Localization, ensemble inequivalence and negative temperatures in the Discrete Non-Linear Schrödinger Equation

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In collaboration with:

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Random Lasers (in progress): Jacopo Niedda, Luca Leuzzi, Giorgio Parisi (Rome)

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Summary

1) Large Deviations and Localization

2) Discrete Non-Linear Schrödinger Equation (DNLSE)

3) DNLSE: State of the art and the problem of ensembles

4) Localization mechanism

5) Ensemble inequivalence, negative temperature, order parameter

6) Open Directions and Conclusions

The ‘Linear Statistic’ problem

Linear Statistic Problem: probability distribution of a sum of random variables

$$P_N(M) = \int \prod_{i=1}^{N} dm_i \; p(m_1, \ldots, m_N) \; \delta\left(M - \sum_{i=1}^{N} m_i \right)$$

Simple case: independent identically distributed **random variables**

$$p(m_1, \ldots, m_N) = \prod_{i=1}^{N} p(m_i)$$

$\langle m \rangle < \infty$ Finite mean

$\langle m^2 \rangle < \infty$ Finite variance

$$|M - N\langle m \rangle| \sim \sqrt{N} \quad \Longrightarrow \quad P_N(M) = \frac{1}{\sqrt{2\pi\sigma N}} e^{-\frac{(X - N\langle m \rangle)^2}{2\sigma^2 N}}$$

Central Limit Theorem

$$|M - N\langle m \rangle| \sim N \quad \Longrightarrow \quad P_N(M) \sim e^{-N \; \underline{\mathcal{I}(m)}} \qquad m = M/N$$

Rate function

Large Deviations

‘Linear Statistic’ and Large Deviations

Linear Statistic Problem: probability distribution of a sum of random variables

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$\langle m \rangle < \infty$ Finite mean

$\langle m^2 \rangle < \infty$ Finite variance

Fat tailed distribution $e^{-m} < p(m) < \dfrac{1}{m^2}$ → ***Localization***

$|M - N\langle m \rangle| \sim N$ ⇒ $P_N(M) \sim p(M)$

Large Deviations

Whole sum is taken up by a single variable

‘Linear Statistic’ and Large Deviations

Mass transport model: stationary partition function

$$\mathcal{Z}_N(M) = \int_0^\infty \prod_{i=1}^{N} dm_i \prod_{i=1}^{N} p(m_i)\ \delta\left(M - \sum_{i=1}^{N} m_i\right)$$

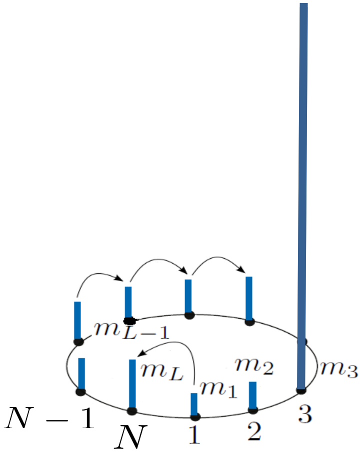
Fat tailed distribution $e^{-m} < p(m) < \dfrac{1}{m^2}$ → ***Localization***

‘*Nature of the condensate in mass transport models*’, Majumdar, Evans, Zia, *PRL* **94**, 180601 (2006)

Partition function

$$\mathcal{Z}_N(M) \sim p(M)$$

Whole sum is taken up by a single variable



$$M \ \sim\ m_i$$

Participation Ratio

$$Y_2(M) = \left\langle \frac{\sum_{i=1}^{n} m_i^2}{\left(\sum_{i=1}^{N} m_i\right)^2} \right\rangle$$

$$M < N\langle m\rangle \implies Y_2(M) \sim 1/N$$

$$M > N\langle m\rangle \implies Y_2(M) = \mathcal{O}(1)$$

Linear statistic: from non-equilibrium …

Important references

‘Nature of the Condensate in Mass Transport Models’
(S.N. Majumdar, M.R. Evans, R. K. P. Zia, PRL **94**, 180601, **2005**)

‘Constraint-Driven Condensation in Large Fluctuations of Linear Statistics’
(J. Stzavits-Nossan, M.R. Evans, S.N. Majumdar, PRL **112**, 020602, **2014**)

Some previous Non-equilibrium results

‘Participation Ratio for Constraint-Driven Condensations with Superextensive Mass’ (G. Gradenigo, E. Bertin, Entropy, **2017, arXiv:1708.08872**)

‘A First-Order Dynamical Transition for a Driven Run-and-Tumble particle’
(G. Gradenigo, S. N Majumdar, JSTAT, **2019, arXiv:1812.07819**)

… to equilibrium statistical mechanics

‘Localization in Discrete Non-Linear Schrödinger Equation’

SYSTEM DESCRIBED: BOSE-EINSTEIN CONDENSATE in a periodic potential (optical traps)

- ‘*Discrete solitons and breathers with dilute Bose-Einstein condensates* ’, Trombettoni, Smerzi, *PRL* **86**, 2353 **(2001)**
- ‘*Discrete Breathers in Bose-Einstein Condensates*’, Franzosi, Livi, Oppo, Politi, *Nonlinearity.* **24**, R89 **(2011)**
- ‘*Non-equilibrium discrete non-linear Schrodinger equation*’, Iubini, Lepri, Politi, *Phys. Rev. E* **86**, 011108 **(2012)**

Linear statistic: from non-equilibrium …

Important references

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‘Localization in Discrete Non-Linear Schrödinger Equation’

‘Condensation transition and ensemble inequivalence in the discrete nonlinear Schrödinger equation’, G. Gradenigo, S. Iubini, R. Livi, S. N Majumdar, *EPJ-E* **44**, 1-6 (**2021**)

‘Localization transition in the discrete nonlinear Schrödinger equation: ensembles inequivalence and negative temperatures’, G. Gradenigo, S. Iubini, R. Livi, S. N Majumdar, *J. Stat. Mech.* 023201 **(2021)**

Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical description of Bose-Einstein condensate

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{x})\right]\Psi(\mathbf{x}) - \frac{\nu}{2}|\Psi(\mathbf{x})|^2\Psi(\mathbf{x}) = 0$$

Gross-Pitaevskii Equation: non-linear equation for the **condensate wavefunction** 'order parameter' of a quantum transition (semiclassical approximation)

$$\Psi(\mathbf{x}) = \langle\hat{\psi}(\mathbf{x})\rangle$$

Classical field

Expectation on ground state of quantum field

HAMILTONIAN ⟺ **EQUILIBRIUM STATISTICAL MECHANICS**

Hamiltonian system on a lattice

$$\mathcal{H} = \sum_{i=1}^{N} \Psi_i^*\Psi_{i+1} + \Psi_{i+1}^*\Psi_i + \frac{\nu}{2}\sum_{i=1}^{N}|\Psi_i|^4$$

Canonical conjugate variables

$$\{\Psi_i^*, \Psi_j\} = i\ \delta_{ij}/\hbar \qquad i\dot{\Psi}_i = -\frac{\partial\mathcal{H}}{\partial\Psi_i^*}$$

Discrete Non-Linear Schrödinger Equation (DNLSE)

Condensate wave-function (order parameter) $\langle \hat{\psi} \rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial \psi_i}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \psi_i^*} = -(\psi_{i+1} + \psi_{i-1}) - \nu |\psi_i|^2 \psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N} (\psi_i^* \psi_{i+1} + \psi_i \psi_{i+1}^*) + \frac{\nu}{2} \sum_{i=1}^{N} |\psi_i|^4$$

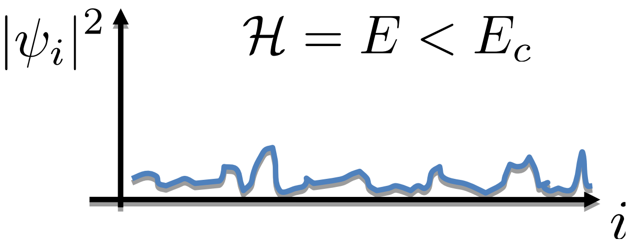
PARTICLES NUMBER (conserved)

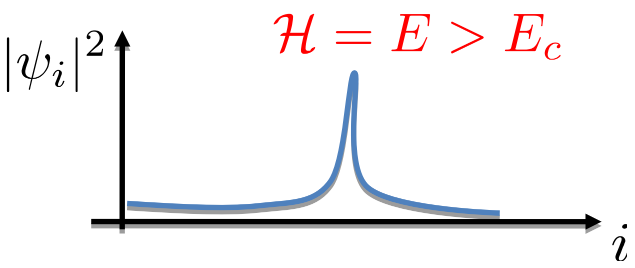
$$A = \sum_{i=1}^{N} |\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





1) WHICH KIND OF PHASE TRANSITION ? **2) WHICH STATISTICAL ENSEMBLE?**

3) LOCALIZATION COMES FROM INTEGRABILITY? (N integrals of motion)

4) IS DISORDER NECESSARY FOR LOCALIZATION?

Discrete Non-Linear Schrödinger Equation (DNLSE)

Condensate wave-function (order parameter) $\langle\hat{\psi}\rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial\psi_i}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\psi_i^*} = -(\psi_{i+1}+\psi_{i-1}) - \nu|\psi_i|^2\psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N}(\psi_i^*\psi_{i+1} + \psi_i\psi_{i+1}^*) + \frac{\nu}{2}\sum_{i=1}^{N}|\psi_i|^4$$

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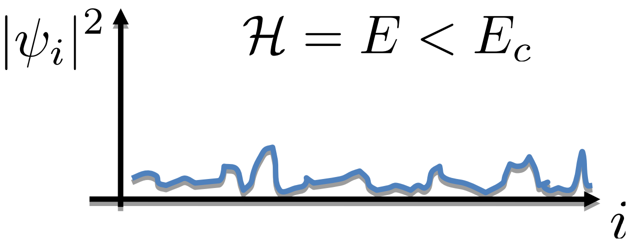
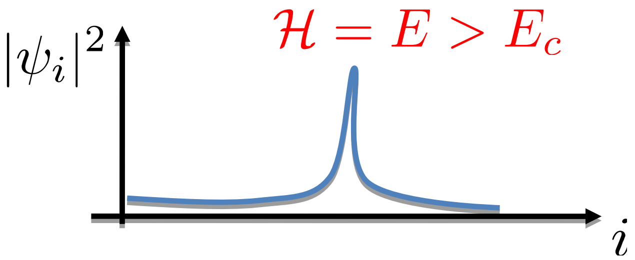




The ‘Fundamental Ensemble’ : MICROCANONICAL

Microcanonical
Partition function

$$\Omega_N(A,E) = \int\prod_{i=1}^{N}d\psi_i\;\underbrace{\delta(A-\sum_{i=1}^{N}|\psi_i|^2)}_{\textbf{Particle number conservation}}\;\underbrace{\delta\left(E-\mathcal{H}[\psi_i^*,\psi_i]\right)}_{\textbf{Energy conservation}}$$

DNLSE theory: equilibrium stat mech state of the art

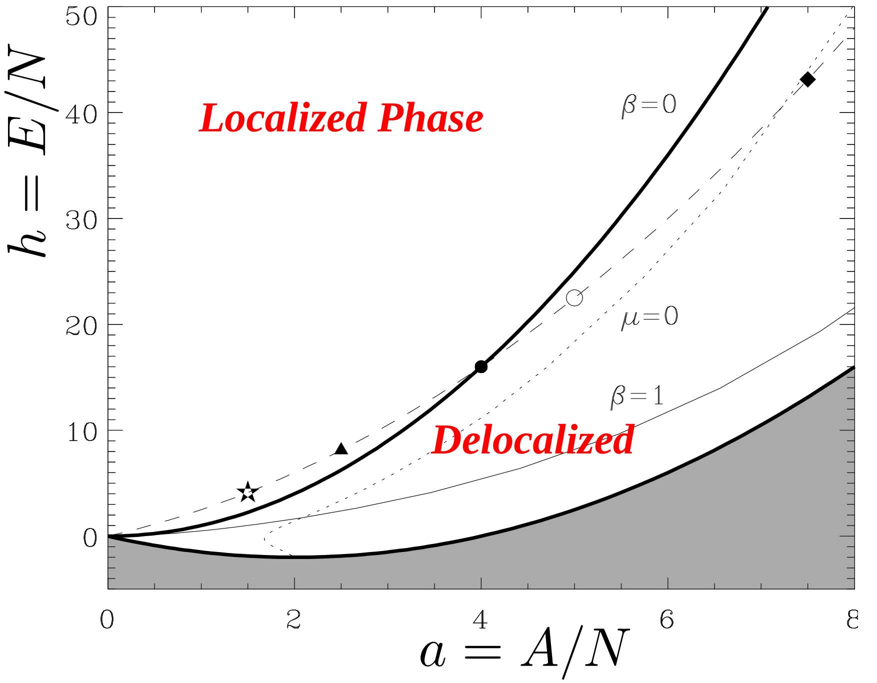
‘Statistical Mechanics of a Discrete Non-Linear System’,
K.O. Rasmussen, T. Cretegny, P.G. Kevridis, N. Gronbech-Jensen, *Phys. Rev. Lett.* **84**, 3740 **(2000)**

Microcanonical → **Grand Canonical** $\mathcal{Z}_N(\mu,\beta) = \int_0^\infty dA\; dE\; e^{-\beta E - \mu A}\; \Omega_N(A,E)$

Grand Canonical: exact solution with trasfer matrix techniques!

