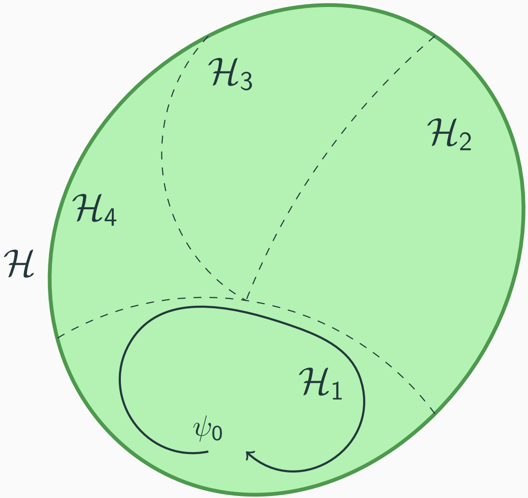

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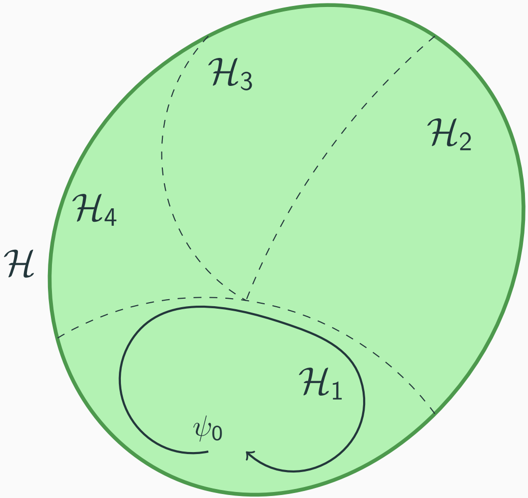

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