Estimation of the Nambu-Goto string thickness using continuous normalizing flows

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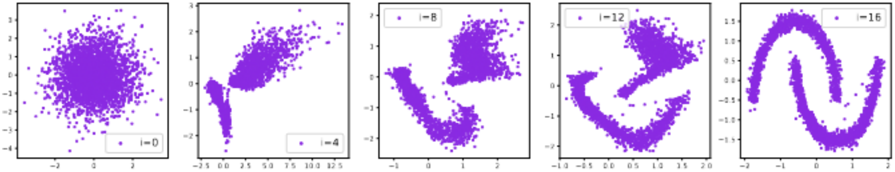
Outline

Normalizing flows



Normalizing flows

Normalizing flows (NFs) **[Rezende and Mohamed; 2015]** are a class of deep learning algorithm recently proposed to sample from Boltzmann distributions.

A NF g_θ is a parametric invertible and differentiable function that maps an easy-to-model prior distribution $q_0(z),\ z \in \mathbb{R}^n$, to an inferred distribution q_θ which approximate the target $p(\phi),\ \phi \in \mathbb{R}^n$.

$$g_\theta :\ q_0 \to q_\theta \simeq p$$

$$\phi = g_\theta(z)$$

$$q_\theta(\phi) = q_0(g^{-1}(\phi))|J_g|^{-1}$$

Targetting Boltzmann distributions

NFs can be trained to $q_\theta \simeq p(\phi)$ with $p(\phi) = \frac{1}{Z}\exp(-S[\phi])$ **[Albergo et al.; 2019]**,**[Noé et al.; 2019]** by minimizing the reverse Kullback-Leibler divergence:

$$D_{KL}(q_\theta || p) = \int d\phi q_\theta(\phi) \log \frac{q_\theta(\phi)}{p(\phi)} \geq 0.$$

Observables can be computed using a re-weighting procedure also called Importance Sampling (IS) in machine learning field **[Nicoli et al.; 2020]**:

$$\langle \mathcal{O} \rangle_{\phi \sim p} = \int D\phi p(\phi) \mathcal{O}(\phi) = \int D\phi q_\theta(\phi) \frac{p(\phi)}{q_\theta(\phi)} \mathcal{O}(\phi) \simeq \frac{1}{\hat{Z}} \langle \mathcal{O} \tilde{w} \rangle_{\phi \sim q_\theta}$$

where

$$\tilde{w} = \frac{e^{-S[\phi]}}{q_\theta(\phi)}$$

and

$$\hat{Z} = \langle \tilde{w} \rangle_{\phi \sim q_\theta}$$

State-of-the-art

The recent preprint **[Abbott et al.; 2211.07541]** reviewed the current status of normalizing flows for lattice field theory.

- Normalizing flows found successful application in toy-model application in $d = 1 + 1$ lattice field theory: **[Albergo et al.; 2019]**,**[Nicoli et al.; 2020]**, **[Albergo et al.; 2022]**,**[Abbott et al.; 2022]**. However, these results cannot be generalized, no general laws for scaling has been found **[Del Debbio et al.; 2021]**, and high dimensional problems requires additional researches.
- Physics-informed approach can be exploited to build efficient algorithms:
  - non-equilibrium thermodynamics: **[Caselle et al.; 2022]** (see **A. Nada's talk**)
  - symmetries: equivariant flows **[Kanwar et al.; 2020]**,**[Nicoli et al.; 2020]**
    - Continuous equivariant normalizing flows **[Gerdes et al.; 2022]** (inspired the models used in this works)

Update: the state-of-the-art for lattice gauge theory has been updated yesterday **[Bacchio et al.; 2022]**.

