Theory and Phenomenology of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

INFN Istituto Nazionale di Fisica Nucleare
Sezione di Bologna

Quantum simulations for Abelian Gauge Theories

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SM&FT Workshop

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OUTLINE

- Quantum Digital Simulation of Lattice Gauge Theories
- Quantum Many Body Hamiltonian on a Lattice
- Goal: nonperturbative phenomena, dynamical effects, …

- First step: to determine the (many body) Ground State with precision
- Fully digital or variational approaches

$\mathbb{Z}_2$ Lattice Gauge Theory in 2D

total Hamiltonian $$H(h) = H_E + h\, H_B$$

electric contribution

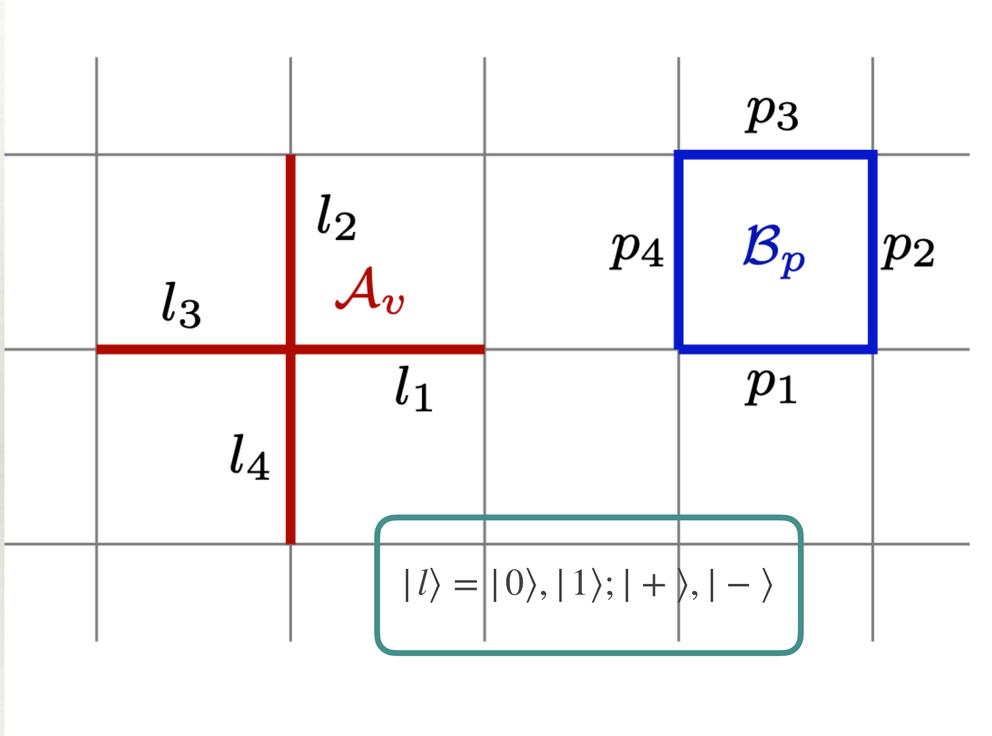
$$H_E = \sum_l (1 - \sigma_l^x)$$

magnetic contribution

$$H_B = -\sum_p \mathscr{B}_p = -\sum_p \sigma_{p_1}^z \sigma_{p_2}^z \sigma_{p_3}^z \sigma_{p_4}^z$$

Gauss law

$$\mathscr{A}_v = \prod_{l \in v} \sigma_l^x \quad \mathscr{A}_v |\psi\rangle_{phys} = |\psi\rangle_{phys}$$



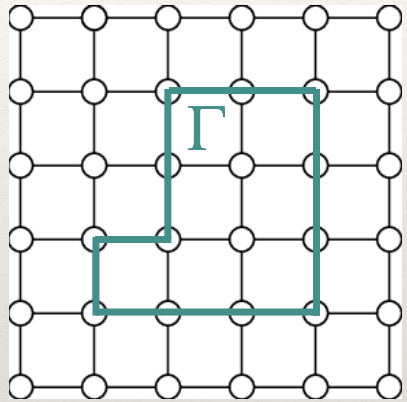
Remark: on dual lattice it becomes an Ising model
(with $S_p^x = \mathscr{B}_p$, $S_p^z S_{p'}^z = \sigma_{l(p,p')}^x$)

- electric limit $h \to 0$ $$|\Omega_E\rangle = \bigotimes_l |+\rangle_l$$

- magnetic limit $h \to \infty$

Toric code $$|\Omega_B\rangle = \mathcal{N} \sum_\Gamma \mathscr{W}_\Gamma |\Omega_E\rangle$$