Transition line at **infinite** temperature: $\beta = 0$

h = 2 a²



PROBLEM

Many numerical evidences that the localized phase has negative temperature, T<0

‘Discrete Breathers and Negative-Temperature States’,
S. Iubini, R. Franzosi, R. Livi, G.-L. Oppo, A. Politi,
New J. Phys. **15**, 023032 **(2013)**

HOW CAN β<0 BE CONSISTENT WITH $e^{-\beta\mathcal{H}}$ **?** → **IT CANNOT!**

Discrete Non-Linear Schrödinger Equation (DNLS)

Condensate wave-function (order parameter) $\langle\hat{\psi}\rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial\psi_i}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\psi_i^*} = -(\psi_{i+1}+\psi_{i-1}) - \nu|\psi_i|^2\psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N}(\psi_i^*\psi_{i+1} + \psi_i\psi_{i+1}^*) + \frac{\nu}{2}\sum_{i=1}^{N}|\psi_i|^4$$

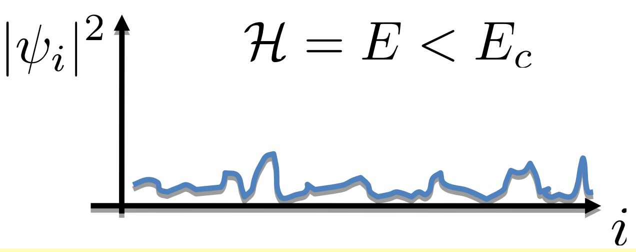
PARTICLES NUMBER (conserved)

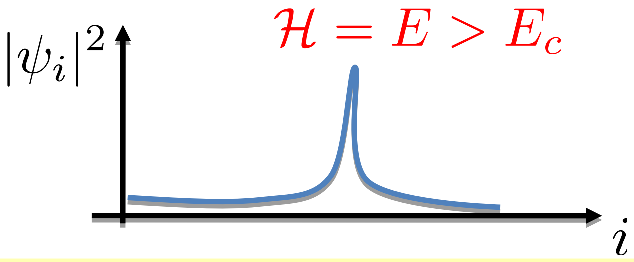
$$A = \sum_{i=1}^{N}|\psi_i|^2$$

PHENOMENON

Condensate wavefunction localize at high enegies

(numerical evidences)





ONLY THE MICROCANONICAL IS CORRECT: GO FOR IT!

$$\Omega_N(A,E) = \int\prod_{i=1}^{N}d\psi_i\;\underbrace{\delta\left(A - \sum_{i=1}^{N}|\psi_i|^2\right)}_{\text{Particle number conservation}}\;\underbrace{\delta\left(E - \sum_{i=1}^{N}|\psi_i|^4\right)}_{\text{Energy conservation}}$$

Neglect hopping terms

(a-posteriori argument)

WHEN ENSEMBLES ARE EQUIVALENT

$$\underbrace{\mathcal{Z}_N(\mu,\beta)}_{\textit{Grand-Canonical}} = \int_0^\infty \underbrace{dA\ dE\ e^{-\mu A}\ e^{-\beta E}}_{\textit{Laplace Transform}}\ \underbrace{\Omega_N(A,E)}_{\textit{Micro-Canonical}} = \int \prod_{i=1}^{N} d\psi_i\ e^{-\mu\sum_{i=1}^{N}|\psi_i|^2}\ e^{-\beta\sum_{i=1}^{N}|\psi_i|^4}$$

$$= [\mathcal{Z}(\mu,\beta)]^N$$

When are they equivalent?

$$\Omega_N(A,E) = \int_{\mu_0-i\infty}^{\mu_0+i\infty} d\mu \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta\ e^{\mu A+\beta E+N\log\mathcal{Z}(\mu,\beta)}$$

$$\mathcal{Z}(\mu,\beta) = 2\pi\int_0^\infty dr\ r\ e^{-(\mu r^2+\beta r^4)}$$

$$\psi = re^{i\phi}$$

$$d\psi = d\varphi\ dr\ r$$

EQUIVALENCE IS WHEN, FOR FIXED ***A*** AND ***E***, YOU HAVE REAL SOLUTIONS $\boldsymbol{\beta_0}$ AND $\boldsymbol{\mu_0}$ FOR SADDLE-POINT EQUATIONS

$$\Omega_N(A,E) = \exp\{\mu_0 A + \beta_0 E + N\log\mathcal{Z}(\mu_0,\beta_0)\}$$

$$\beta_0,\mu_0 \in \mathbb{R}$$

$$\frac{A}{N} = -\frac{\partial}{\partial\mu}\log[\mathcal{Z}(\mu,\beta)] = \frac{\langle\mathcal{A}\rangle_{\beta_0,\mu_0}}{N}$$

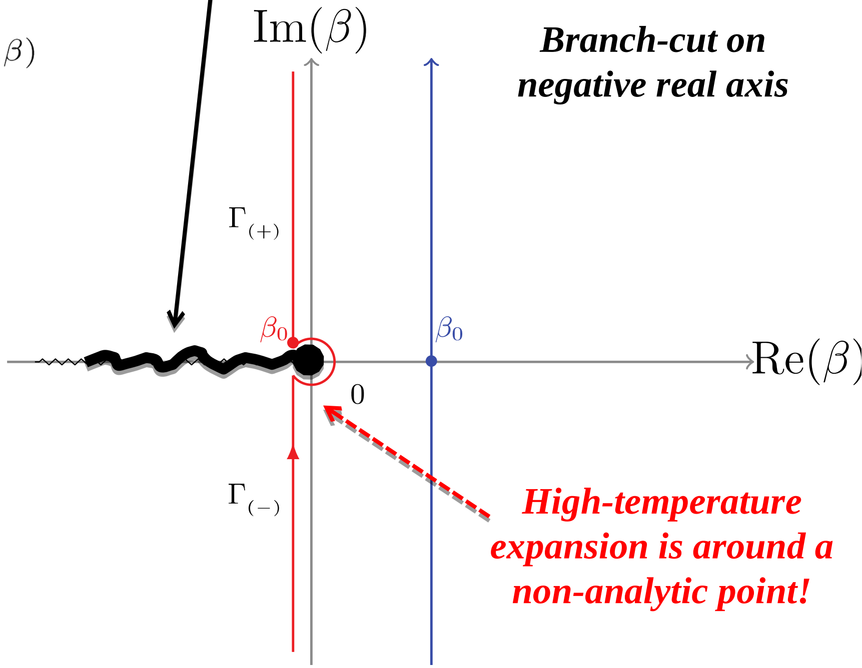
$$\frac{E}{N} = -\frac{\partial}{\partial\beta}\log[\mathcal{Z}(\mu,\beta)] = \frac{\langle\mathcal{H}\rangle_{\beta_0,\mu_0}}{N}$$

WHY ENSEMBLES ARE NOT EQUIVALENT

$$\underset{\textit{Grand-Canonical}}{\underline{\mathcal{Z}_N(\mu,\beta)}} = \int_0^\infty \underbrace{dA\; dE\; e^{-\mu A}\; e^{-\beta E}}_{\textit{Laplace Transform}}\; \underset{\textit{Micro-Canonical}}{\underline{\Omega_N(A,E)}} = \int \prod_{i=1}^{N} d\psi_i\; e^{-\mu\sum_{i=1}^{N}|\psi_i|^2}\; e^{-\beta\sum_{i=1}^{N}|\psi_i|^4}$$

$$\mathcal{Z}(\mu,\beta) = 2\pi\int_0^\infty dr\; r\; e^{-(\mu r^2+\beta r^4)} = \frac{e^{\mu^2/(4\beta)}\sqrt{\pi}\;\mu\;\mathrm{Erfc}\left(\frac{\mu}{2\sqrt{\beta}}\right)}{2\sqrt{\beta}}$$

$$\Omega_N(A,E) = \int_{\mu_0-i\infty}^{\mu_0+i\infty} d\mu \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta\; e^{\mu A+\beta E+N\log\mathcal{Z}(\mu,\beta)}$$



Branch-cut on negative real axis

High-temperature expansion is around a non-analytic point!

$E > E_{\rm th} \implies$ ***No real saddle point***
Must take analytic prolongation

$$\lim_{\substack{\beta_0\to 0\\ \mu_0=1/a}} \langle\mathcal{H}\rangle_{\beta_0,\mu_0} = E_{\rm th} < \infty$$

$$\frac{E}{N} = -\frac{\partial}{\partial\beta}\log[\mathcal{Z}(\mu,\beta)] = \frac{\langle\mathcal{H}\rangle_{\beta_0,\mu_0}}{N}$$

WHY ENSEMBLES ARE NOT EQUIVALENT

$$\underline{\mathcal{Z}_N(\mu,\beta)} = \int_0^\infty dA\, dE\, e^{-\beta E-\mu A}\, \Omega_N(A,E) = \left[\int d\psi\, e^{-\mu|\psi|^2-\beta|\psi|^4}\right]^N$$

Grand-Canonical

Every degree of freedom contributes identically to the partition function

$E > E_{\rm th} \quad \Longrightarrow \quad$ ***LOCALIZATION***

$$\lim_{\substack{\beta_0\to 0\\ \mu_0=1/a}} \langle \mathcal{H} \rangle_{\beta_0,\mu_0} = E_{\rm th} < \infty$$

Our Microcanonical Calculation

$$E_{\rm th} = 2A^2$$

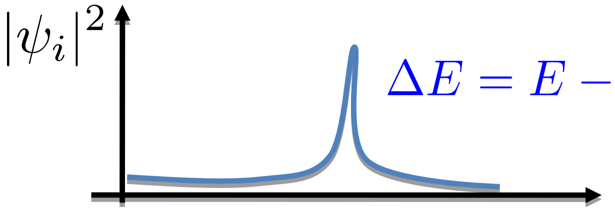
(a-posteriori argument to neglect hopping terms)

Sketchy mechanism of localization $E > E_{\rm th}$

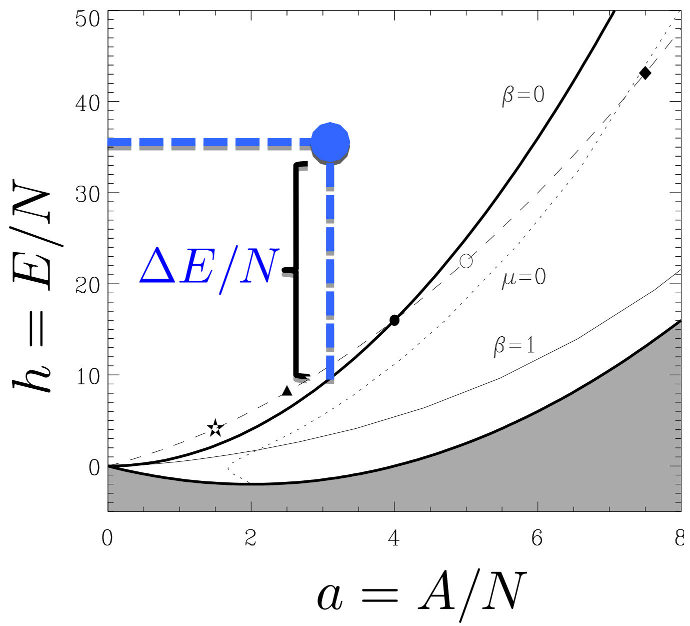
1) Cannot reach such energy by equal sharing among d.o.f.

