From classical to quantum Markov chains: known and new results

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Joint work with P. Facchi and A. Konderak (UNIBA)

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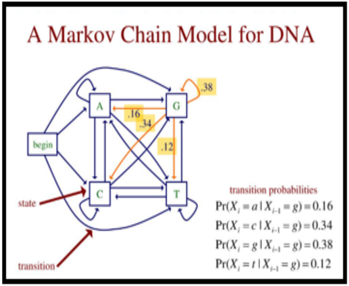
Markov chains ⇒ **memoryless** process

Classical Markov chains: an appetizer

Markov chains ⇒ **memoryless** process

Applications:

- physics
- economics
- biology
- chemistry
- ......



Goal: sample a known probability distribution π

Solution: sample a Markov chain with stationary distribution π

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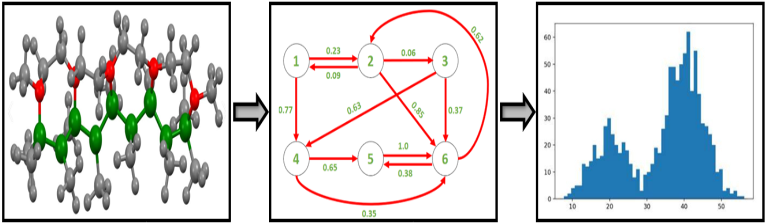

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Markov chain Monte Carlo methods

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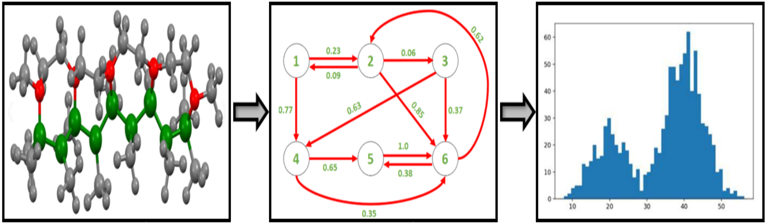

Main message

$$\text{Markov chain Monte Carlo} = (\text{Markov chain problem})^{-1}$$

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Discrete-time (classical) Markov chain

$X_1, X_2, \dots$ random variables in $(\Omega, \mathcal{F}, \mathbb{P})$

Ω **finite** state space, $\mathrm{card}(\Omega) = d$

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$$\mathbb{P}(X_n = x | X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = x | X_{n-1} = x_{n-1})$$

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- $Q_{ij} := \mathbb{P}(X_{n+1} = x_i | X_n = x_j)$ **stochastic** (or **transition**) matrix

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  - $Q_{ij} \geq 0$ (**non-negativity**)
  - $\sum_i Q_{ij} = 1$ (**normalization**)

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Normalization $\Rightarrow Q\pi = \pi$ **stationary** distribution

$\mathbb{P}(X_n = x) = Q^n p(x)$ probability distribution at $t = n$

Long-time behaviour of Markov chains

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Asymptotic limit: $n \to \infty$

Q stochastic matrix, $\sigma(Q)$ spectrum of Q

$\sigma_P(Q) = \{\lambda \in \sigma(Q) \,|\, |\lambda| = 1\}$ **peripheral** spectrum of Q

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Peripheral eigenvalues (eigenvectors) related to the asymptotic behaviour

Irreducible and primitive stochastic matrices

Q stochastic matrix

- Q **irreducible** $\Leftrightarrow$ 1 simple eigenvalue with positive eigenvector
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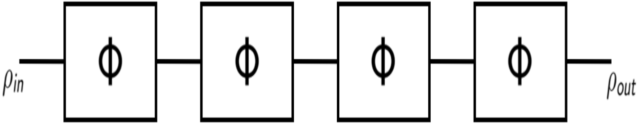
Quantum Markov chains: motivation

1) Quantum information theory: **circuit** notation



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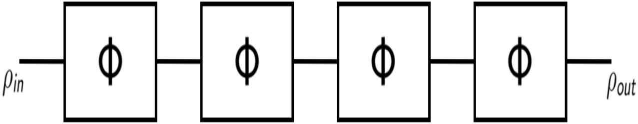
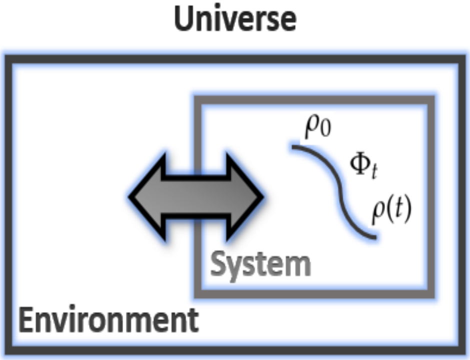


2) Attractor quantum neural networks (aQNNs):

Storage capacity $\Leftrightarrow$ maximum # **stationary** states noisy dynamics

M. Lewenstein et al., Quantum Sci. Technol. **6**, 045002 (2021).
M. Lewenstein et al., Quantum Sci. Technol. **7**, 029502 (2022).

Open quantum systems



Quantum Markov chains: basics

Classical Markov chain $\leftrightarrow$ Quantum Markov chain

Finite state space $\leftrightarrow$ d-dimensional **Hilbert** space

T. Heinosaari, and M. Ziman, The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement, Cambridge University Press (2011).