Wilson loop operator $$\mathscr{W}_\Gamma = \prod_{l \in \Gamma} \sigma_l^z$$



Phase transition $h_c = 3.04438(2)$

$h < h_c$ *confined phase*

$h > h_c$ *deconfined phase (topological)*
with long range entanglement

Fully Digital Simulation

- **ENCODING** of the degrees of freedom

on each link, the Hilbert space is $\mathscr{H}_l = L^2(G)$, and to represent the elements of the basis $\{|g\rangle\}_{g\in G}$ we need means a register of n **qubits**, with $|G| \leq 2^n$

for the total Hilbert space $\mathscr{H} = \otimes_{links} L^2(G)$ we have, with L=#links: $dim\mathscr{H} = 2^{nL}$

e.g., in a square lattice with PBC, L=2V (V=#vertices), clearly showing the *exponential growth*

What about the gauge invariant subspace?

$$\mathscr{H}_{phys} = \{|\psi\rangle \in \mathscr{H} \ : G|\psi\rangle = |\psi\rangle\}$$

$$dim\mathscr{H}_{phys} = |G|^{L-V+1} \simeq \sqrt{dim\mathscr{H}} \quad \text{for PBC}$$

❖ **STATE PREPARATION via ADIABATIC THEOREM**

Starting from the known GS of $H(g = g^0)$, the we use the adiabatic theorem to find the GS for a different value of coupling constant.

EVOLUTION: Trotter approximation

$$U(t) = e^{-itH} = \left(e^{-iHt/m}\right)^m \approx \left(e^{-iH_E t/m} e^{-iH_B t/m}\right)^m$$

H_E, H_B do not commute, they are diagonal in the representation/group basis respectively

but both H_E, H_B are (separately) gauge-invariant

The evolution operators : $e^{-iH_E t/m}$, $e^{-iH_B t/m}$ are going to be implemented via a

QUANTUM CIRCUIT

QUANTUM CIRCUIT

Standard Gates: phase gate $\mathcal{U}_{ph}(\phi)$ + other elementary logic gates

"Group operations gates"

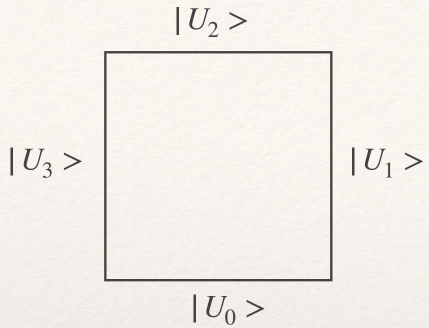
inversion	*multiplication*	*trace*
$\mathcal{U}_{-1}\,\lvert g\rangle = \lvert g^{-1}\rangle$	$\mathcal{U}_{\times}\,\lvert g\rangle\,\lvert h\rangle = \lvert g\rangle\,\lvert gh\rangle$	$\mathcal{U}_{tr}(\theta)\,\lvert g\rangle = \lvert g\rangle\, e^{i\theta Re(tr[g])}$

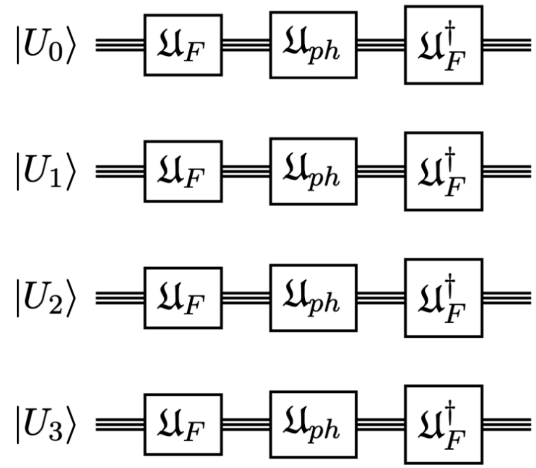
G-Fourier transform

$$\mathcal{U}_F \sum_{g\in G} f(g)\,\lvert g\rangle = \sum_{J\in UIR} \sum_{mn} \hat{f}(J)_{mn}\,\lvert J, mn\rangle$$