2) The amount $E_{\rm th}$ is identically distributed among the degrees of freedom (infinite temperature background)

3) Excess energy is put into the localized phase



$$\Delta E = E - E_{\rm th}$$



THE LARGE DEVIATIONS APPROACH

Microcanonical Ensemble

$$\Omega_N(A,E) = \int \prod_{i=1}^{N} d\psi_i \; \delta\left(A - \sum_{i=1}^{N} |\psi_i|^2\right) \; \delta\left(E - \sum_{i=1}^{N} |\psi_i|^4\right)$$

Release constraint on 'particle number'

$$\Omega_N(\mu,E) = \int \prod_{i=1}^{n} d\psi_i \; e^{-\mu \sum_{i=1}^{N} |\psi_i|^2} \delta\left(E - \sum_{i=1}^{N} |\psi_i|^4\right)$$

Change of variables

$$\Omega_N(\mu,E) \approx \int \prod_{i=1}^{n} \left[d\varepsilon_i \; \frac{e^{-\mu\sqrt{\varepsilon_i}}}{\sqrt{\varepsilon_i}}\right] \delta\left(E - \sum_{i=1}^{N} \varepsilon_i\right)$$

1) $\psi = re^{i\phi}$

2) $r_i^4 = \varepsilon_i$

Partition Function **=** **Probability distribution of fat tailed variables sum**

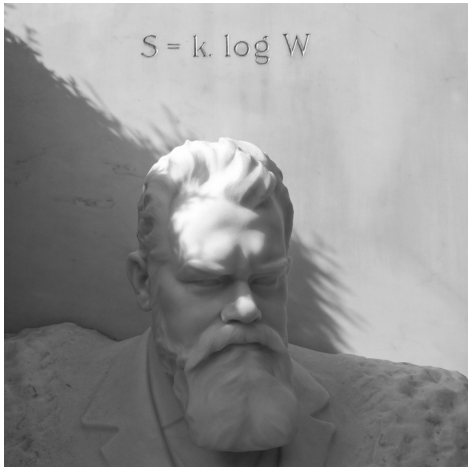
$$e^{-\varepsilon_i} < \frac{e^{-\mu\sqrt{\varepsilon_i}}}{\sqrt{\varepsilon_i}} < \frac{1}{\varepsilon_i^2}$$

→ ***Localization***

$$E > N\langle\varepsilon\rangle_\mu = E_{\rm th}$$

$$\Omega_N(\mu,E) \sim \frac{e^{-\mu\sqrt{E-E_{\rm th}}}}{\sqrt{E-E_{\rm th}}}$$

THE MAIN RESULT: MICROCANONICAL ENTROPY

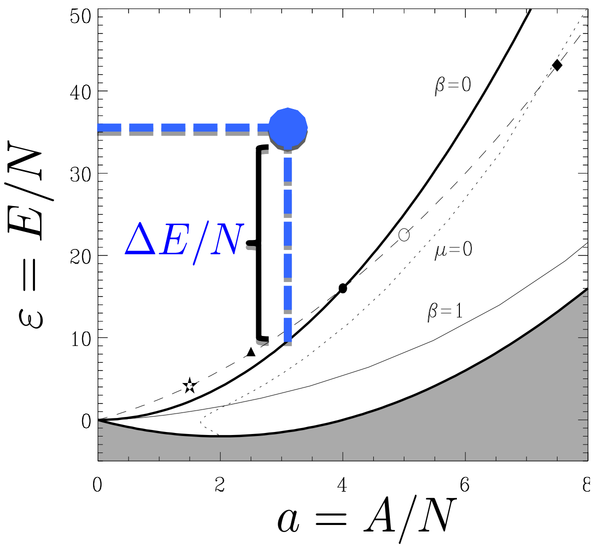


Microcanonical Entropy

$$S_N(A,E) = \mathrm{k}\ \log[\Omega_N(A,E)]$$

The first, the one … and the ONLY

$$\varepsilon_{\mathrm{th}} = 2\ a^2$$



$$\Delta E = E - E_{\mathrm{th}}$$

Localized Phase Entropy

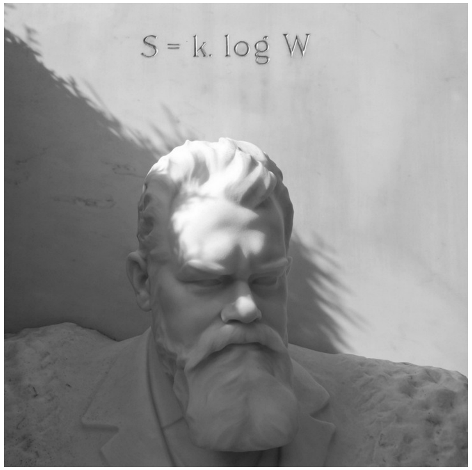
$$S_N(A,E) = \underbrace{\Sigma_0(A)} + \overbrace{\Sigma_1(E,A)}$$

Background Entropy (energy indipendent)

$$\Sigma_0(A) = N[1 + \log(\pi a)]$$

THE MAIN RESULT: MICROCANONICAL ENTROPY

Microcanonical Entropy



$$S_N(A,E) = N[1 + \log(\pi a)] + \Sigma_1(\Delta E, A)$$

Homogeneous background
ENTROPY
(EXTENSIVE)

Localized Phase
ENTROPY
(SUBEXTENSIVE)

Three regimes $\varepsilon = E/N$

$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon - \varepsilon_{\mathrm{th}})^2 & \textbf{Gaussian} \quad \varepsilon - \varepsilon_{\mathrm{th}} \sim 1/\sqrt{N} \\ -N^{1/3}\Psi(\zeta) & \textbf{Matching} \quad \varepsilon - \varepsilon_{\mathrm{th}} \sim 1/N^{1/3} \\ -N^{1/2}\sqrt{\varepsilon - \varepsilon_{\mathrm{th}}} & \textbf{Large Deviations} \quad \varepsilon - \varepsilon_{\mathrm{th}} \sim 1 \end{cases}$$

CONDENSATE
ENTROPY

$$\varepsilon_{\mathrm{th}} = 2\, a^2 \qquad \zeta = N^{1/3}(\varepsilon - \varepsilon_{\mathrm{th}})$$

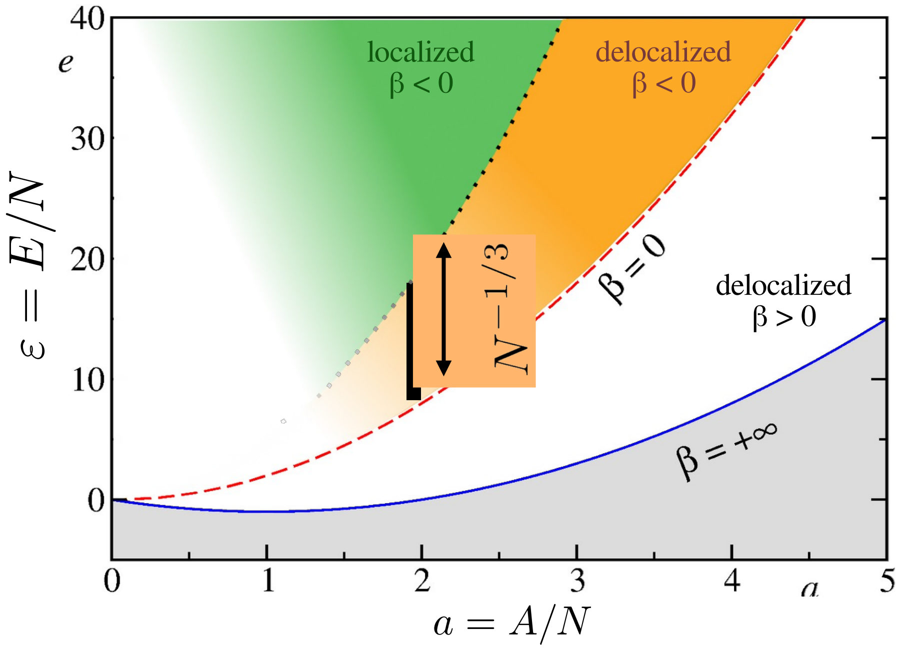
THE MAIN RESULT: MICROCANONICAL ENTROPY

$$\Psi'(\zeta_c) = \text{jump}$$

$$\zeta_c = N^{1/3}(\varepsilon_c - \varepsilon_{th})$$

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

Finite-size correction to the critical line



$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon - \varepsilon_{\rm th})^2 & \textbf{Gaussian} \quad \varepsilon - \varepsilon_{\rm th} \sim 1/\sqrt{N} \\ -N^{1/3}\Psi(\zeta) & \textbf{Matching} \quad \varepsilon - \varepsilon_{\rm th} \sim 1/N^{1/3} \\ -N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}} & \textbf{Large Deviations} \quad \varepsilon - \varepsilon_{\rm th} \sim 1 \end{cases}$$

CONDENSATE ENTROPY

Exact calculation of this function

$$\varepsilon_{\rm th} = 2\,a^2 \qquad \zeta = N^{1/3}(\varepsilon - \varepsilon_{\rm th})$$

