Lattice determination of the topological susceptibility slope χ' of $2d$ CP^{N-1} models at large N

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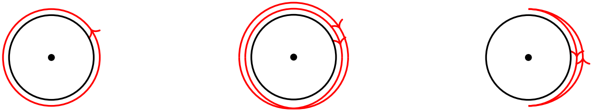
Talk based on:

C. Bonanno, "*Lattice determination of the topological susceptibility slope χ' of $2d$ CP^{N-1} models at large N*", arXiv:2212.02330 [hep-lat]

The topological susceptibility slope χ'

In gauge theories, the **integer** topological charge $Q = \int d^d x \, q(x) \in \mathbb{Z}$ is a quantity corresponding to the number of windings of the gauge field around the group manifold at $x \to \infty$:



$Q = 1$ $Q = -2$ $Q = 0$

Let us consider the **momentum expansion** of the Topological Charge Density Correlator up to $\mathcal{O}(p^4)$:

$$\widetilde{G}(p^2) = \int d^d x \, e^{ip\cdot x} \, \langle q(x)q(0)\rangle = \chi - \chi' p^2 + \mathcal{O}(p^4),$$

$$\chi = \int d^d x \, \langle q(x)q(0)\rangle = \frac{\langle Q^2 \rangle}{V}, \qquad \chi' = \frac{1}{2d} \int d^d x |x|^2 \, \langle q(x)q(0)\rangle \, .$$

LO term $\to$ well-known topological susceptibility (see next talk by F. D'Angelo).

NLO term $\to$ topological susceptibility slope χ', subject of this talk.

Physical relevance of the susceptibility slope χ'

Interesting implications of χ' in QCD and QCD-like theories

- $\mathrm{U}(1)_\mathrm{A}$ **anomaly**: Witten–Veneziano mechanism relates η' mass to the large-N limit of the top. susceptibility $\overline{\chi}_{\mathrm{YM}}$ of $\mathrm{SU}(N)$ pure-gauge theories. This relation holds if $|\overline{\chi}'_{\mathrm{YM}}| \ll \overline{\chi}_{\mathrm{YM}}/m^2_{\eta'}$ in the large-N limit
(Di Vecchia & Veneziano, Nucl. Phys. B **171**, 253, 1980)
- **Proton "spin"**: in QCD the chiral limit of χ'_{QCD} is related to the spin-polarized proton matrix element of the axial current $J^a_{5,\mu}$
(Shore & Veneziano, Phys. Lett. B **244**, 75, 1990)
- **Lower dim. theories**: in $2d$ CP^{N-1} models, it is possible to compute the large-N limit of χ' analytically within the $1/N$ expansion scheme:

$$\chi' = -\frac{3}{10\pi}\frac{1}{N} + 1.53671\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

(Campostrini & Rossi, Phys. Lett. B **272**, 305, 1991)

Only preliminary lattice attempts to compute χ' (Boyd, Alles, D'Elia & Di Giacomo, 1997 [hep-lat/9711025]). Recently revived by QCD Sum Rule (Narison, 2022 [arXiv:2111.02873])

$\longrightarrow$ **Goal**: determine χ' on the lattice with state-of-the-art techniques.

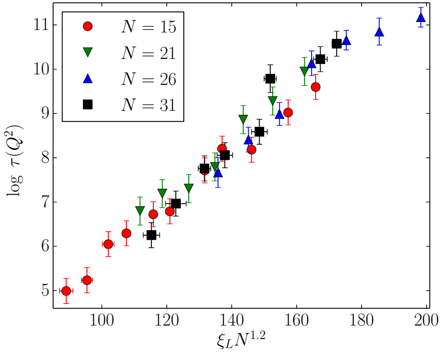
We start by investigating the simpler case of lattice $2d$ CP^{N-1} models at large N.
Main issues to be addressed: **topological freezing** and **impact of smoothing**.

Critical Slowing Down and Topological Freezing

Approaching the continuum limit $a \to 0$ ($\xi_L = \xi/a \to \infty$), Monte Carlo Markov Chains experience a **Critical Slowing Down** (CSD) when local updating algorithms (e.g., heat-bath) are employed.

CSD = autocorrelation time $\tau(\mathcal{O})$, i.e., number of updating steps to generate two gauge configurations with uncorrelated values of $\mathcal{O}$, grows with $1/a \sim \xi_L$.

For topological quantities, **CSD** is particularly severe, further worsens increasing N.