$$\hat{f}(J)_{mn} = \langle J, mn \,\lvert\, f\rangle$$

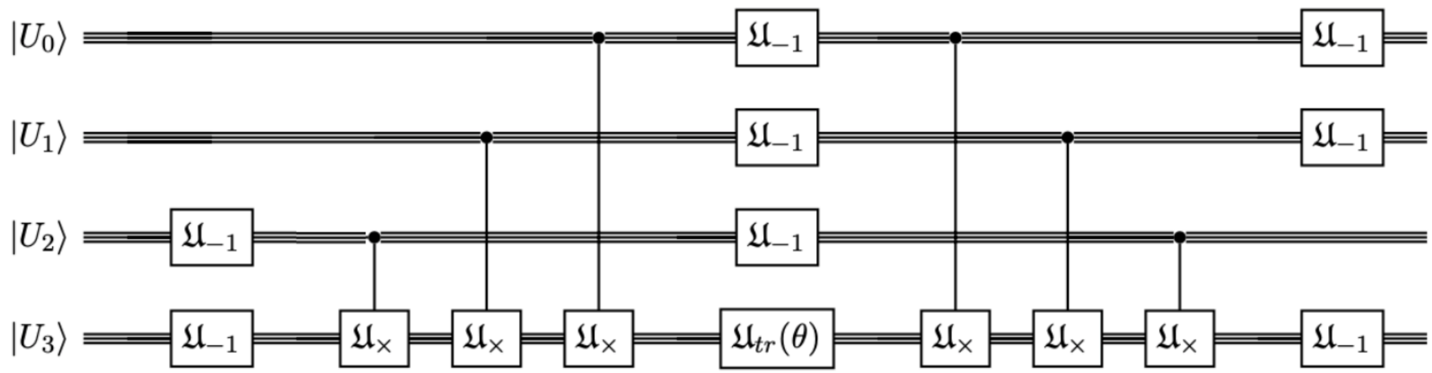
e.g. for $G = \mathbb{Z}_2$





phase gate
$\phi \propto \lambda_E \Delta t$

Circuit realizing the time evolution of 4 links generated by H_E.



phase gate
$\theta \propto \lambda_B \Delta t$

Quantum circuit realizing the time evolution of a single plaquette generated by the magnetic term of the hamiltonian.

In summary, the steps of the calculations are:

1. prepare the system in the $|GS>_{g=0}$ and choose the final value of the coupling h
2. construct the circuit that adiabatically prepares the ground state of $H(h)$, through a Trotter approximation

(with N_s, t_s as numbers and duration of time-steps, which introduce a *systematic error*)

3. measure the expectation value of the chosen observable

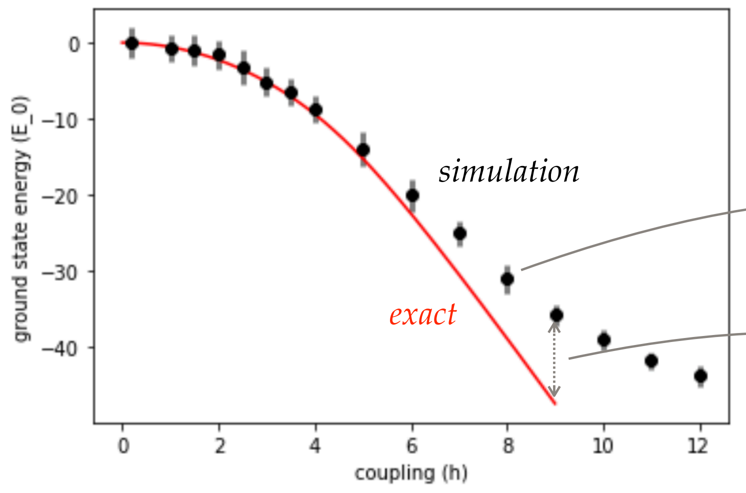
(e.g. Hamiltonian or Wilson loops)

This has to be repeated a sufficient number of times to extract the probabilities of possible outcomes of the observable.
This introduces a *statistical error*.

$G = \mathbb{Z}_2$

3x3 lattice with PBC

(with L. Lumia, Master thesis, 2021)