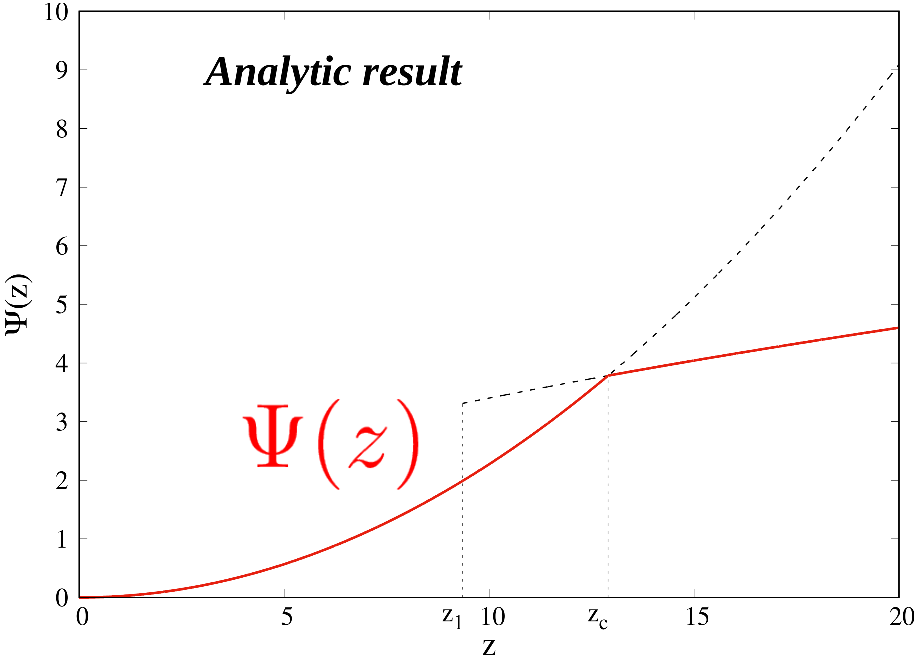
FIRST-ORDER MECHANISM

$$\Psi'(\zeta_c) = \text{jump}$$

$$\zeta_c = N^{1/3}(\varepsilon_c - \varepsilon_{th})$$

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

Finite-size correction to the critical line



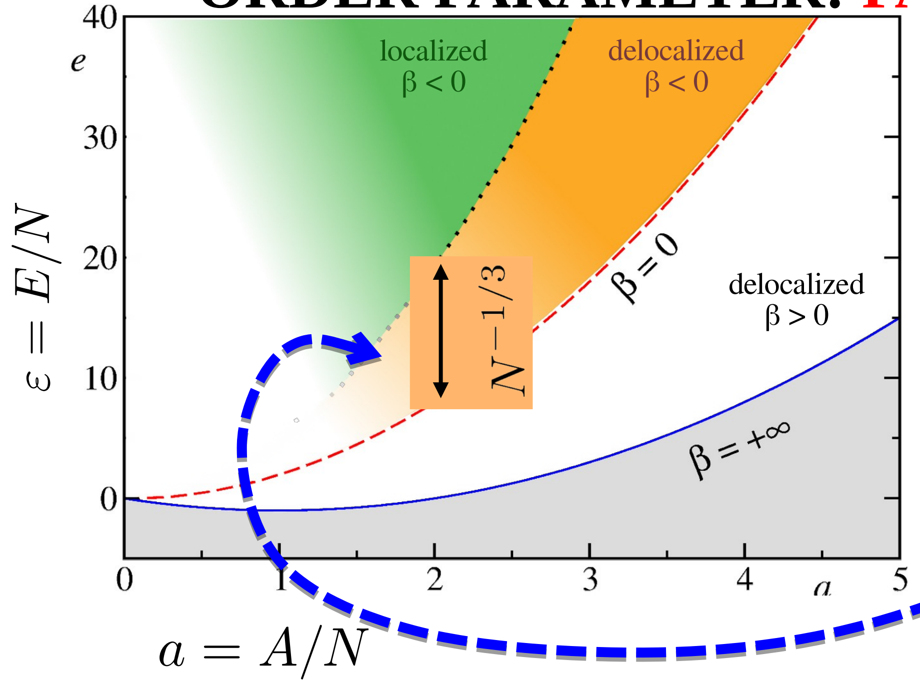
$$\Omega(A,E) \propto \underbrace{\exp\left\{-N^{1/3}\zeta^2/(2\sigma^2)\right\}}_{\Omega_{\mathrm{deloc}}(\zeta,N)} + \underbrace{\exp\left\{-N^{1/3}\chi(\zeta)\right\}}_{\Omega_{\mathrm{loc}}(\zeta,N)}$$

$$P_{\mathrm{loc}}(\zeta,N) = \frac{\Omega_{\mathrm{loc}}(\zeta,N)}{\Omega_{\mathrm{loc}}(\zeta,N) + \Omega_{\mathrm{deloc}}(\zeta,N)}$$

$$\zeta > \zeta_c \implies \lim_{N\to\infty} P_{\mathrm{loc}}(\zeta,N) = 1$$

$$\zeta < \zeta_c \implies \lim_{N\to\infty} P_{\mathrm{loc}}(\zeta,N) = 0$$

ORDER PARAMETER: PARTICIPATION RATIO



Order parameter : participation ratio

$$\varepsilon > \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N = c > 0$$

$$\varepsilon < \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N \sim 1/N$$

The order parameter jumps at the dotted blue line!

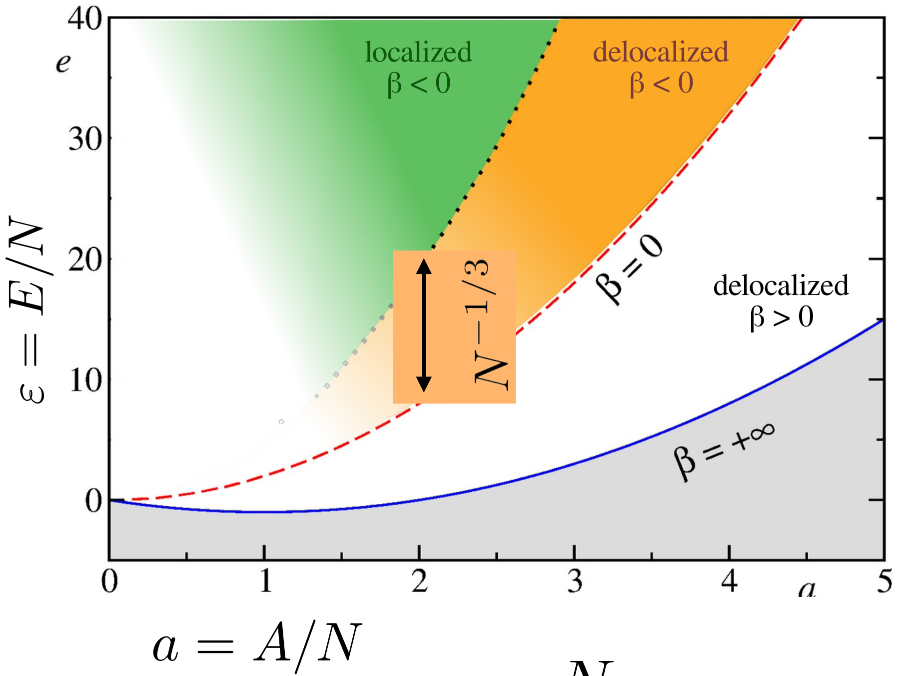
$$\Psi'(\zeta_c) = \text{jump}$$

$$\zeta_c = N^{1/3}(\varepsilon_c - \varepsilon_{th})$$

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

Finite-size correction to the critical line

NEGATIVE TEMPERATURE – SUBEXTENSIVE ENTROPY



$$\varepsilon_{\rm th} < \varepsilon < \varepsilon_c \quad = \quad \textbf{Uninteresting ?}$$

Not really…

$$\varepsilon > \varepsilon_{\rm th} \quad \Longrightarrow \quad \frac{\partial S}{\partial E} = \frac{1}{T} < 0$$

NEGATIVE TEMPERATURE

$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon - \varepsilon_{\rm th})^2 \\ -N^{1/3}\Psi(\zeta) \\ -N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}} \end{cases}$$

CONDENSATE ENTROPY

Gaussian $\varepsilon - \varepsilon_{\rm th} \sim 1/\sqrt{N}$

Matching $\varepsilon - \varepsilon_{\rm th} \sim 1/N^{1/3}$

Large Deviations $\varepsilon - \varepsilon_{\rm th} \sim 1$

$$\varepsilon_{\rm th} = 2\,a^2 \qquad \zeta = N^{1/3}(\varepsilon - \varepsilon_{\rm th}) \qquad T = N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}}$$

ORDER PARAMETER: PARTICIPATION RATIO

$$\Psi'(\zeta_c) = \text{jump}$$

Order Parameter = Participation Ratio

$$\mathcal{P}_{_N} = \left\langle \frac{\sum_{i=1}^{N} \varepsilon_i^2}{\left(\sum_{i=1}^{N} \varepsilon_i\right)^2} \right\rangle_{micro}$$

Consistent with non-analyticity of Entropy

$$\boxed{\begin{array}{lcl} \varepsilon > \varepsilon_c & \Longrightarrow & \lim\limits_{N\to\infty} \mathcal{P}_{_N} = c > 0 \\ \varepsilon < \varepsilon_c & \Longrightarrow & \lim\limits_{N\to\infty} \mathcal{P}_{_N} \sim 1/N \end{array}}$$

'Pseudo-condensate' **Localization**

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim\limits_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ensembles inequivalence

ORDER PARAMETER: PARTICIPATION RATIO

$$\Psi'(\zeta_c) = \text{jump}$$

Consistent with non-analyticity of Entropy

$$\boxed{\begin{array}{lcl} \varepsilon > \varepsilon_c & \Longrightarrow & \lim_{N\to\infty} \mathcal{P}_{_N} = c > 0 \\ \varepsilon < \varepsilon_c & \Longrightarrow & \lim_{N\to\infty} \mathcal{P}_{_N} \sim 1/N \end{array}}$$

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'Pseudo-condensate' **Localization**

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

ORDER PARAMETER: PARTICIPATION RATIO

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

In the thermodynamic limit the two values coincide and the order parameter is continuous at the transition

Consistent with non-analyticity of Entropy

$$\varepsilon > \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N = (\varepsilon - \varepsilon_{th})^2/\varepsilon^2$$

$$\varepsilon < \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N \sim 1/N$$

'Pseudo-condensate' **Localization**

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

ORDER PARAMETER: PARTICIPATION RATIO

Merging at N=∞ into a mixed-order transition?

2nd Order **1st Order**

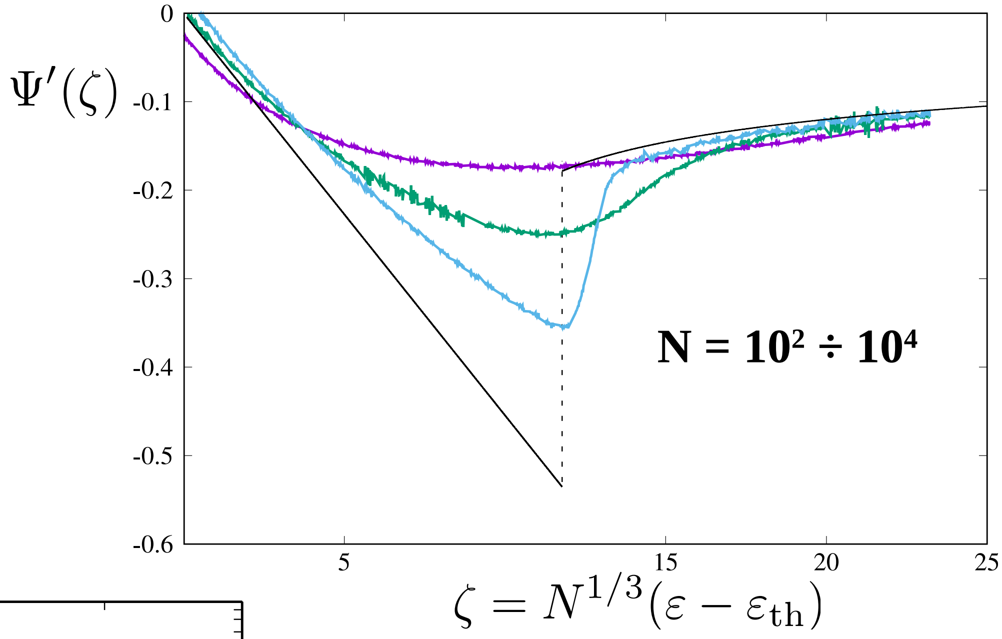
‘Pseudo-condensate’ **Localization**

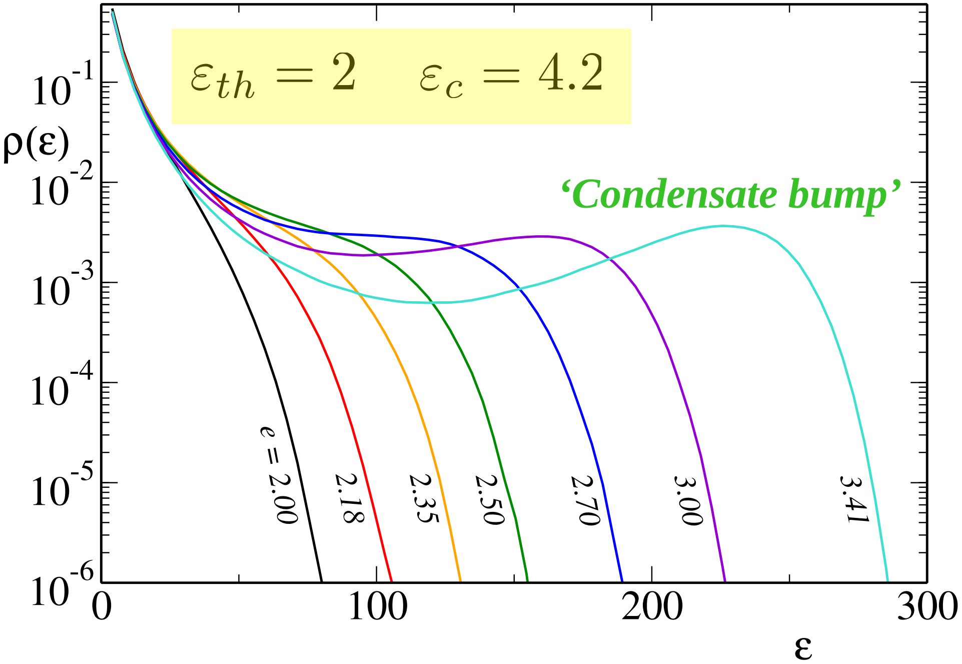
	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

