On the origin of the correspondence between integrable models and differential equations.

A possible explanation of the ODE/IM correspondence

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State of the art: the ODE/IM correspondence

• Consider the ODE (Schroedinger)

$$-\frac{d^2}{dx^2}\psi(x)+\left(\frac{\ell(\ell+1)}{x^2}+x^{2M}\right)\psi(x)=E\psi(x)$$

A solution $y(x)\sim x^{-M/2}\exp\left(-\frac{x^{M+1}}{M+1}\right)$ as $x\to+\infty$

Two solutions $\chi_+\sim x^{\ell+1}$, $\chi_-\sim x^{-\ell}$ as $x\to 0$

Connection coefficients $Q_\pm(E)$: $y(x)=Q_+(E)\chi_-(x)+Q_-(E)\chi_+(x)$

$Q_\pm(E)$ are vacuum eigenvalues of Q-operators (Q-functions) of CFT minimal models Dorey,Tateo; Bazhanov, Lukyanov, Zamolodchikov '98

• Generalisation: PDEs $(\partial_w+V(w,\bar w))\Psi=(\partial_{\bar w}+\bar V(w,\bar w))\Psi=0$, $V,\bar V$ 2x2 matrices: a Lax pair. The compatibility condition $\partial_w\bar V-\partial_{\bar w}V+[V,\bar V]=0$ defines classical equations for the entries of $V,\bar V$.

A particular choice of V and $\bar V$ depends on a field $\hat\eta$, solution of the classical sinh-Gordon equation. In this case connection coefficients between different vector solutions Ψ are Q-functions of sine-Gordon model Gaiotto-Moore-Neitzke '08,'09; Lukyanov, Zamolodchikov '10

• Infinite number of conserved charges: vacuum eigenvalues $I_n,\bar I_n$ appear in asymptotic expansion at $|\mathrm{Re}\theta|\to+\infty$ of Q-functions $Q_\pm(\theta)$

Plan: give an explanation for ODE/IM

Why do ODE/IM appear? To answer this question, we reverse the arrow. We start from quantum integrable field theories: for a large class of them a state is characterised by Baxter's *TQ*-relations (T is the eigenvalue of the transfer matrix)

$$T(\theta)Q_{\pm}(\theta) = \phi_1(\theta)Q_{\pm}(\theta + i\gamma) + \phi_2(\theta)Q_{\pm}(\theta - i\gamma)\,,$$

with T, $Q_{\pm}$ entire (state dependent) functions and ϕ_i given functions. When $\theta = \theta_n^+$ ($\theta = \theta_n^-$) zero of Q_+ (or Q_-), a *TQ*-relation implies Bethe equations

$$\phi_1(\theta_n^{\pm})Q_{\pm}(\theta_n^{\pm} + i\gamma) + \phi_2(\theta_n^{\pm})Q_{\pm}(\theta_n^{\pm} - i\gamma) = 0\,.$$

We want to associate to a state of a quantum integrable model a classical model: two PDEs (Lax pair). Tool: A Marchenko-like equation Marchenko '55

We will discuss the example of sine-Gordon model in the vacuum, but the discussion can be made more general.

Functional relations

Example of sine-Gordon model on a cylinder

$$\mathcal{L} = \frac{1}{16\pi}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] + 2\mu\cos\beta\varphi\,, \quad \varphi(x+R,t) = \varphi(x,t)$$

Different k-vacua: $\varphi \to \varphi + 2\pi/\beta \Rightarrow |\Psi_k\rangle \to e^{2\pi i k}|\Psi_k\rangle$. Q-functions are $Q_\pm(\theta)$ ($\pm$ sign of k). Some properties of $Q_\pm$.