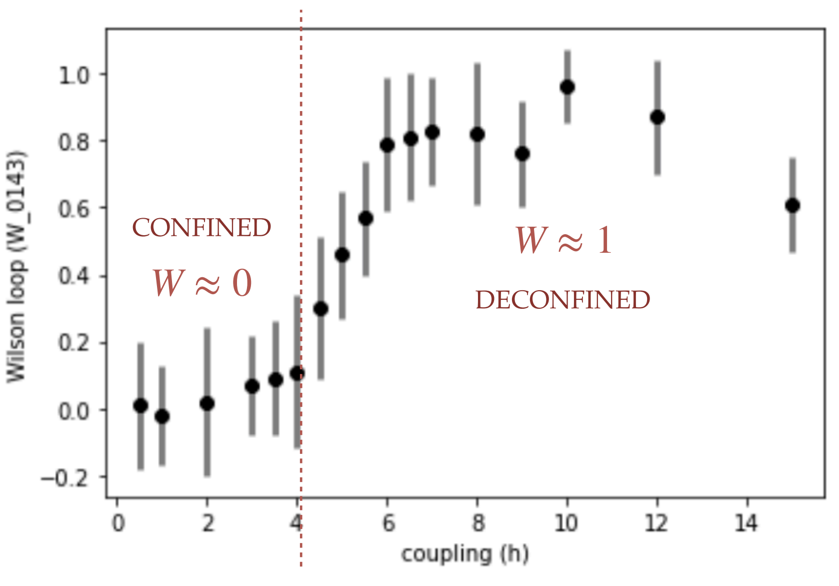


(b) $t_s = 0.0008, \ N_s = 600$

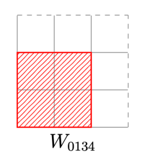
GROUND STATE ENERGY

error bar: statistical

systematic:
Trotter approximation +
adiabatic evolution



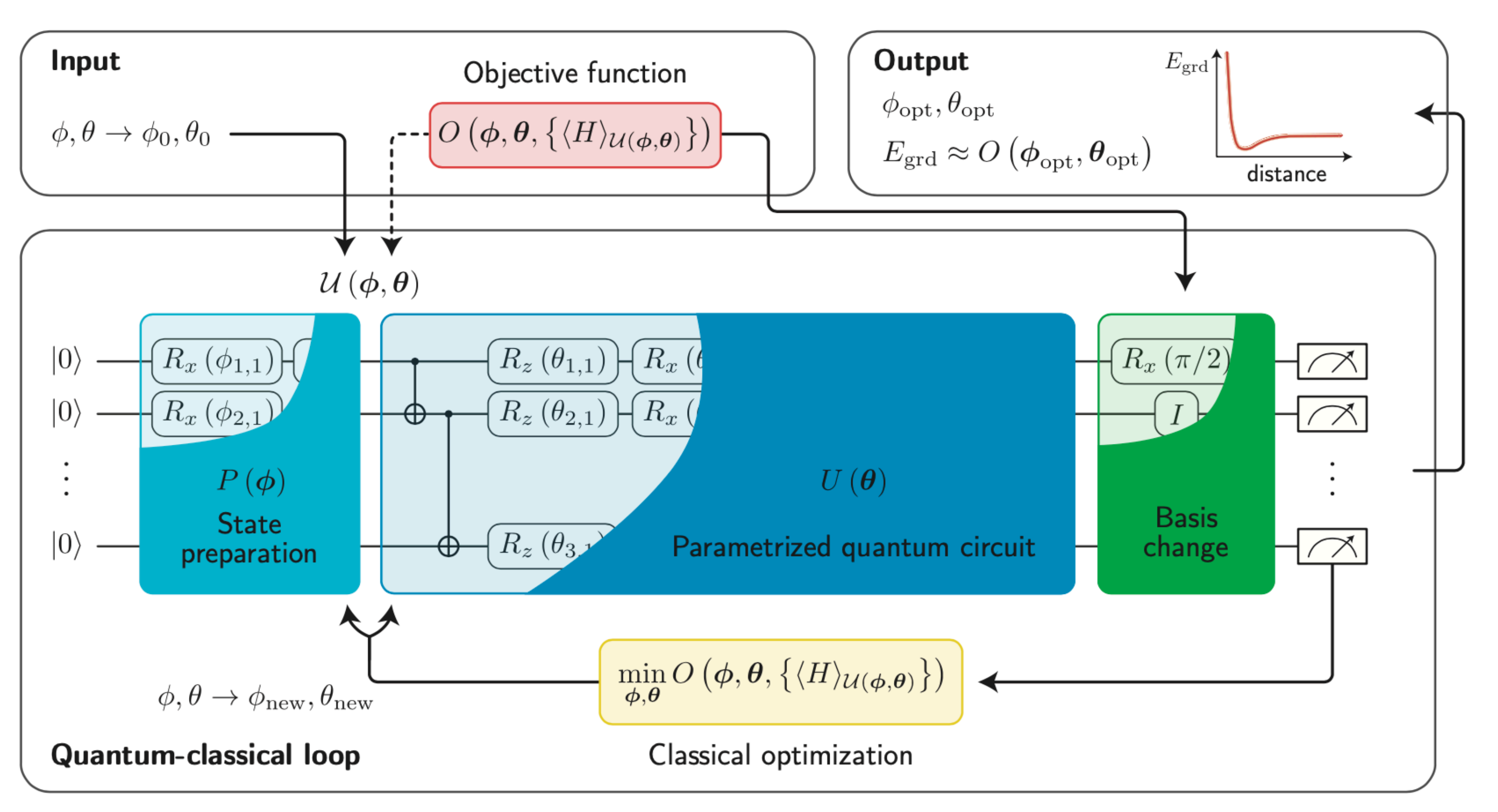
2x2 WILSON LOOP



QAOA

Quantum Approximate Optimization Algorithm

hybrid classical-quantum protocol, in which an input state is manipulated via a parametrised quantum circuit,
to be update and optimised by means of classical optimization protocols