FINALLY SOME FIGURES : Monte Carlo sampling of rare events

Entropy of the condensate
As a function of size





Marginal distribution of energy on a single site (microcanonical)

Delocalized phase

Monotonous exponential decay

Localized & pseudo-localized

Formation of a secondary peak, the "condensate bump"

Discrete Non-Linear Schrödinger Equation (DNLSE)

QUITE OFTEN LOCALIZATION IS RELATED TO INTEGRABILITY

'Integrals of motion in the many-body localized phase', Valentina Ros, M. Müller, A. Scardicchio, *Nuclear Physics B* **891**, 420-465 **(2015)**

They compute explicitly the N integrals of motion!

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N} (\psi_i^* \psi_{i+1} + \psi_i \psi_{i+1}^*) + \frac{\nu}{2} \sum_{i=1}^{N} |\psi_i|^4$$

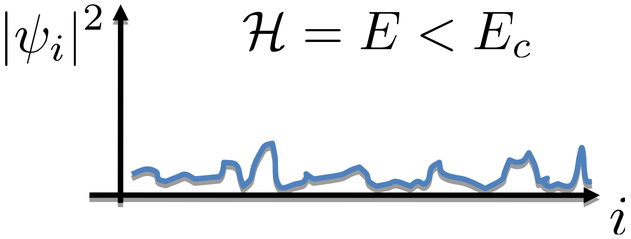
PARTICLES NUMBER (conserved)

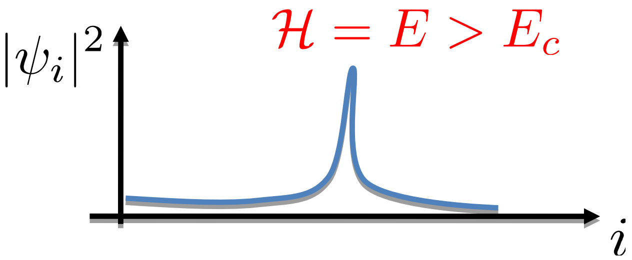
$$A = \sum_{i=1}^{N} |\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





FIRST ORDER!

1) WHICH KIND OF PHASE TRANSITION ?

MICROCANONICAL

2) WHICH STATISTICAL ENSEMBLE?

3) LOCALIZATION COMES FROM INTEGRABILITY? (N integrals of motion) ***NO!***

4) IS DISORDER NECESSARY FOR LOCALIZATION? ***NO!***

Discrete Non-Linear Schrödinger Equation (DNLSE)

Anderson Localization

One-body localization due to quenched disorder

$$\mathcal{H} = J \sum_{\langle ij \rangle} \hat{c}_i^\dagger \hat{c}_j + \sum_{i=1}^{N} h_i \; \hat{c}_i^\dagger \hat{c}_i$$

Many-body Localization (MBL)

Disorder + **many-body interactions**.

$$\mathcal{H} = J \sum_{\langle ij \rangle} \hat{c}_i^\dagger \hat{c}_j + \sum_{i=1}^{N} h_i \; \hat{c}_i^\dagger \hat{c}_i + k \sum_{i=1}^{N} \hat{c}_i^\dagger \hat{c}_i \; \hat{c}_{i+1}^\dagger \hat{c}_{i+1}$$

STATE of THE ART

1) **Localized phase is stable with respect to (weak) non-linearities.**

2) **Role of disorder in presence of many-body interactions?**

3) **Does localization survives without disorder?**

This work contribution

1) **We do find localization in absence of disorder! (known numerically)**

2) **NON-LINEAR terms (many-body) are the source of localization! (outcome of the exact calculation)**

What about Localization of Glassy Light in Random Lasers?

'Glassines and the lack of equpartition in random lasers',
G. Gradenigo, F. Antenucci, L. Leuzzi, Phys. Rev. Research 2, 023399 (2020)

'Universality class in the mode-locked random laser',
J. Niedda, G. Gradenigo, L. Leuzzi, G. Parisi, ***arXiv:2210.04362*** (2022)

'Intensity pseudo-localized phase in the glassy random laser',
J. Niedda, L. Leuzzi, G. Gradenigo, ***arXiv:2212.05106*** (2022).

Signatures of the pseudo-localized phase in spin glass model of random lasers

CONCLUSIONS - PERSPECTIVES

1) We provided the first fully consistent description of the **localization transition** in the Discrete Non-Linear Schrödinger Equation **(DNLSE)**

2) Localization in the DNLSD can only described within the **Microcanonical Ensemble**

3) We put in evidence the existence, at large but finite N, of a delocalized (presumably non ergodic) state at negative temperature, the **pseudo-condensate** (relevant for experiments).

Further investigations: multifractal wave function: $I(q) = N\langle|\psi_i|^{2q}\rangle$

4) We clarified that the transition has a **mixed first/second order**, similarly to the **ergodicity breaking** transition in **glasses** (not spin glasses!): **Random First-Order transition**.

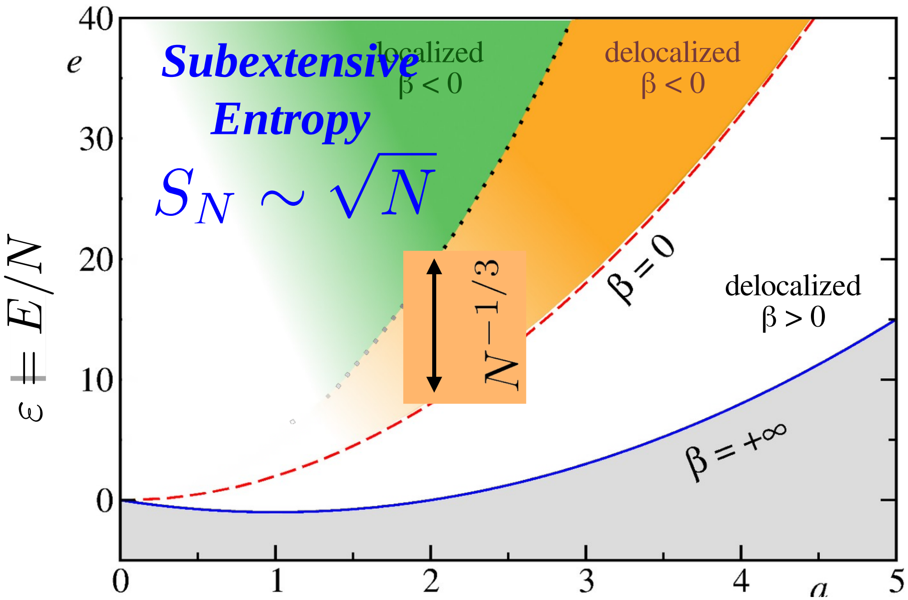
Further investigations: pseudo-localization/localization in models of glasses (in progress).

5) We clarified a mechanism for localization/ergodicity-breaking in the strong-coupling regime:

- Not related to integrability (only two conserved quantities, perhaps **emergent** integrability?)
- Straighforwad extension to D > 1 (further investigations)

THANKS FOR YOUR ATTENTION

THE MAIN RESULT: MICROCANONICAL ENTROPY



1) Microcanonical and canonical ensembles are not equivalent

2) Localization looks like a a mixed order transition in the microcanonical ensemble

3) Negative temperature ONLY in microcanonical ensemble (zero for N=∞).

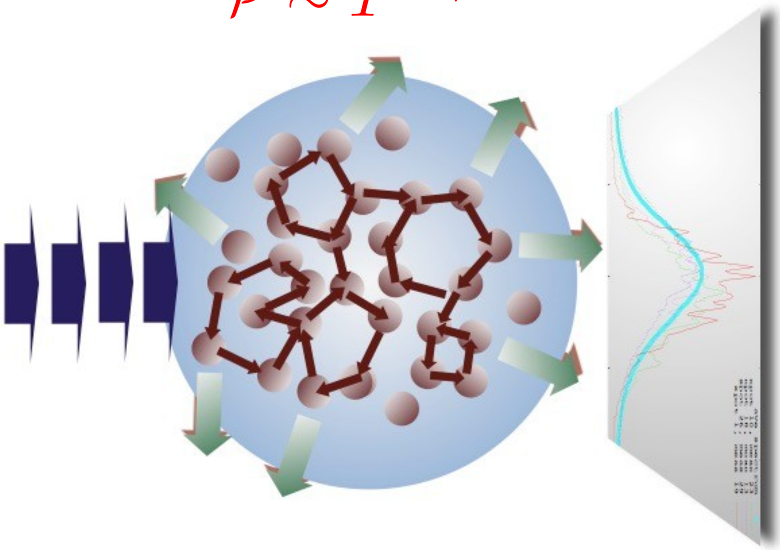
4) Localized solution has subextensive entropy (area law?, entaglement?)

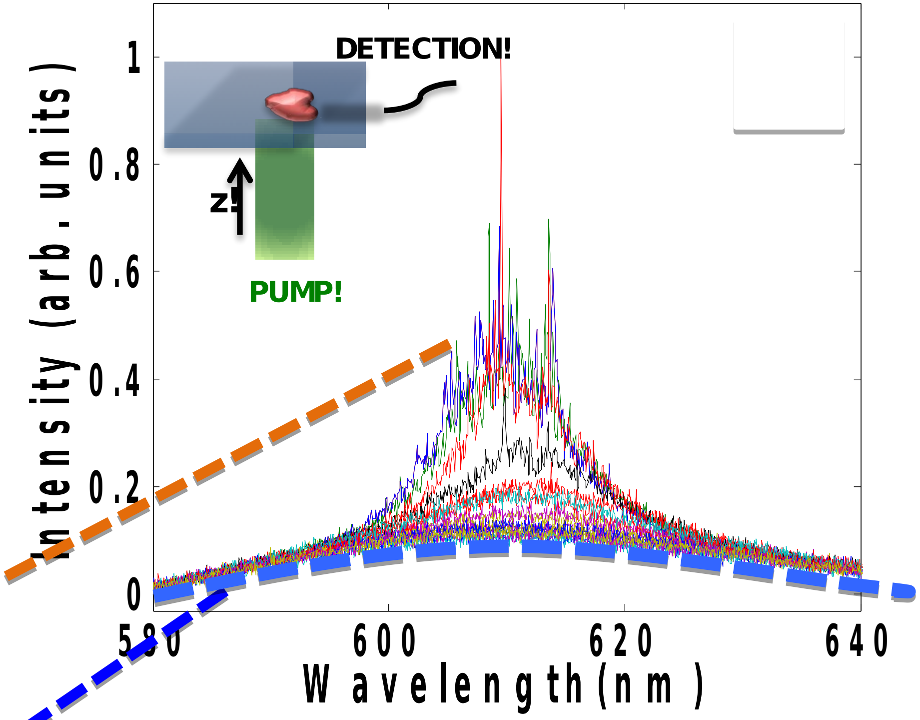
Phenomenon: from Fluorescence to Random Lasing