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Complete positivity is a **purely quantum** concept!

T. Heinosaari, and M. Ziman, The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement, Cambridge University Press (2011).

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Main message

Every quantum channel may be decomposed in terms of unitary and irreducible ones in the asymptotic limit

D. Amato, P. Facchi, and A. Konderak, arXiv:2210.17513 [quant-ph] (2022).

Concluding remarks

- **Long-time** behaviour of **classical** Markov chains related to **irreducible** stochastic matrices

- **Asymptotics** of **quantum** Markov chains linked to **unitary** and **irreducible** quantum channels

- Quantum **Channels** (stochastic matrices) with a **given asymptotics** may be constructed

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J. J. McDonald, Linear Algebra Appl. **363**, 217 (2003).

D. Amato, P. Facchi, and A. Konderak, arXiv:2210.17513 [quant-ph] (2022).

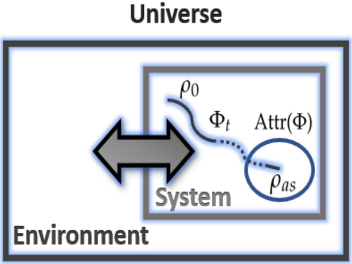
Thanks for your attention.

Attractor subspace and asymptotic map

Asymptotic or **attractor** subspace $\mathrm{Attr}(\Phi)$

$$\mathrm{Attr}(\Phi) = \mathrm{span}\{X \in \mathcal{M}_d \,|\, \Phi(X) = \lambda X \text{ for some } \lambda \in \sigma_P(\Phi)\}$$

$\hat{\Phi}_P = \Phi|_{\mathrm{Attr}(\Phi)}$ **asymptotic map** of Φ



M. M. Wolf, Quantum channels & operations: Guided tour, Lecture Notes (2012).

Asymptotic dynamics: structure theorem

$$\mathrm{Attr}(\Phi) \ni X = 0_{d_0} \oplus \bigoplus_{k=1}^{M} X_k \otimes \rho_k$$

- $X_k \in \mathcal{M}_{d_k}$
- $0 < \rho_k \in \mathcal{M}_{m_k}$ density matrix
- $d_0 + \sum_{k=1}^{M} d_k m_k = d$

$$\hat{\Phi}_P(X) = 0_{d_0} \oplus \bigoplus_{k=1}^{M} U_k X_{\pi(k)} U_k^{\dagger} \otimes \rho_k$$

- $U_k \in \mathcal{M}_{d_k}$ unitary
- π **permutation** on $\{1, \ldots, M\}$

M. M. Wolf, and D. Perez-Garcia, arXiv:1005.4545 [quant-ph] (2010).

Asymptotic dynamics: cyclic decomposition

$$\mathrm{Attr}(\Phi) \ni X = 0_{d_0} \oplus \bigoplus_{c=1}^{L} X_c \otimes Y_c$$

- $X_c \in \mathcal{M}_{\tilde{d}_c}$
- $Y_c \in \mathrm{Attr}(\Phi_c)$, Φ_c **irreducible** channel on $\mathcal{M}_{\tilde{m}_c}$
- $d_0 + \sum_{c=1}^{L} \tilde{m}_c \tilde{d}_c = d$

$$\hat{\Phi}_P(X) = 0_{d_0} \oplus \bigoplus_{c=1}^{L} \tilde{U}_c X_c \tilde{U}_c^\dagger \otimes \Phi_c(Y_c)$$

- $\tilde{U}_c \in \mathcal{M}_{\tilde{d}_c}$ unitary
- **Factorization** of permutations and unitary evolutions

D. Amato, P. Facchi, and A. Konderak, arXiv:2210.17513 [quant-ph] (2022).

Asymptotic dynamics of unitary and irreducible channels

Unitary channel

$$\text{Attr}(\Phi) = \mathcal{M}_d, \quad \hat{\Phi}_P = \Phi$$

Irreducible channel

$$\text{Attr}(\Phi) \ni X = \bigoplus_{k=1}^{M} c_k \rho_k, \quad \hat{\Phi}_P(X) = \bigoplus_{k=1}^{M} c_{\pi(k)} \rho_k$$

- $c_k \in \mathbb{C}$
- π **cyclic** permutation on $\{1, \dots, M\}$

Quantum channels with a given asymptotics

Given

$$\mathcal{K} = 0_{d_0} \oplus \bigoplus_{k=1}^{M} \mathcal{M}_{d_k} \otimes \rho_k$$

and $\Psi : \mathcal{K} \mapsto \mathcal{K}$ of the form

$$\Psi(X) = 0_{d_0} \oplus \bigoplus_{k=1}^{M} U_k X_{\pi(k)} U_k^\dagger \otimes \rho_k$$

then there exists a quantum channel Φ such that

$$\mathrm{Attr}(\Phi) = \mathcal{K}$$

$$\hat{\Phi}_P = \Psi$$

The proof is **constructive**!

D. Amato, P. Facchi, and A. Konderak, arXiv:2210.17513 [quant-ph] (2022).