- Entire quasi-periodic functions: $Q_\pm(\theta + i\tau) = e^{\pm i\pi\left(\ell+\frac{1}{2}\right)} Q_\pm(\theta)$, $\ell = 2|k| - 1/2$, quasi-period $\tau = \pi/(1-\beta^2)$
- TQ-relation

$$T(\theta)Q_\pm(\theta) = e^{\mp i\pi\left(\ell+\frac{1}{2}\right)}Q_\pm(\theta+i\pi) + e^{\pm i\pi\left(\ell+\frac{1}{2}\right)}Q_\pm(\theta-i\pi)$$

- Asymptotics: $\ln Q_\pm(\theta + i\tau/2) \simeq -w_0 e^{\theta} - \bar{w}_0 e^{-\theta}\,, \quad w_0 = -\frac{MR}{4\cos\frac{\pi\beta^2}{2(1-\beta^2)}}$
- Extensions: Homogeneous sine-Gordon model (many masses)

From functional relations to integral equations

- $Q_\pm$ are the unique entire functions solutions of the integral equation

$$Q_\pm(\theta+i\tau/2)=q_\pm(\theta)\pm\int_{-\infty}^{+\infty}\frac{d\theta'}{4\pi}\tanh\frac{\theta-\theta'}{2}T\left(\theta'+i\frac{\tau}{2}\right)e^{-w_0(e^{\theta}+e^{\theta'})-\bar{w}_0(e^{-\theta}+e^{-\theta'})}\cdot$$
$$\cdot e^{\pm(\theta-\theta')\ell}Q_\pm\left(\theta'+i\frac{\tau}{2}\right),\quad q_\pm(\theta)=e^{\pm\frac{i\pi}{4}\pm\left(\theta+\frac{i\pi}{2}\right)\ell}e^{-w_0e^{\theta}-\bar{w}_0e^{-\theta}}$$

- The TQ-relation holds due to the property (of the kernel on continuous functions):

$$\lim_{\epsilon\to 0^+}\left[\tanh\left(x+\frac{i\pi}{2}-i\epsilon\right)-\tanh\left(x-\frac{i\pi}{2}+i\epsilon\right)\right]=2\pi i\delta(x),\quad x\in\mathbb{R}.$$

- Define the functions $X_\pm(\theta)$: $q_\pm(\theta)X_\pm(\theta)=Q_\pm(\theta+i\tau/2)$
- Make $w_0,\bar{w}_0$ dynamical: $w_0\to -iw'$, $\bar{w}_0\to i\bar{w}'$, $X_\pm(\theta)\to X_\pm(w',\bar{w}'|\theta)$
- Integral equation satisfied by $X_\pm(w',\bar{w}'|\theta)$, $\lambda=e^{\theta}$:

$$X_\pm(w',\bar{w}'|\theta)=1\pm\int_0^{+\infty}\frac{d\lambda'}{4\pi\lambda'}\frac{\lambda-\lambda'}{\lambda+\lambda'}T(\lambda'e^{\frac{i\tau}{2}})e^{-2iw'\lambda'+2i\frac{\bar{w}'}{\lambda'}}X_\pm(w',\bar{w}'|\theta')$$

Getting a Marchenko-like equation

- We define a Fourier transform of $X_\pm - 1$ (with an active role for w')

$$K_\pm(w', \xi; \bar{w}') = \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\lambda e^{i(\xi - w')\lambda} [X_\pm(w', \bar{w}'|\theta) - 1] \,.$$

- Let us take the Fourier transform of the integral equation for $X_\pm$. We get

$$K_\pm(w', \xi; \bar{w}') \pm F(w' + \xi; \bar{w}') \pm \int_{w'}^{+\infty} \frac{d\xi'}{2\pi} K_\pm(w', \xi'; \bar{w}') F(\xi' + \xi; \bar{w}') = 0 \,, \quad \xi > w' \,,$$

with $F(x; \bar{w}') = i \int_0^{+\infty} d\lambda' e^{-ix\lambda' + 2i \frac{\bar{w}'}{\lambda'}} T(\lambda' e^{i\frac{\tau}{2}})$.

- This has the structure of a Marchenko equation appearing in quantum inverse scattering (from scattering data and bound states to Schroedinger). However for usual Marchenko $F(x) = \int_{-\infty}^{+\infty} d\lambda e^{-ix\lambda} (S(\lambda) - 1) + \sum_n S(\lambda_n)$: S=S-matrix, λ_n bound states
- In our construction scattering data and bound states are encoded in T, vacuum eigenvalue of the transfer matrix of a quantum integrable model.