L. Lumia et al, PRX Quantum 3 (2022) 020320

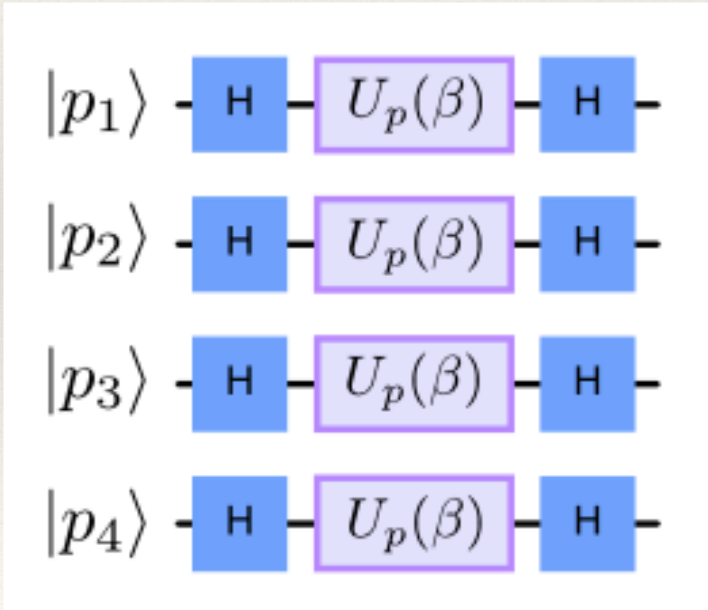
$$|\psi_P(\gamma,\beta)\rangle = \left(\prod_{m=1}^{P} e^{-i\beta_m H_E} e^{-i\gamma_m H_B}\right)|\psi_0\rangle$$

- quantum circuit for each step ($m = 1,\cdots,P$) of the QAOA to implement the evolutions through H_E , H_B
- classical minimisation of $E_P\left(\gamma,\beta\right) = \langle\psi_P(\gamma,\beta)\,|\,H(h)\,|\,\psi_P(\gamma,\beta)\rangle$

Electric part evolution

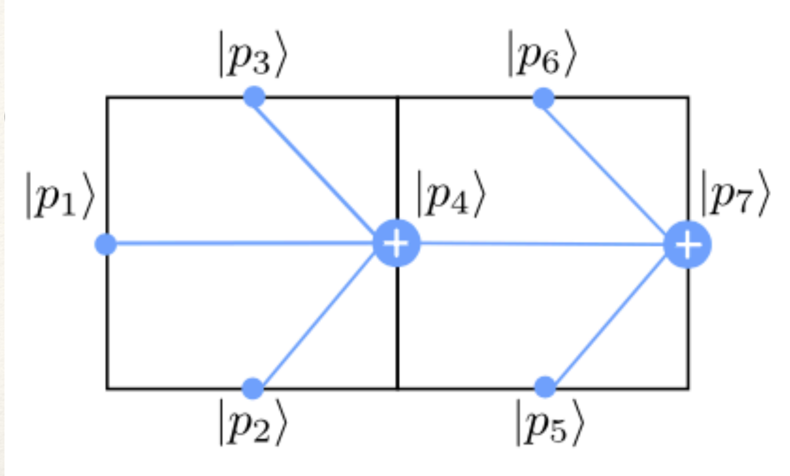
for each step m, we get a product of all single qubit rotations around the x axis by an angle β_m