Nambu-Goto string

Effective string theory

Effective string theory is a non-perturbative framework that provide an effective description of the confining flux tube in term of vibrating string (see **[Caselle; 2021]** for a recent review)

$$\langle PP^{\dagger} \rangle = \int D\phi e^{-S_{eff}} \equiv Z(L, R, \sigma).$$

The most natural choice for S_{eff} is the Nambu-Goto string **[Nambu; 1974]**,**[Goto; 1971]**. In the $d = 2 + 1$ case, using a "physical gauge" the action can be defined as:

$$S_{NG}[\phi] = \sigma \sum_{x \in \Lambda} \left[\sqrt{1 + (\partial_{\mu}\phi)^2/\sigma} - 1 \right]$$

where Λ is a square lattice of size $L \times R$ with step $a = 1$, $\phi(x) = \phi(\tau, \epsilon) \in \mathbb{R}$ and boundary conditions: $\phi(\tau + L, \epsilon) = \phi(\tau, \epsilon)$ and $\phi(\tau, 0) = \phi(\tau, R) = 0$.

The choice of the physical gauge is anomalous at quantum level, however, the anomaly vanishes at large distance. **[Aharony and Komargodski; 2013]**.

Free Boson expansion

In the limit $\sigma \to \infty$ the Nambu-Goto action can be expanded in series:

$$S_{NG} \sim S_{FB} + S_{FB^4} + O(\sigma^{-2})$$

where:

$$S_{FB} = \frac{1}{2} \sum_{x \in \Lambda} (\partial_\mu \phi)^2$$

and

$$S_{2nd} = -\frac{1}{8\sigma} \sum_{x \in \Lambda} \left[(\partial_\mu \phi)^2 \right]^2$$

with

$$(\partial_\mu \phi)^2 = (\partial_\tau \phi)^2 + (\partial_\epsilon \phi)^2$$

Finite size analytical solutions

The renormalized analytical solution of the partition function is well know for all the actions previusly introduced **[Lüsher et al.; 1980]**,**[Lüscher; 1981]**,**[Alvarez; 1981][Caselle and Pinn; 1996]**,**[Billo' and Caselle; 2005]**.
However, the finite size solution can be found only for S_{FB} and S_{FB^4}.

Both solutions exploit a diagonalization with eigenvalues:

$$\lambda_{m,n} = 4\sin^2\left(\frac{\pi m}{L}\right) + 4\sin^2\left(\frac{\pi n}{2R}\right)$$

Thus:

$$\beta F_{FB} = -\log Z_{FB} = \frac{1}{2}\sum_{m=1,n=1}^{L,R-1} \log\left(\frac{\lambda_{m,n}}{2\pi}\right)$$

and:

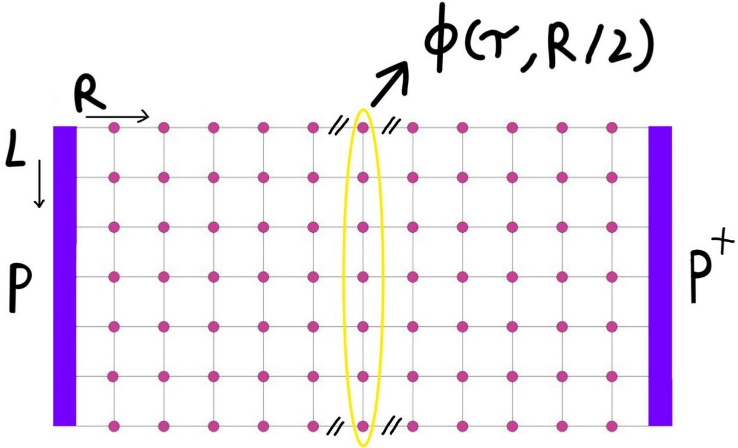
$$\beta F_{FB^4} = \beta F_{FB} - \frac{\mathcal{R}(\lambda)}{\sigma LR}$$