CB, Bonati, D'Elia, 2018 [arXiv:1807.11357]

Numerical evidence that $\tau(Q^2)$ diverges as $\sim \exp N$ at fixed ξ_L and vice-versa.

Adopted solution: **Parallel Tempering on Boundary Conditions**.

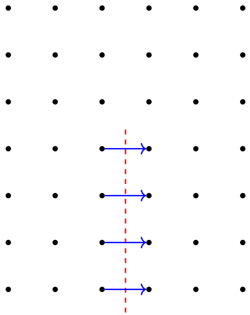
Proposed for $2d$ CP^{N-1} models (Hasenbusch, 2017 [1706.04443]) and employed to study topology and θ-dependence in these theories (Berni, CB, D'Elia, 2019 [1911.03384]).

Recently implemented also in $4d$ SU(N) **pure-gauge theories** (CB, Bonati, D'Elia, 2021 [2012.14000]; CB, D'Elia, Lucini, Vadacchino, 2022 [2205.06190])

The Algorithm

- consider a collection of N_r lattice replicas
- replicas differ for boundary conditions on **small** sub-region: *the defect*
- each replica is updated with standard methods
- after updates, propose swaps among configurations via Metropolis test

The Defect



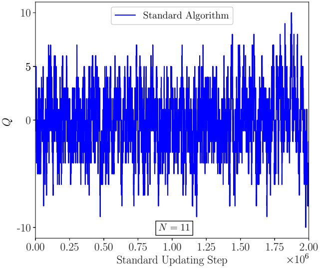
• Links crossing the defect: $\beta \to \beta \cdot c(r)$.

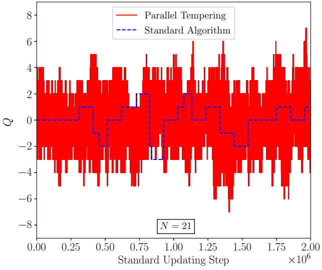
• Periodic: $c = 1$. Open: $c = 0$.
Interpolating replicas: $0 < c(r) < 1$.

• Thanks to swaps, configuration does *random walk* through replicas $\Longrightarrow$ Faster decorrelation of Q in open replica is transferred to the periodic one.

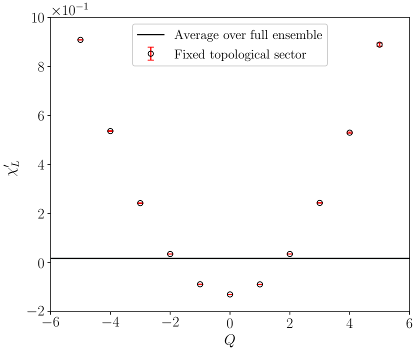
• Observables are computed on periodic replica $\to$ easier to have finite-size effects under control (no boundary effects on the correlator).

Improvement at large N





Adopted lattice topological charge $Q \longrightarrow$ geometric lattice definition
$Q_L = \sum_x q_L(x) = \frac{1}{2\pi}\sum_x \arg \Pi_{01}(x) \in \mathbb{Z}$
computed after cooling



Very important to ensure that topological sectors are properly sampled.
The lattice value of the susceptibility slope

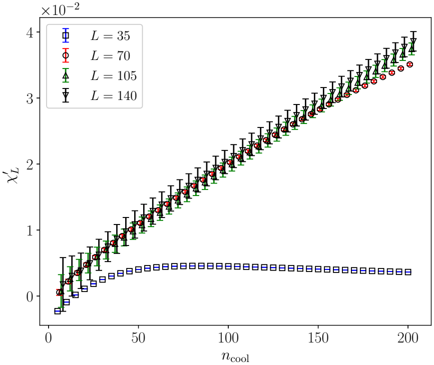
$$\chi'_L = \left\langle \sum_x d^2_{x,0}\, q_L(x) q_L(0) \right\rangle$$

exhibits a non-trivial dependence on the fixed topological sector Q within which it is computed.