$$U_p(\beta) = e^{i\beta\sigma^z}$$



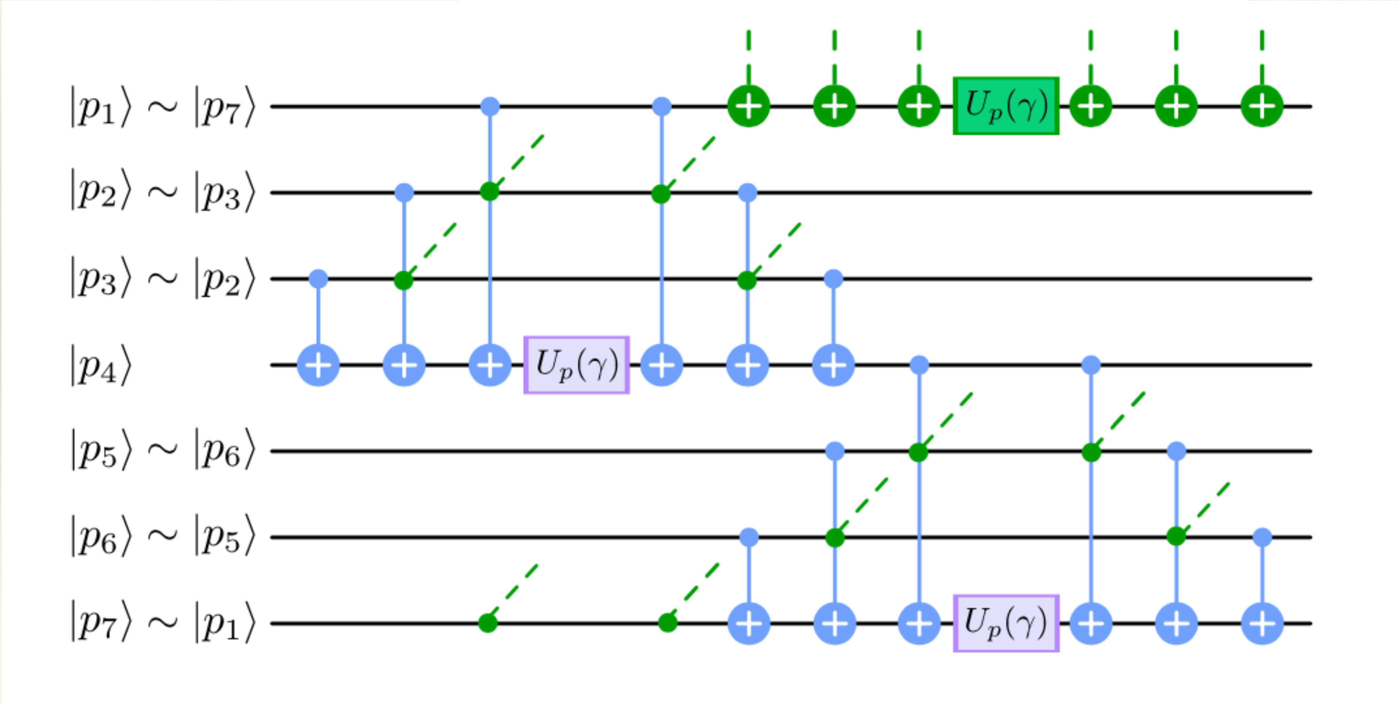
Magnetic part evolution

for all 4 links on a plaquette; it involves a rotation around the z axis by an angle γ_m on one (vertical) link, controlled by the other three links $U_p(\gamma) = e^{i\gamma\sigma^z}$



each (vertical) link acts as a target for the left plaquette and as control for the right one:
each elementary part of the circuit acts on a pair of plaquettes (7 links)

it can be run in parallel on each pair of (horizontal) plaquettes
(with care to boundary conditions)



Preparation of initial state

$$|\Omega_E\rangle = \bigotimes_l |+\rangle_l$$ simple product state, prepared by Hadamard gate

$$|\Omega_B\rangle = \mathcal{N} \sum_\Gamma \mathscr{W}_\Gamma |\Omega_E\rangle$$ more complicated, circuit similar to that of magnetic evolution plus Hadamard, in parallel on columns

(consistent with results that O(L) circuit depth to prepare states with long range entanglement)

Y.-J. Liu, K. Shtengel, A. Smith, and F. Pollmann, Methods for simulating string-net states and anyons on a digital quantum computer, arXiv:2110.02020

CLASSICAL OPTIMIZATION

Energy landscape -> rugged; barren plateaus

1) Two-step protocol, inspired by adiabatic quantum computation

- first optimising an annealing schedule dt=dt* by means of a linear protocol of time $\tau = Pdt$:

$$\gamma_m = \frac{m\,dt}{P}h \quad \beta_m = dt$$

- then 10 local optimisations: $\gamma = \gamma(dt^*) + \epsilon \;\; \beta = \beta(dt^*) + \delta$

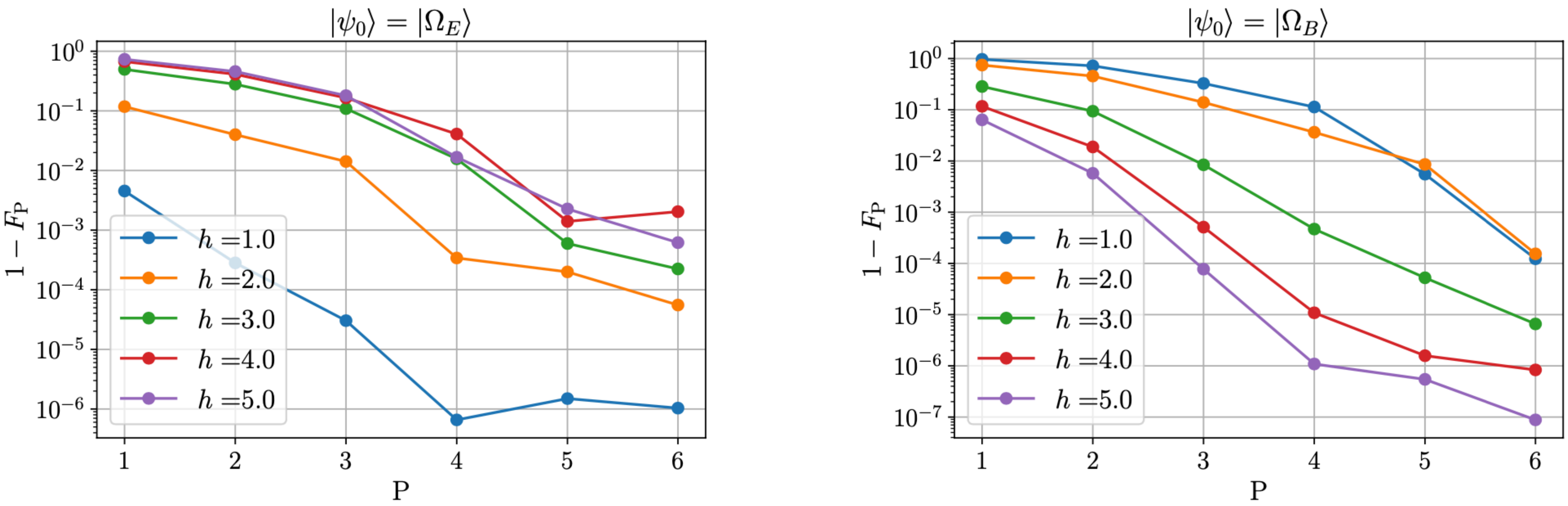
ϵ, δ are P-dimensional vectors with random numbers uniformly distributed in $[-0.025, 0.025]$