String thickness

The thickness of the strings can be computed as:

$$\sigma w^2 = \langle h^2 \rangle$$

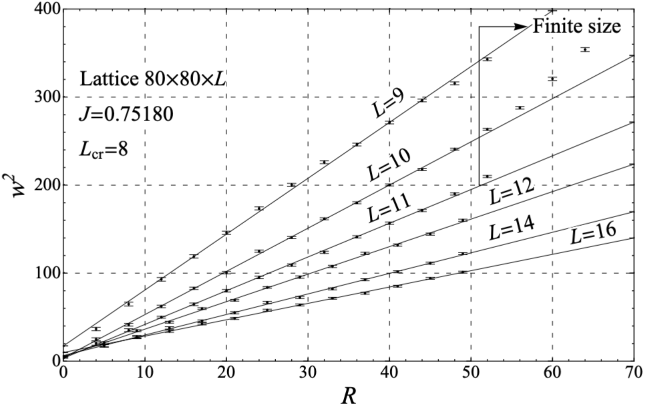
where $h^2 = \langle \phi^2(\tau, R/2) \rangle_\tau$



Free boson thickness

The analytical solution for σw^2 of the Free Boson is well known **[Lüscher et al.; 1981][Caselle and Allais; 2009]**:

- $L \gg R$: $\sigma w^2 = \frac{1}{2\pi} \log \frac{R}{Rc} + \log \frac{R}{1+4\exp[-\frac{\pi T}{2R}]} + \dots$
- $R \gg L$: $\sigma w^2 = \frac{1}{2\pi} \log \frac{L}{Lc} + \frac{R}{4L} + \dots$



Numerical results

Goals of the numerical studies

The goal of our studies is provide a numerical analysis of the Nambu-Goto string thickness.
To validate our results, we checked:

- Test of the performance of the algorithms using the Effective Sample Size:

$$ESS = \frac{\langle \tilde{w} \rangle^2}{\langle \tilde{w}^2 \rangle}$$

  We accepted all the models with $ESS > 0.1$

- Benchmark of the Nambu-Goto numerical free energy using F_{FB} and F_{FB^4}.
- Derivation of the FB thickness from NG simulations (focus on $R \gg L$).

We run the models on NVIDIA Tesla V100 GPU of the Marconi100 (CINECA).

Continuous normalizing flows

Exploiting Neural Ordinary Differential Equations (NODE) **[Chen et al.; 2018]** is possible to build Continuous NFs (CNFs) in which g_θ is the solution of an ODE parameterized by a neural network V_θ:

$$\frac{d\phi(t)}{dt} = V_\theta(\phi(t), t)$$

with

$$\phi(t=0) = z \sim \mathcal{N}(0, \mathbb{1}/2) \;\; and \; \phi(t=T) = \phi$$

Thus:

$$\phi(T) = ODESOLVER(V_\theta, \phi(0), [0, T])$$

The density of the generated samples can be computed through the ODE:

$$\frac{d \log q_\theta(\phi(t))}{dt} = -(\nabla \cdot V_\theta)(\phi(t), t)$$

CNF: architectures

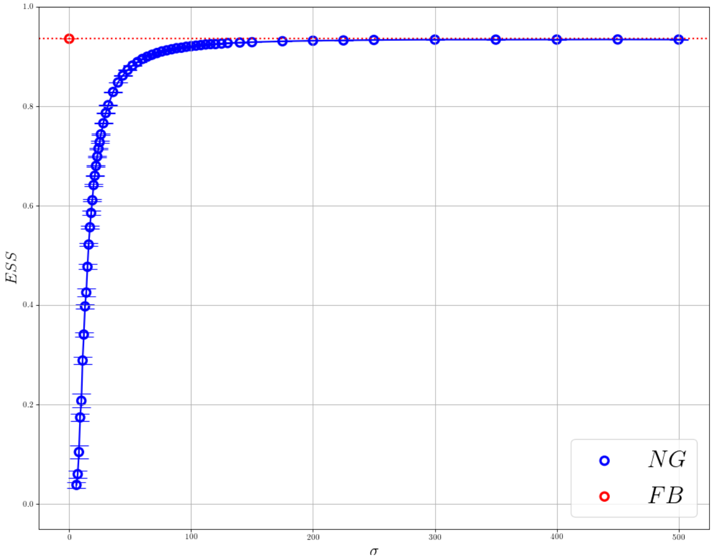
The architecture used is a continuous in time linear model inspired by **[Gerdes et al.; 2022]** and **[Köhler et al.; 2020]**:

$$V_\theta(\phi(t), t) = \sum_{y,d} W_{x,y,d} K(t)_d \phi(t)_y$$

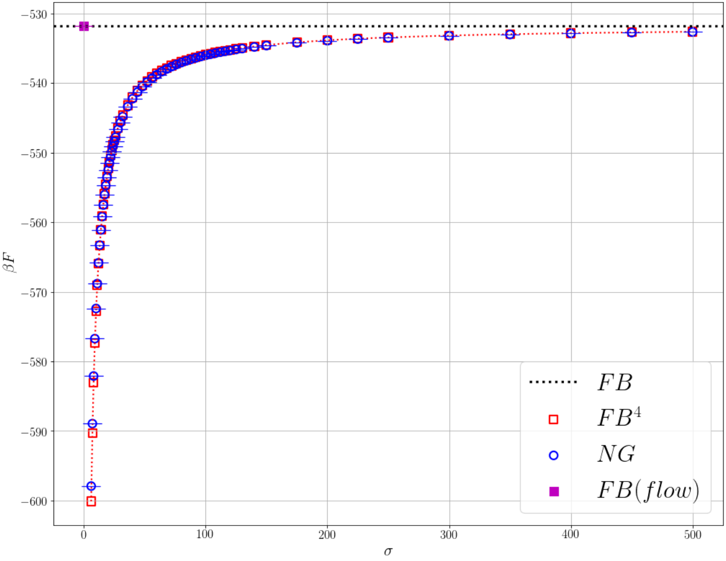
$$(\nabla \cdot V_\theta)(\phi(t), t) = \mathrm{Tr}\left[\sum_d W_d K(t)_d\right]$$

Where $K(t) \in \mathbb{R}^D$ is a temporal kernel of D Fourier coefficients, $W \in \mathbb{R}^{A \times A \times D}$ is a linear neuron with $A = L \times R$.