Optical power (inverse temperature)
is pumped into an optically active
random medium

$\mathcal{P} \sim T^{-1/2}$

→ **EMISSION SPECTRUM** $\langle |A_k|^2 \rangle$



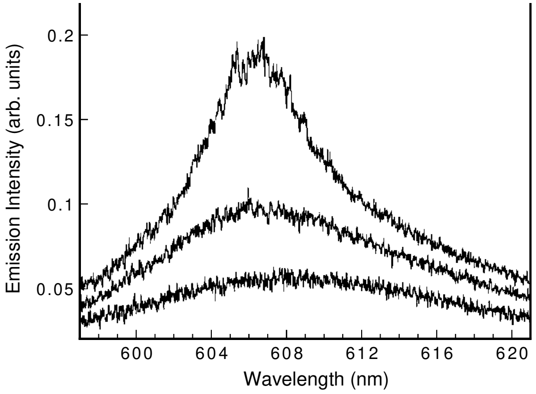


$\mathcal{P} > \mathcal{P}_{\rm th}$ **Random Lasing:** **coherent** anisotropic emission **(NON-ERGODIC)**

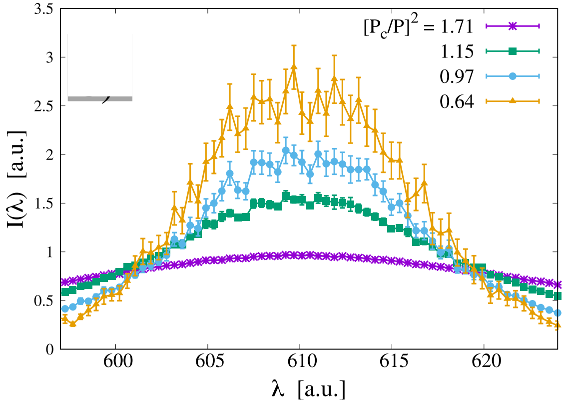
$\mathcal{P} < \mathcal{P}_{\rm th}$ **Fluorescence:** **incoherent** isotropic emission **(ERGODIC)**

Glass Transition

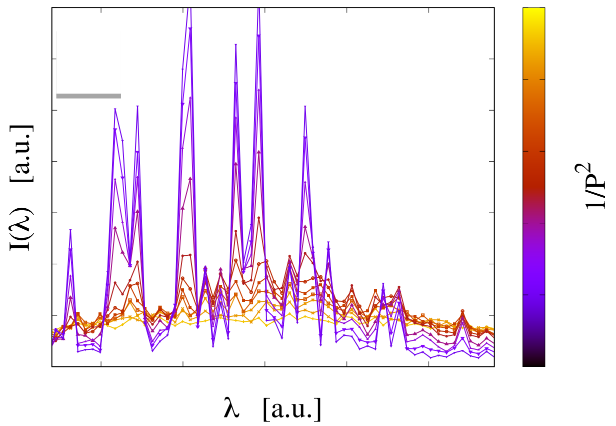
Experiments (not mine!)

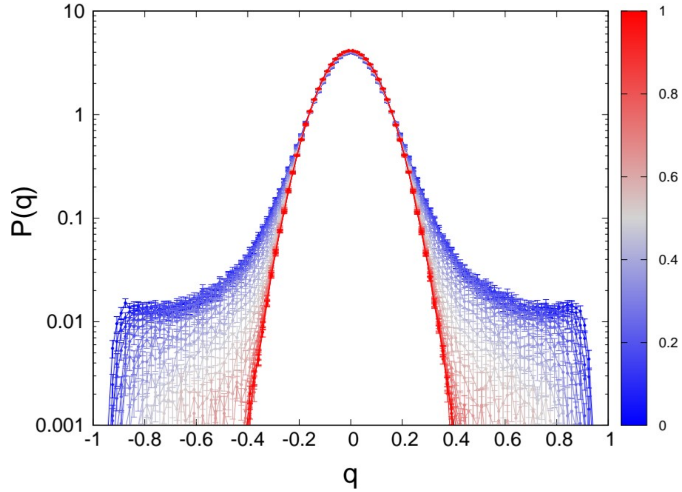


Simulations: average over disorder



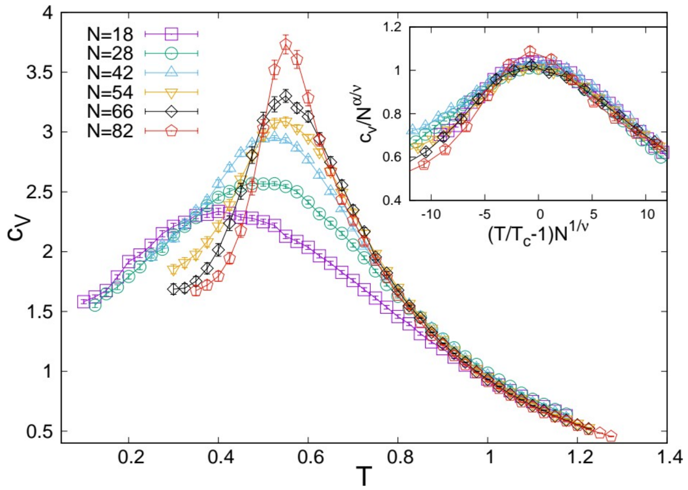
One istance of disorder





OVERLAP (Glass order parameter)

HEAT CAPACITY

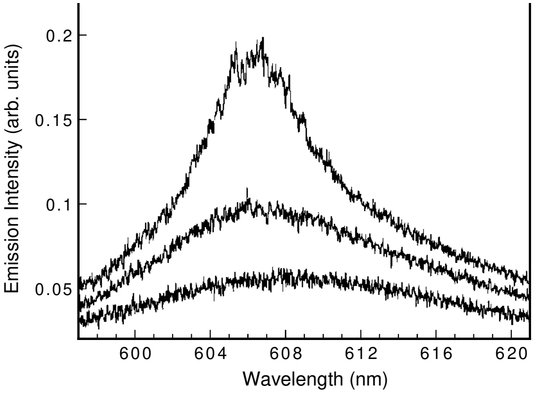


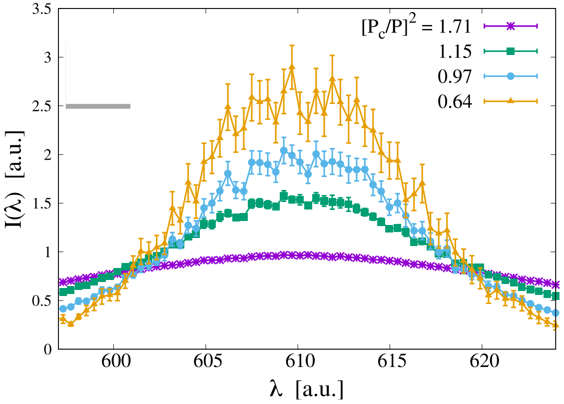
‘Universality class in the mode-locked random laser’,

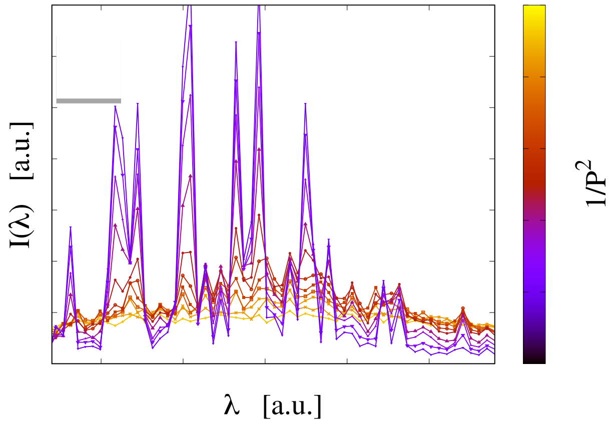
J. Niedda, G. Gradenigo, L. Leuzzi, G. Parisi, ***arXiv:2210.04362*** (2022)

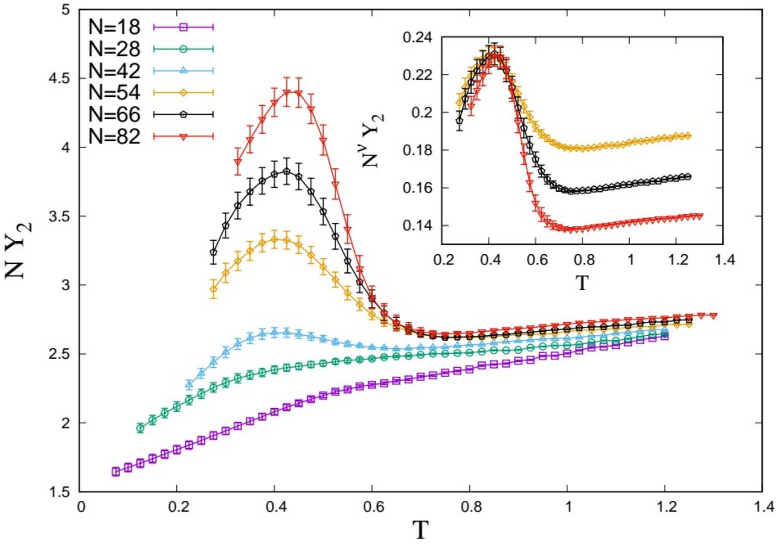
Glass phase in Random Lasers is pseudo-localized

(no localization no equipartition)









PARTICIPATION RATIO

$$Y_2 = \left\langle \frac{\sum_k m_k^2}{\left(\sum_k m_k\right)^2} \right\rangle \qquad m_k = |A_k|^2$$

Deloc $Y_2 \sim 1/N$

Loc $Y_2 = \mathcal{O}(1)$

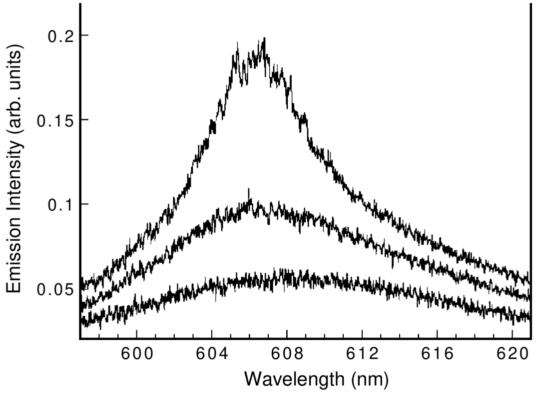
$Y_2 = 1/N^{1-\nu}$

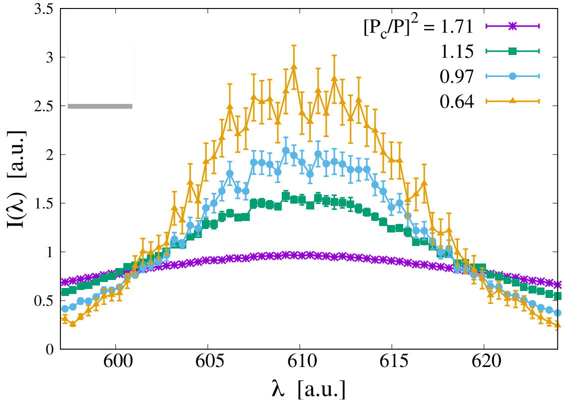
Pseudo-Localized

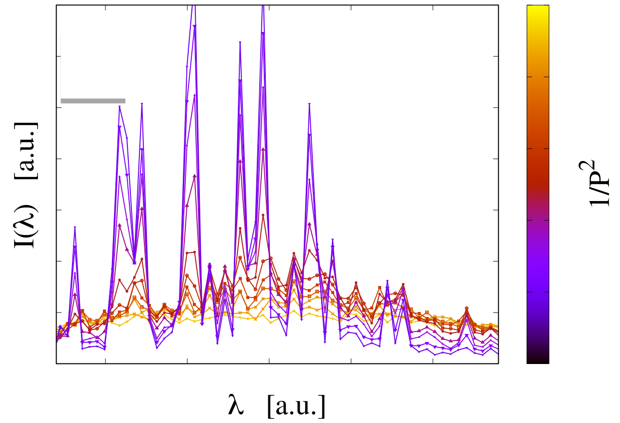
‘Pseudo-localized phase in the mode-locked p-spin’,
J. Niedda, L. Leuzzi, G. Gradenigo, *in preparation* (2022).