2) global optimisation (using basin-hopping) starting from both $|\Omega_E\rangle, |\Omega_B\rangle$

Cases considered L=3 (4,5) P=1,…, 8 h=0, …, 10

Numerical results

(IN)FIDELITY (w.r.t. exact ground state evaluated numerically)



WILSON LOOPS

$\mathscr{W}_{l_x,l_y}$ defined over rectangles of size $l_x \times l_y$

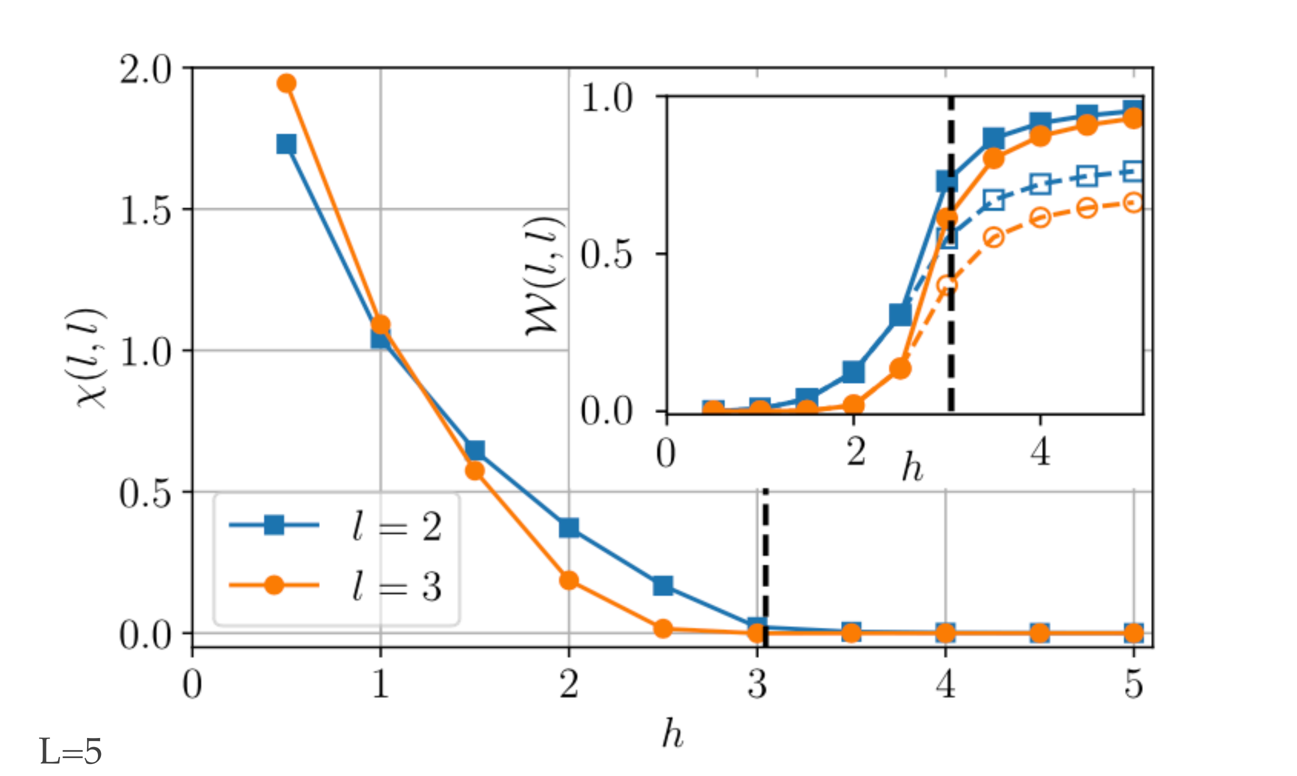
$$\langle \mathscr{W}_\Gamma \rangle \propto e^{-\chi A[\Gamma] - \delta P[\Gamma]}$$

A=area , P=perimeter

If $\chi > 0$, the exponential decay with the area dominates for large loops ⇒ confined phase
If instead $\chi = 0$, then the decay is dictated by the perimeter law only ⇒ deconfined phase