Nambu-Goto: Scaling $L \times R = 40 \times 41$

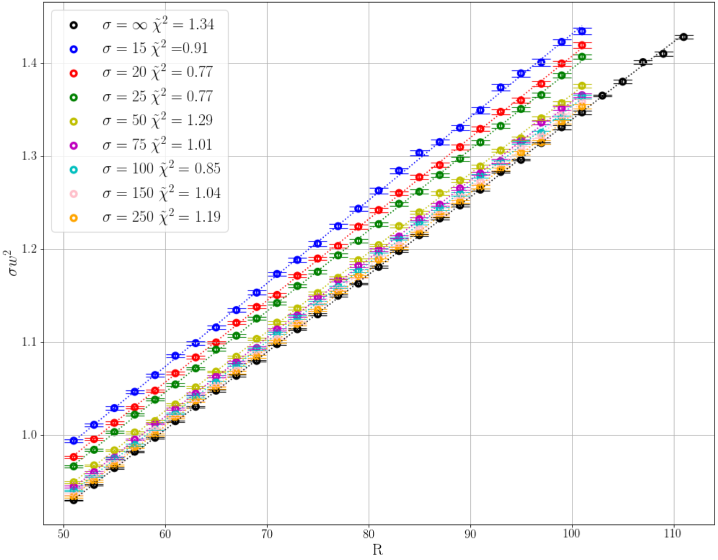


Nambu-Goto: Free energy $L \times R = 40 \times 41$



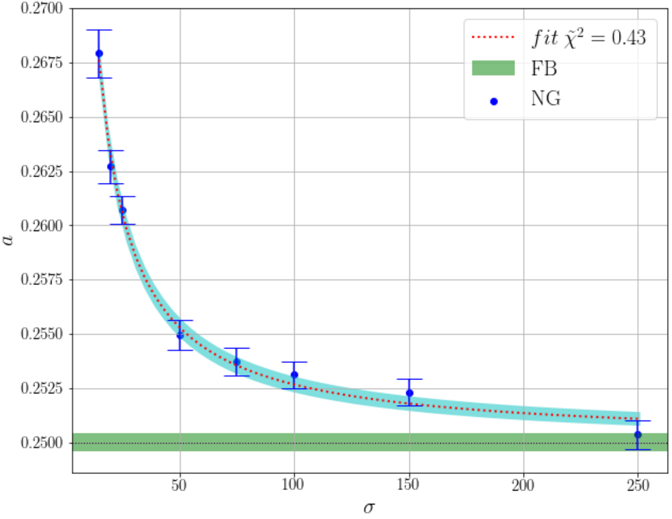
Nambu-Goto: Thickness $R \gg L = 30$

Fit: $ax + b$



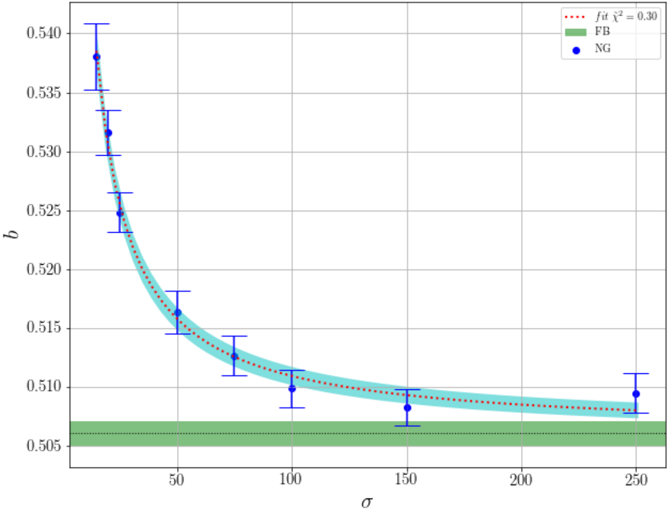
Nambu-Goto: $a(\sigma)$

Fit: $\frac{A}{x} + B$, $A = 0.264 \pm 0.008$, $B = 0.2500 \pm 0.0003$ (expected 0.25) $\tilde{\chi}^2 = 0.43$



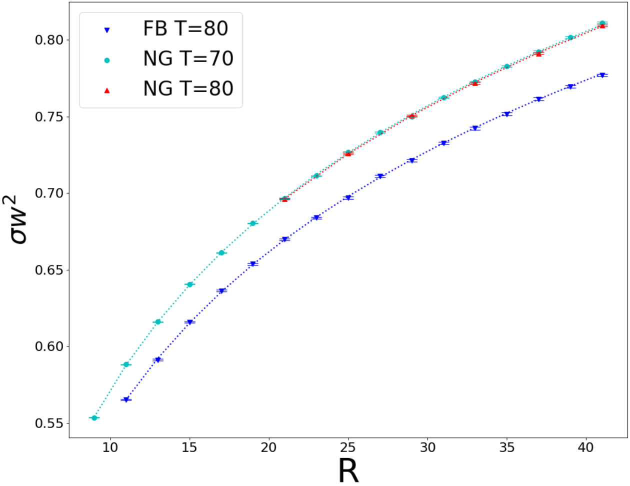
Nambu-Goto: $b(\sigma)$

Fit: $\frac{A}{x} + B$, $A = 0.49 \pm 0.02$, $B = 0.5061 \pm 0.0006$ (expected 0.506 ± 0.001)
$\tilde{\chi}^2 = 0.30$



Nambu-Goto: Thickness $L \gg R$

Preliminary results



Outlook

Outlook

In this contribution we show how to compute the thickness of confining flux tube using effective string theory and normalizing flows. Natural future developments of this contribution will concern:

- Analysis and interpretation of the results using **[Gliozzi, Pepe and Wiese; 2010]** as benchmark.
- Physics-informed customization:
  - exploit the so called "$T\bar{T}$" irrelevant perturbation to obtain Nambu-Goto samples using the free boson as prior **[Cavaglià et al.; 2016]**,**[Smirnov and Zamolodchikov; 2016]**

Thank you for your attention!

Hyperparameters

We trained our algorithms for 10000 epochs with 10000 samples using ADAM ($lr = 0.0001$) as optimizer and cosine annealing scheduler.

We computed the observables using jackknife with 500000 samples and 5000 as JK batch size.