Glass phase in Random Lasers is pseudo-localized

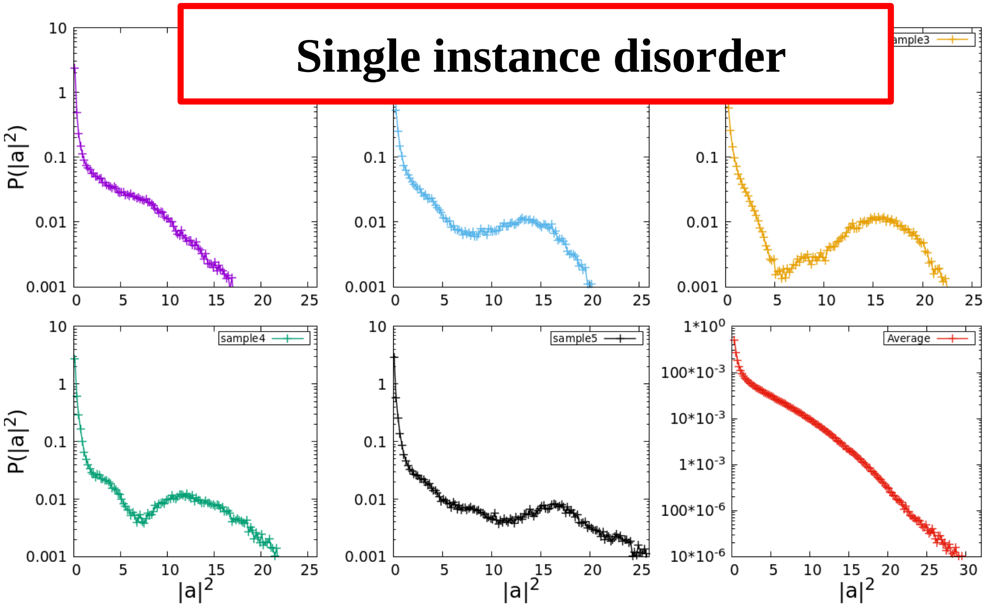
(no localization no equipartition)



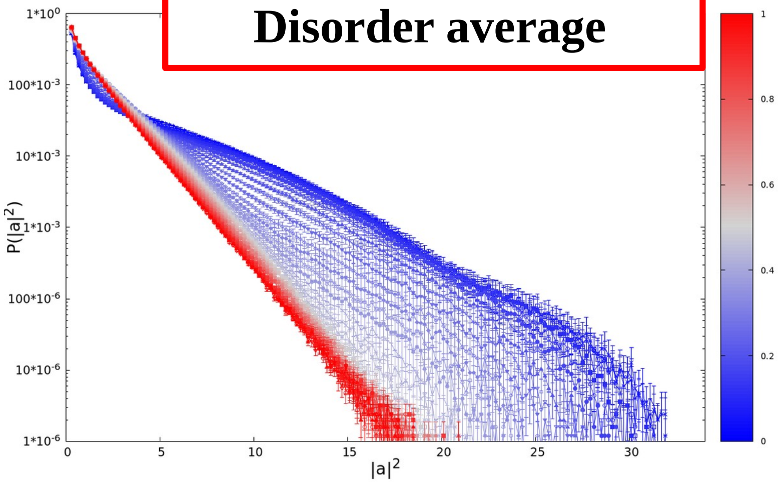




Single instance disorder



Disorder average



'Pseudo-localized phase in the mode-locked p-spin',
J. Niedda, L. Leuzzi, G. Gradenigo, *in preparation* (2022).

AMPLITUDE LOCAL MARGINALS

Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical Approximation

$$\hat{H} = \int d^3x\ \hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} \right] \hat{\psi}(\mathbf{x}) + \frac{4\pi\hbar^2 a_s}{2m} \int d^3x\ \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x})$$

‘*Discrete Breathers in Bose-Einstein Condensates*’, Franzosi, Livi, Oppo, Politi, *Nonlinearity.* **24**, R89 **(2011)**

Second-quantization Hamiltonian of interacting bosons condensate

$V(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$ **Repulsive contact interactions**

Bogoliubov approximation $\hat{\psi}(\mathbf{x}) = \Psi(\mathbf{x}) + \hat{\varphi}(\mathbf{x})$

$\Psi(\mathbf{x}) = \langle \hat{\psi}(\mathbf{x}) \rangle$ **Condensate wave-function** (c-number)

$\hat{\varphi}(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \langle \hat{\psi}(\mathbf{x}) \rangle$ **Deviation opeartor**

Expand the Hamiltonian up to second order in powers of $\hat{\varphi}(\mathbf{x}),\ \hat{\varphi}^\dagger(\mathbf{x})$
(small quantum fluctuations around the mean-field solution)

$$\hat{H} = K_0 + \hat{K}_1 + \hat{K}_2 + \ldots \qquad \hat{K}_1 = \mathcal{O}(\hat{\varphi}) \qquad \hat{K}_2 = \mathcal{O}(\hat{\varphi}^2)$$

Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical Approximation

$$\hat{K}_1 = 0 \quad \Longleftrightarrow \quad \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\mathrm{ext}}(\mathbf{x}) \right] \Psi(\mathbf{x}) - \frac{\nu}{2} |\Psi(\mathbf{x})|^2 \Psi(\mathbf{x}) = 0$$

Gross-Pitaevskii Equation: non-linear equation for the ‘order parameter’ of a quantum transition (semiclassical approximation)

Bogoliubov approximation $\hat{\psi}(\mathbf{x}) = \Psi(\mathbf{x}) + \hat{\varphi}(\mathbf{x})$

$\Psi(\mathbf{x}) = \langle \hat{\psi}(\mathbf{x}) \rangle$ **Condensate wave-function** (c-number)

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Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical Approximation

$$\hat{K}_1 = 0 \quad \Longleftrightarrow \quad \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) \right] \Psi(\mathbf{x}) - \frac{\nu}{2} |\Psi(\mathbf{x})|^2 \Psi(\mathbf{x}) = 0$$

Gross-Pitaevskii Equation: non-linear equation for the 'order parameter' of a quantum transition (semiclassical approximation)

$$V_{\text{ext}}(\mathbf{x}) = \underbrace{\frac{\hbar^2\omega^2}{4E_r} \; \sin^2(k_{\text{L}} x)}_{\text{Periodic modulation - x}} + \underbrace{\frac{m\Omega^2}{2} \; (y^2 + z^2)}_{\text{Harmonic traps (y,z)-plane}}$$

Effectively on a
1-dimensional lattice

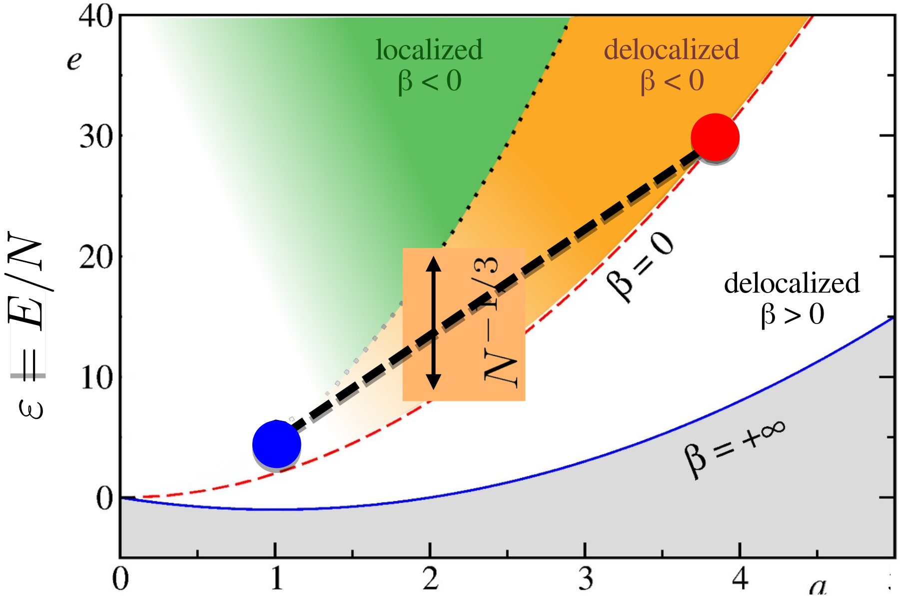
Hamiltonian system on a lattice

$$\mathcal{H} = \sum_{i=1}^{N} \Psi_i^* \Psi_{i+1} + \Psi_{i+1}^* \Psi_i + \frac{\nu}{2} \sum_{i=1}^{N} |\Psi_i|^4$$

Canonical conjugate variables

$$\{\Psi_i^*, \Psi_j\} = i \; \delta_{ij}/\hbar \qquad i\dot{\Psi}_i = -\frac{\partial \mathcal{H}}{\partial \Psi_i^*}$$

PROBING THE NEGATIVE TEMPEATURE



$\varepsilon_{\rm th} < \varepsilon < \varepsilon_c \quad = \quad$ **Uninteresting ?**

Not really…

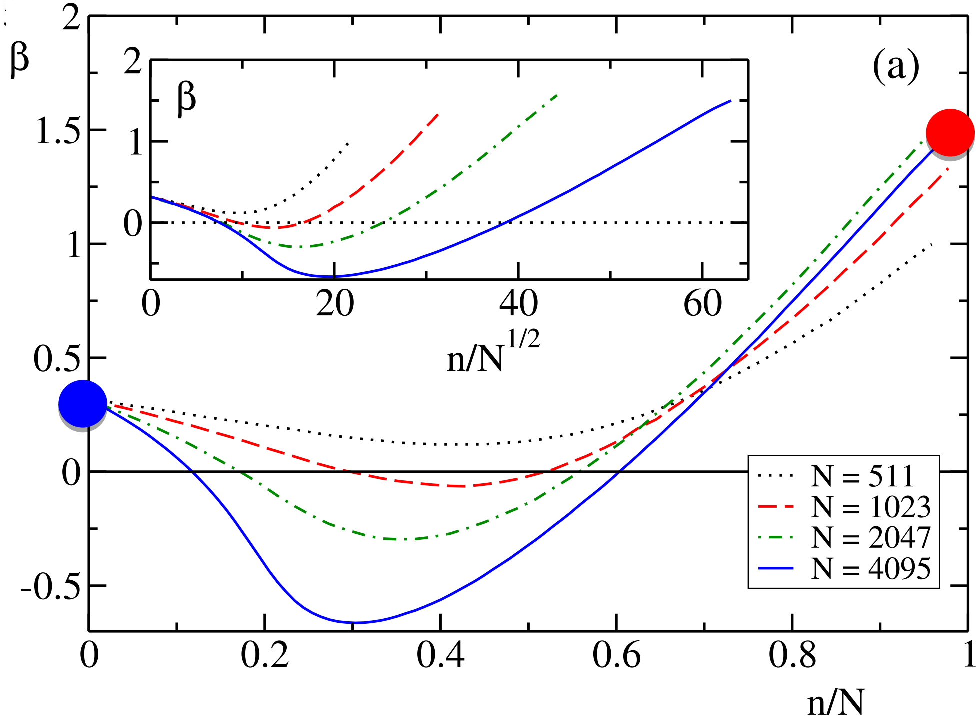
$$\varepsilon > \varepsilon_{\rm th} \quad \Longrightarrow \quad \frac{\partial S}{\partial E} = \frac{1}{T} < 0$$