Wilson loop ratio

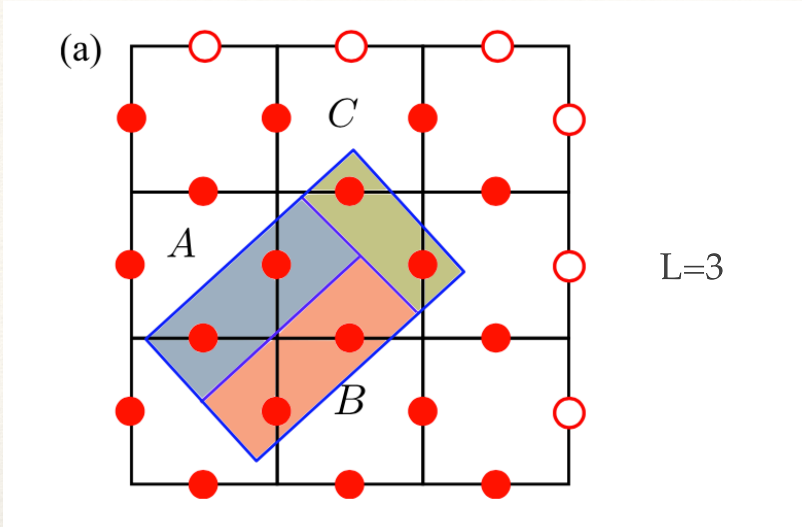
$$\chi(l,l) = -\log \frac{\langle W_{l,l} \rangle \langle W_{l-1,l-1} \rangle}{\langle W_{l,l-1} \rangle \langle W_{l-1,l} \rangle}$$

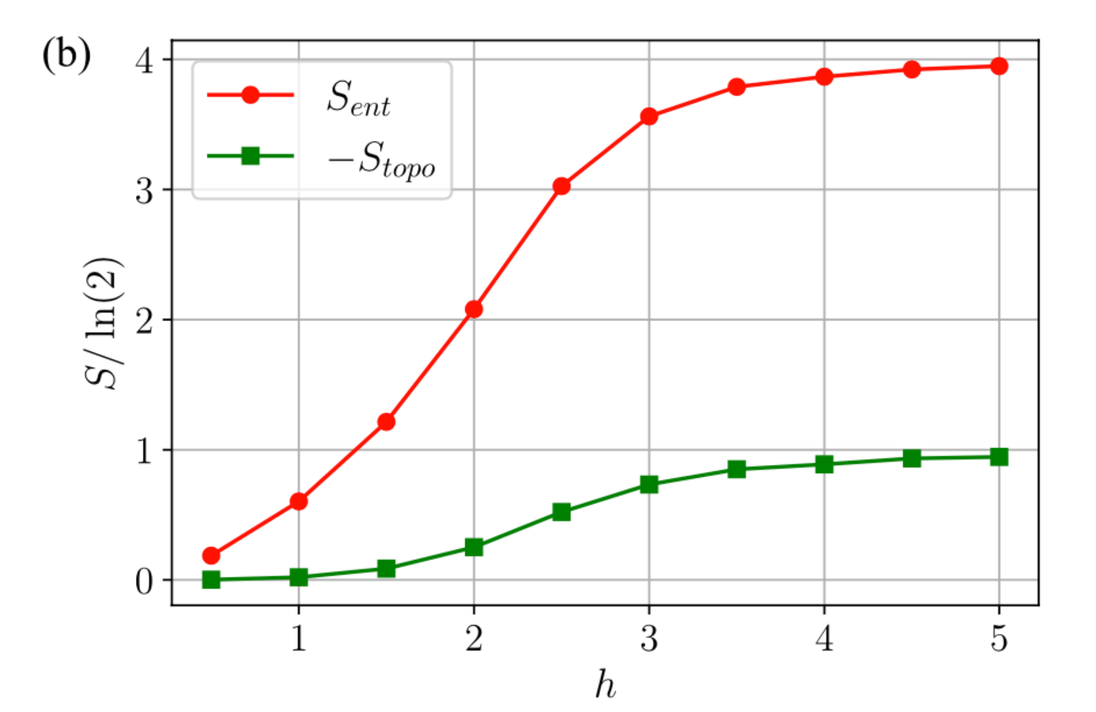


TOPOLOGICAL ENTROPY

tripartite region ABC

$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$





despite the small dimension of the lattice and its subsystem, our results agree perfectly with the theoretical predictions

for $h = 0$, $\quad S_{ABC} = S_{top} = 0$

for $h = \infty$, $\quad S_{ABC} = (N_v - 1)\ \ln 2 \quad S_{top} = -\ln 2$

OUTLOOKS

- Extend the preparation of the ground state to $\mathbb{Z}_N$ to higher N, e.g. N=3 and N=4
- Study the phase diagram of non-abelian groups as D_N
- Addition of matter

✓Real computer
IBM (within INFN-CERN agreement)