From Marchenko-like to Schroedinger

- Define the wave function $\psi_\pm(w', \bar{w}'|\theta) = e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}} X_\pm(w', \bar{w}'|\theta)$,

$$X_\pm(w', \bar{w}'|\theta) - 1 = \int_{w'}^{+\infty} \frac{d\xi}{2\pi} e^{-i(\xi - w')\lambda} K_\pm(w', \xi; \bar{w}'), \quad \lambda = e^\theta$$

- Differentiate (twice) and use our Marchenko-like equation: we get

$$-\frac{\partial^2}{\partial w'^2}\psi_\pm(w', \bar{w}'|\theta) + u_\pm(w'; \bar{w}')\psi_\pm(w', \bar{w}'|\theta) = e^{2\theta}\psi_\pm(w', \bar{w}'|\theta),$$

i.e. Schroedinger equations with potentials

$$u_\pm(w'; \bar{w}') = -2\frac{d}{dw'}\frac{K_\pm(w', w'; \bar{w}')}{2\pi}.$$

- Explicit solution of Marchenko equation gives access to the potential and the (Jost) wave function. The potential is

$$u_\pm(w'; \bar{w}') = \mp\partial_{w'^2}\hat{\eta} + (\partial_{w'}\hat{\eta})^2, \quad \hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})$$

$$V(\theta, \theta') = \frac{T\left(\theta + i\frac{\tau}{2}\right)}{4\pi}\frac{e^{-2iw'e^\theta + 2i\bar{w}'e^{-\theta}}}{\cosh\frac{\theta - \theta'}{2}}$$

Wave function and first Lax

- The wave function is $\psi_\pm(w', \bar{w}'|\theta) = X_\pm(w', \bar{w}'|\theta) e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}}$,

$$X_\pm(w', \bar{w}'|\theta) = -2 \mp \int \frac{d\theta'}{4\pi} e^{\frac{\theta-\theta'}{2}} V(\theta,\theta') X_\pm(w', \bar{w}'|\theta')$$

- To summarise, we have obtained two Schroedinger equations

$$-\frac{\partial^2}{\partial w'^2}\psi_\pm(w', \bar{w}'|\theta) + u_\pm(w'; \bar{w}')\psi_\pm(w', \bar{w}'|\theta) = e^{2\theta}\psi_\pm(w', \bar{w}'|\theta) .$$

- Introduce $D_{\hat\eta} = \partial_w + \frac{1}{2}\partial_w\hat\eta\, \sigma^3 - e^{\theta+\hat\eta}\sigma^+ - e^{\theta-\hat\eta}\sigma^-$.

$$\mathbf{D} = \begin{pmatrix} D_{\hat\eta} & 0 \\ 0 & D_{-\hat\eta} \end{pmatrix} , \quad \Psi = \begin{pmatrix} e^{\frac{\theta+\hat\eta}{2}}\psi_+ \\ e^{-\frac{\theta+\hat\eta}{2}}(\partial_w + \partial_w\hat\eta)\psi_+ \\ e^{\frac{\theta-\hat\eta}{2}}\psi_- \\ e^{-\frac{\theta-\hat\eta}{2}}(\partial_w - \partial_w\hat\eta)\psi_- \end{pmatrix}$$

- The first order matrix equation $\mathbf{D}\Psi = 0$ (the first Lax) is equivalent to Schroedinger equations in w'

Second Lax

- A differential equation in $\bar{w}'$ is defined by using the Fourier transform

$$K_{\pm}^{bis}(\bar{w}',\xi;w')=\int_{-\infty-i\epsilon}^{+\infty-i\epsilon}d\lambda^{-1}e^{i(\xi+\bar{w}')\lambda^{-1}}[X_{\pm}(w',\bar{w}'|\theta)-1]\,,$$