Linear model: $D = 3$

$$K(t) = \left[1, \cos\left(\frac{2\pi t}{T}\right), \sin\left(\frac{2\pi t}{T}\right)\right]$$

Analytical solution (1)

Consider the free-boson action:

$$S[\phi] = \frac{1}{2}\sum_x (\partial_\mu \phi(x))^2 = \frac{1}{2}\sum_x \phi(x)\Box\phi(x)$$

where $x = (\tau, \epsilon)$ and

$$\Box\phi(x) = \sum_\mu \Big(\phi(x) - \phi(x+\hat{\mu}) - \phi(x-\hat{\mu})\Big)$$

It is possible to diagonalize this action using the orthormal basis:

$$\Psi(m,n,\tau,\epsilon) = \frac{2}{\sqrt{2LR}}\left(\cos\left(\frac{2m\pi\tau}{L}\right) + \sin\left(\frac{2m\pi\tau}{L}\right)\right)\sin\left(\frac{n\pi\tau}{R}\right)$$

with

$$y(k) = \sum_x \Psi(k,x)\phi(x)$$

thus:

$$S[y] = \frac{1}{2}\sum_k \lambda_k y(k) \; and \; \lambda_k = \sum_x \Psi(k,x)\Box\Psi(k,x)$$

Analytical solution (2)

The eigenvalues take the form of:

$$\lambda_{m,n} = 4\sin^2\left(\frac{m\pi}{L}\right) + 4\sin^2\left(\frac{2\pi}{2R}\right)$$

Finally:

$$Z = \int D\phi e^{-S[\phi]} = \prod_k \sqrt{\frac{2\pi}{\lambda_k}}$$

Consider now the action:

$$S_4[\phi] = -\frac{1}{8\sigma}\sum_{x\in\Lambda}\left[(\partial_\mu\phi)^2\right]^2$$

The corresponding free-energy is **[Caselle and Pinn; 1996]**:

$$F_4 = -\frac{\mathcal{R}}{\sigma LR}$$

Analytical solution (3)

Where

$$\mathcal{R} = \mathcal{R}_\infty + \mathcal{R}_f$$

with:

$$\mathcal{R}_\infty = \frac{1}{4}(G_1 + G_2)^2$$

and

$$\mathcal{R}_f = \frac{1}{8}(G_1 - G_2)^2$$

G_μ is defined as:

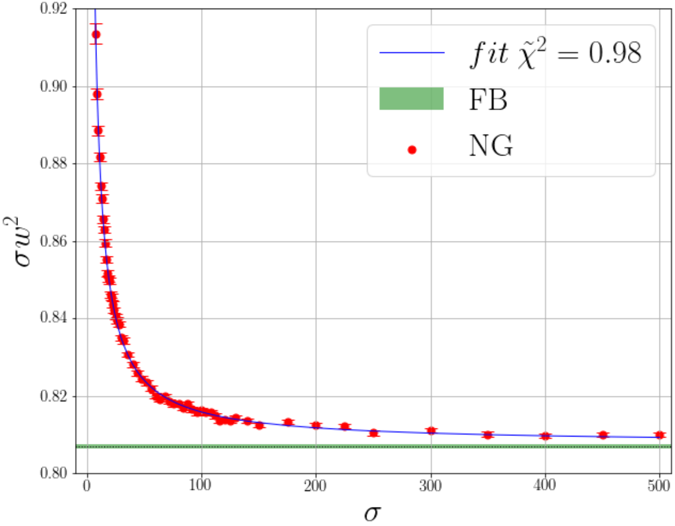
$$G_1 = -\sum_{m=1}^{L}\sum_{n=1}^{R-1} \frac{sin^2(\frac{m\pi}{L})}{sin^2(\frac{m\pi}{L}) + sin^2(\frac{n\pi}{2R})}$$