Smoothing

- Smoothing algorithms damp short-distance fluctuations up to a **smoothing radius** $r_s \propto \sqrt{\text{amount of smoothing}}$
- Leaves global topological charge Q unaltered $\implies$ typically amount of smoothing not critical for susceptibility $\chi \propto \langle Q^2 \rangle$
- Smoothing **modifies short-distance behavior** of Topological Charge Density Correlator $\implies$ quantities like χ' exhibit a non-trivial dependence on the amount of smoothing

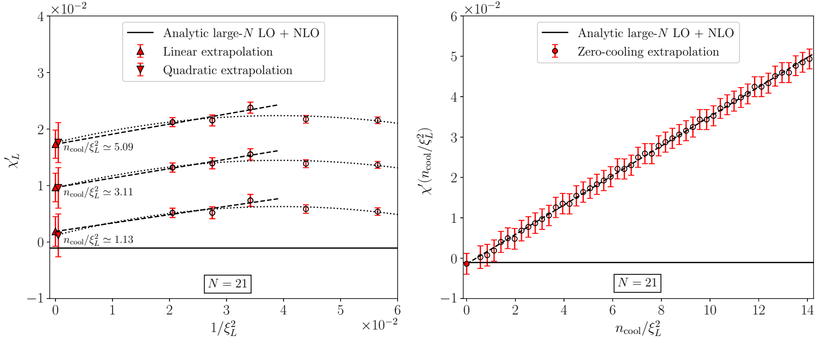
Example below: smoothing method = cooling, $\chi'_L(n_{\text{cool}})$ for several lattice sizes ($N = 11$)



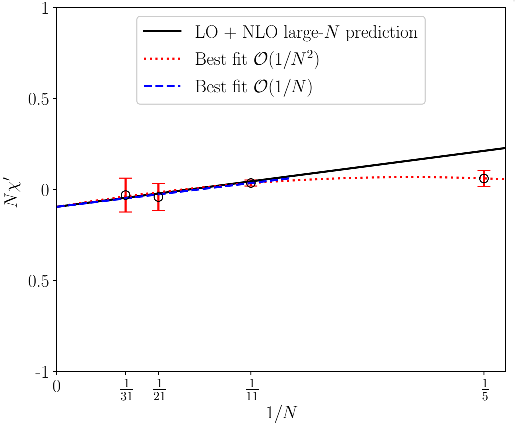
Double extrapolation

Strategy to compute χ' (following the steps of Altenkort et al., 2020 [arXiv:2012.08279])

1. Continuum limit at fixed smoothing radius r_s
   - consider determinations of χ'_L with same value of $n_{\text{cool}}/\xi_L^2 \propto (r_s/\xi)^2$
   - $\chi'(n_{\text{cool}}/\xi_L^2) = \chi'_L\big|_{n_{\text{cool}}/\xi_L^2} + c_1/\xi_L^2 + \dots$
2. Zero-smoothing-limit $r_s \to 0$
   - gradient flow formalism predicts (Altenkort et al., 2020 [arXiv:2012.08279]) $\langle q(x)q(0)\rangle(\tau_{\text{flow}}) = \langle q(x)q(0)\rangle + c_1\tau_{\text{flow}} + c_2\tau_{\text{flow}}^2 + \dots$
   - numerical equivalence between cooling and gradient flow: $n_{\text{cool}} \propto \tau_{\text{flow}}$ (Bonati, D'Elia, 2014 [arXiv:1401.2441])
   - We can expect $\implies \chi'(n_{\text{cool}}/\xi_L^2) = \chi' + c_1 n_{\text{cool}}/\xi_L^2 + \dots$



Large-N limit of χ' (after double extrapolation)



Analytic:

$$N\chi' = -\frac{3}{10\pi} + 1.53671\frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

Best fits:

$$N\chi' = -\frac{3}{10\pi} + 1.44(18)\frac{1}{N}$$

$$N\chi' = -\frac{3}{10\pi} + 1.97(37)\frac{1}{N} - 5.9(2.6)\frac{1}{N^2}$$

Coefficients grow in abs. value with alternating sign, similarly to what happens to χ and b_2 (Berni, CB, D'Elia, 2019 [arXiv:1911.03384]) $\Longrightarrow$ no surprise, $1/N$ series is asymptotic

Parting Remarks

Summary & Conclusions

- Combining state-of-the-art algorithms (Parallel Tempering on Boundary Conditions) and numerical techniques (double continuum + zero-smoothing extrapolations) it is possible to reliably determine topological susceptibility slope χ' from lattice Monte Carlo simulations
- results obtained for $2d$ CP^{N-1} models in the large-N limit are in perfect agreement with analytic predictions obtained with the $1/N$ expansion

Future Outlooks

- Currently in progress: investigation of χ' in SU(3) pure-gauge theory, in view of a study of its large-N limit
- Near future: investigation of χ' in full QCD (trickier computation, now χ' is no more a Renormalization-Group Invariant)