(with active role for $\bar{w}'$) on the equation for $X_{\pm}$. Following the Marchenko procedure, we end up with the 'conjugate' differential equation

$$-\frac{\partial^2}{\partial\bar{w}'^2}\psi_{\pm}^{bis}(w',\bar{w}'|\theta)+\bar{u}_{\mp}(w',\bar{w}')\psi_{\pm}^{bis}(w',\bar{w}'|\theta)=e^{-2\theta}\psi_{\pm}^{bis}(w',\bar{w}'|\theta)$$

for

$$\psi_{\pm}^{bis}(w',\bar{w}'|\theta)=e^{-iw'\lambda+i\bar{w}'\lambda^{-1}}\left[1+\int_{-\bar{w}'}^{+\infty}\frac{d\xi}{2\pi}e^{-i(\xi+\bar{w}')\lambda^{-1}}K_{\pm}^{bis}(\bar{w}',\xi;w')\right]\,.$$

- Introduce $\bar{D}_{\hat{\eta}}=\partial_{\bar{w}}-\frac{1}{2}\partial_{\bar{w}}\hat{\eta}\,\sigma^3-e^{-\theta+\hat{\eta}}\sigma^- -e^{-\theta-\hat{\eta}}\sigma^+$ and

$$\bar{\mathbf{D}}=\begin{pmatrix}\bar{D}_{\hat{\eta}} & 0\\ 0 & \bar{D}_{-\hat{\eta}}\end{pmatrix}\,,\quad \Psi^{bis}=\begin{pmatrix}e^{\frac{\theta-\hat{\eta}}{2}}(\partial_{\bar{w}}+\partial_{\bar{w}}\hat{\eta})\psi_{-}^{bis}\\ e^{-\frac{\theta-\hat{\eta}}{2}}\psi_{-}^{bis}\\ e^{\frac{\theta+\hat{\eta}}{2}}(\partial_{\bar{w}}-\partial_{\bar{w}}\hat{\eta})\psi_{+}^{bis}\\ e^{-\frac{\theta+\hat{\eta}}{2}}\psi_{+}^{bis}\end{pmatrix}$$

- The first order matrix equation $\bar{\mathbf{D}}\Psi^{bis}=0$ is equivalent to Schroedinger equations in $\bar{w}'$.

The classical model

- Let us compare the two vectors Ψ and Ψ^{bis}.
- By examining the solutions we constructed we find that $\psi^{bis}_{\mp}(w', \bar{w}'|\theta) = \psi_{\mp}(w', \bar{w}'|\theta) e^{\mp\hat{\eta}(w,\bar{w})}$.
- On the four-vectors this connection implies $\Psi = -e^{\theta}\Psi^{bis}$. Then, we can write $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$: from this relations we get that $[\mathbf{D}, \bar{\mathbf{D}}]\Psi = 0$, which means for $\hat{\eta}$
$$\partial_w \partial_{\bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta} \, ,$$
i.e. that $\hat{\eta}$ satisfies the classical sinh-Gordon equation.
- The two Lax problems $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$ coincide with the starting point of usual ODE/IM construction (Lukyanov and Zamolodchikov). We have completed our inverse construction.

Conformal limit

- Potentials $u_{\pm}(w', \bar{w}')$ of Schroedinger equations are complicated functions (Fredholm determinants)
- Simplifications occur in the conformal limit, when masses $(w_0) \to 0$, $\bar{w}' \to 0$ and w' scales as

$$\frac{dw'}{dx} = \sqrt{p(x)}e^{-\theta} \quad \theta \to +\infty$$

with $p(x) = x^{2M} - E$, $M = 1/\beta^2 - 1$ (θ 'rapidity').

- Then, the new wave function $\psi^{cft}(x) = \psi_+(w')p(x)^{-\frac{1}{4}}$ satisfies the ODE

$$-\frac{d^2}{dx^2}\psi^{cft}(x) + \left(p(x) + \frac{\ell(\ell+1)}{x^2}\right)\psi^{cft}(x) = 0$$

which is ODE considered by Dorey and Tateo and Bazhanov, Lukyanov, Zamolodchikov in their '98 papers.