and

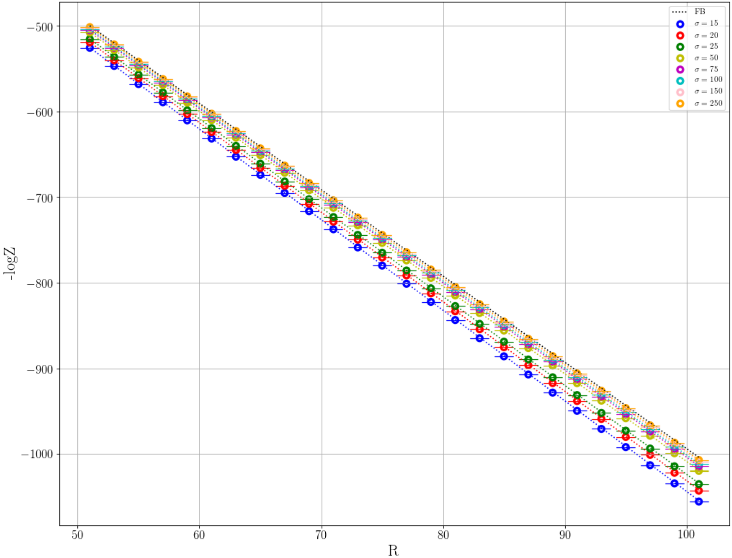
$$G_2 = -\sum_{m=1}^{L}\sum_{n=1}^{R-1} \frac{sin^2(\frac{n\pi}{2R})}{sin^2(\frac{m\pi}{L}) + sin^2(\frac{n\pi}{2R})}$$

Nambu-Goto: Thickness $L \times R = 40 \times 41$

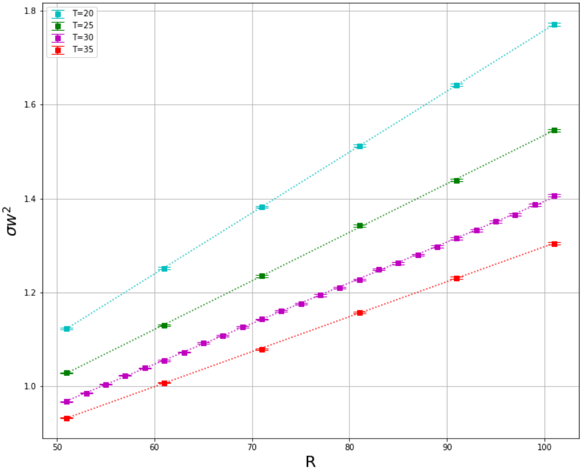
Fit: $\frac{a}{x} + b$



Nambu-Goto Free Energy: $R \gg L = 30$

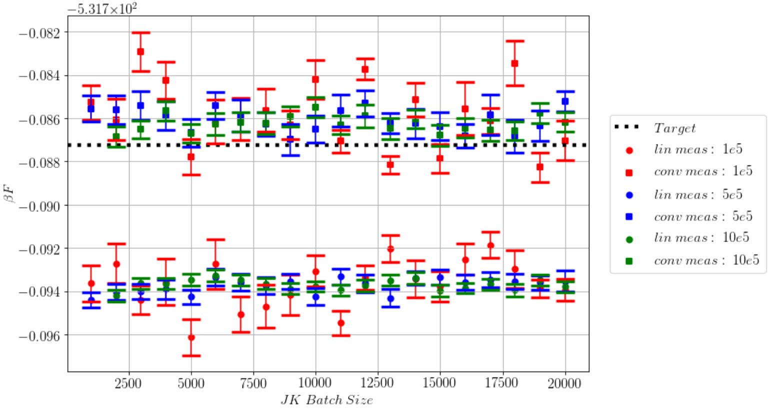


Nambu-Goto Thickness: $R \gg L$



Free Boson: Bias $L \times R = 40 \times 41$

ESS: Linear model$\simeq 0.9$, Convolutional$\simeq 0.8$. Estimated relative percentile bias: $b^L_{\%} = 0.00121 \pm 0.00002$ and $b^C_{\%} = 0.00024 \pm 0.00003$.



We choose the Linear model to target the Nambu-Goto actions; time for 100 epoch of 10000 configurations: Linear=135 seconds, Convolutional=586 seconds