Summary and Perspectives

- We have given a possible explanation for the occurrence of the ODE/IM correspondence. The idea is that the *TQ*-functional relation is equivalent to a an equation with the form of a Marchenko equation. From this Marchenko-like equation one gets Schroedinger equations.
- We have discussed the case of vacuum eigenvalues of $\hat{T}$, $\hat{Q}$ for sine-Gordon model.
- However, *TQ*-relations are common in quantum integrable models (they are equivalent to Bethe Ansatz). The same connection to a classical model can be found for other quantum models (e.g. spin chains).

Back to quantum (usual path)

- As in usual ODE/IM, in the classical model we constructed we find Q functions of (Homogeneous) sine-Gordon as connection coefficients between different solutions.
- In the Wick rotated new variable $w = iw'$, when $w \to w_0$ the potentials

$$u_\pm \simeq -\ell(\ell \pm 1)/(w - w_0)^2$$

- We have solutions (Frobenius) that when $w \to w_0$

$$f_+^{(-\ell)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{-\ell}, \quad f_+^{(\ell+1)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{\ell+1},$$
$$f_-^{(\ell)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{\ell}, \quad f_-^{(-\ell+1)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{-\ell+1}.$$

In terms of f we expand $\psi_\pm$

$$\psi_+(w', \bar{w}'|\theta) = -e^{\theta(\ell+1)} Q_-(\hat{\theta}) f_+^{(\ell+1)}(w', \bar{w}') + e^{-\theta\ell} Q_+(\hat{\theta}) f_+^{(-\ell)}(w', \bar{w}')$$
$$\psi_-(w', \bar{w}'|\theta) = e^{\theta\ell} Q_-(\hat{\theta}) f_-^{(\ell)}(w', \bar{w}') - e^{-\theta(\ell-1)} Q_+(\hat{\theta}) f_-^{(-\ell+1)}(w', \bar{w}')$$

- Connection coefficients contain Q-functions of the quantum model:

$$\lim_{w \to w_0} (w - w_0)^{\pm\ell} \psi_\pm(w', \bar{w}'|\theta) = D_\pm e^{\mp\theta\ell} Q_\pm \left(\hat{\theta} = \theta + i\frac{\tau}{2}\right).$$

Special case

- Particular case: $\beta^2 = 2/3, \ell = 0$ which imply $T = 1$
- Now $\hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})$, with

$$V(\theta, \theta') = \frac{e^{-2iw' e^{\theta} + 2i\bar{w}' e^{-\theta}}}{4\pi \cosh \frac{\theta - \theta'}{2}}$$

- The field $\hat{\eta}$ depends only on $t = 4\sqrt{w'\bar{w}'}$, $w' = \frac{t}{4} e^{i\varphi}$ and the sinh-Gordon equation $\partial_w \partial_{\bar{w}} \hat{\eta} = 2\sinh 2\hat{\eta}$ reduces to the Painlevè III_3 equation:

$$\frac{1}{t}\frac{d}{dt}\left(t\frac{d}{dt}\hat{\eta}(t)\right) = \frac{1}{2}\sinh 2\hat{\eta}(t)$$

- The wave functions $\psi_\pm(w', \bar{w}'|\theta)$ depend only on $t, \theta + i\varphi$. Then, they satisfy differential equations in t and θ. This means that $Q_\pm(\theta) = \psi_\pm(t = 4w_0|\theta)$ satisfy also differential equations (in θ).

$$\frac{d^2 Q_\pm(\theta)}{d\theta^2} + \tanh(\theta \pm \hat{\eta}_0)\left[-\frac{dQ_\pm(\theta)}{d\theta} \mp 2w_0\hat{\eta}_0' Q_\pm(\theta)\right] - 4w_0^2(\hat{\eta}_0')^2 Q_\pm(\theta) +$$
$$+ \quad 2w_0^2[\cosh 2\theta + \cosh 2\hat{\eta}_0]Q_\pm(\theta) = 0\,,$$

where $\hat{\eta}_0 = \hat{\eta}(t = 4w_0)$.