NEGATIVE TEMPERATURE

Discrete Non-Linear Schrödinger Equation coupled at the boundaries with reservoirs at different temperature

'A chain, A bath, A sink and a Wall',

S. Iubini, S. Lepri, R. Livi, G.-L. Oppo, A. Politi, *Entropy* **(2017)**



Discrete Non-Linear Schrödinger Equation (DNLSE)

Condensate wave-function (order parameter) $\langle\hat{\psi}\rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial\psi_i}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\psi_i^*} = -(\psi_{i+1}+\psi_{i-1}) - \nu|\psi_i|^2\psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N}(\psi_i^*\psi_{i+1}+\psi_i\psi_{i+1}^*) + \frac{\nu}{2}\sum_{i=1}^{N}|\psi_i|^4$$

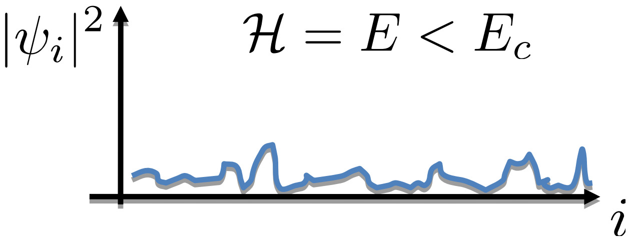
PARTICLES NUMBER (conserved)

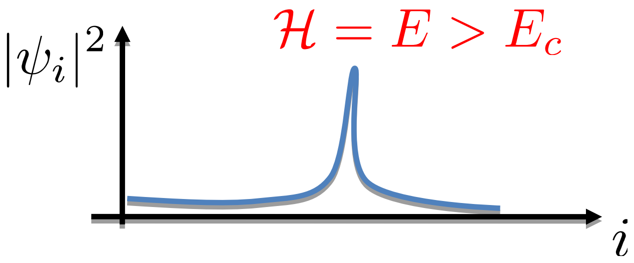
$$A = \sum_{i=1}^{N}|\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





1) WHICH KIND OF PHASE TRANSITION ? **2) WHICH STATISTICAL ENSEMBLE?**

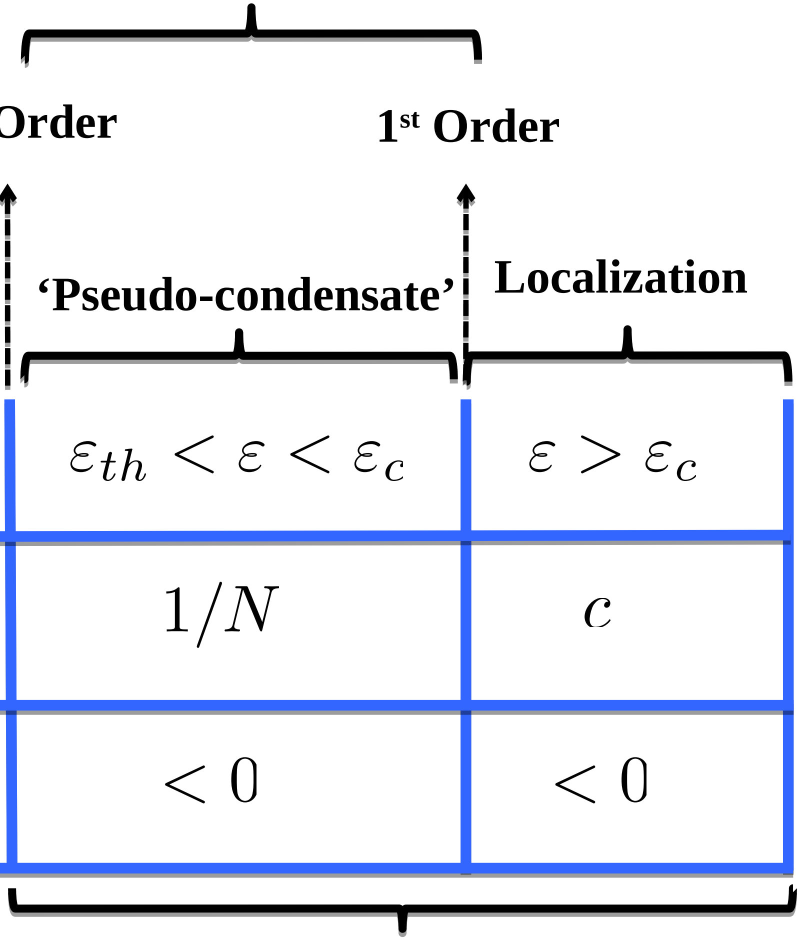
3) LOCALIZATION COMES FROM INTEGRABILITY? (N integrals of motion)

4) IS DISORDER NECESSARY FOR LOCALIZATION?

ORDER PARAMETER: PARTICIPATION RATIO

Merging at N=∞ into a mixed-order transition?

Is there any known example of such a transition?



2^{nd} Order

1^{st} Order

‘Pseudo-condensate’

Localization

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

A VERY WELL KNOWN MIXED ORDER TRANSITION:
RANDOM FIRST-ORDER or IDEAL GLASS TRANSITION

P-spin model $\mathcal{H} = -\sum_{ijkl} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$ $\sum_{i=1}^{N}\sigma_i^2 = N$

#-interactions = N^4 $J_{ijkl} =$ **iid Gaussian variates** $\langle J^2\rangle \sim N^{-3}$

GLASS TRANSITION = ERGODICITY BREAKING TRANSITION

IMPORTANT SIMILARITIES WITH DNLS

- ✓ ***Locally unbounded continuous variables***
- ✓ ***Non-linear interactions***
- ✓ ***Global spherical constraint***

… NOT SHARED BY MODELS LIKE SHERRINGTON-KIRKPATRICK

- ✓ ***Discrete spins***
- ✓ ***Linear interactions***

A VERY WELL KNOWN MIXED ORDER TRANSITION:
RANDOM FIRST-ORDER or IDEAL GLASS TRANSITION

P-spin model $\mathcal{H} = -\sum_{ijkl} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$ $\sum_{i=1}^{N}\sigma_i^2 = N$

#-interactions = N^4 $J_{ijkl} =$ **iid Gaussian variates** $\langle J^2\rangle \sim N^{-3}$

GLASS TRANSITION = ERGODICITY BREAKING TRANSITION

FIRST-ORDER FEATURES

Order Parameter: ***OVERLAP*** = Similarity among two configurations chosen at random in the equilibrium ensemble

$$\boxed{q^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{\alpha}\sigma_i^{\beta}}$$

$q \approx 0$ **different**

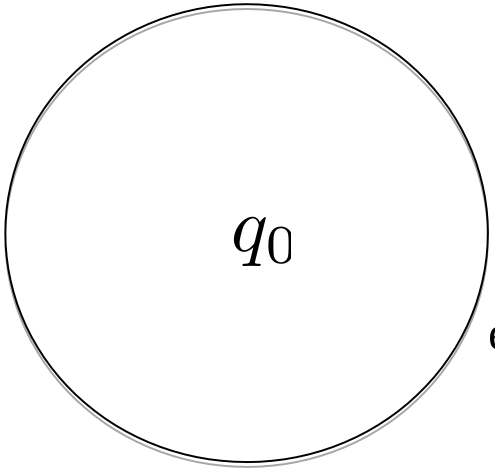
$q \approx 1$ **similar**

Can be measured in simulations

$$P(q) = (1-m)\ \delta(q-q_1) + m\ \delta(q-q_0)$$

Ergodicity Breaking: Parisi's order parameter

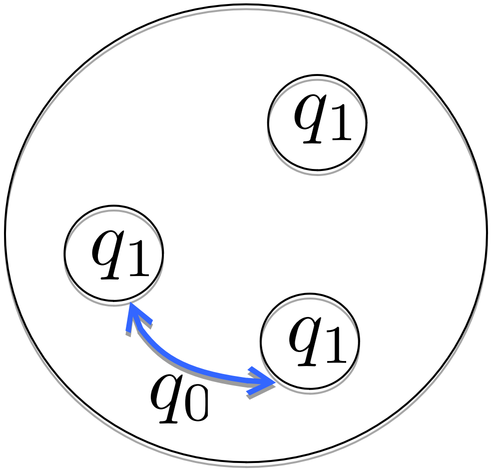
High Temperature



ERGODIC

all regions of
phase space are
equally available

Low Temperature

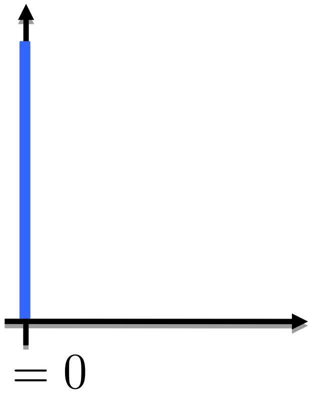


NON-ERGODIC

Phase-space partitioning in **disjoint ergodic components** with self overlap q_1 (mutual q_0)

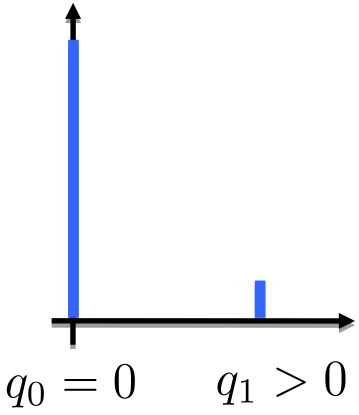
$P(q)$

'First-order like' behaviour

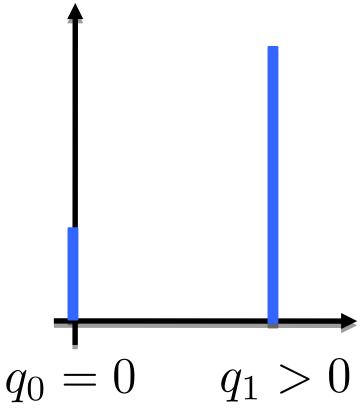


Ergodic T > T_K

Typically confs are different



T = T_K



Glass T < T_K

Typically confs are similar

Ergodicity Breaking: Parisi’s order parameter

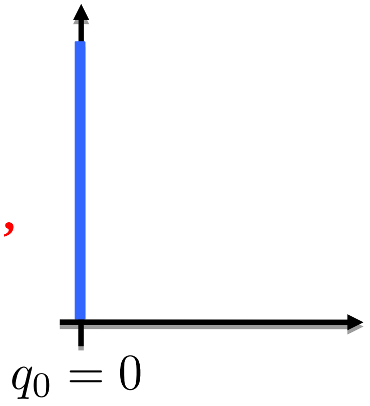
…BUT STILL IS NOT A FIRST-ORDER TRANSITION

- **NO LATENT HEAT AT THE CRITICAL TEMPERATURE T_K**
- **AVERAGE VALUE OF ORDER PARAMETER CONTINUOUS AT THE TRANSITION**

$$\int dq\ P(q)\ q = (1-m)\ q_1$$

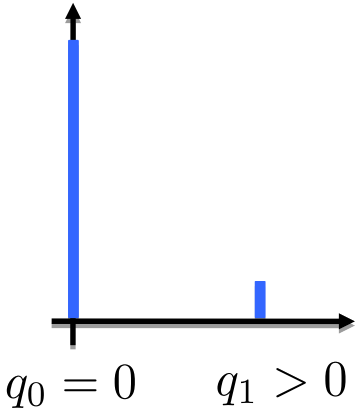
$P(q)$

‘First-order like’ behaviour

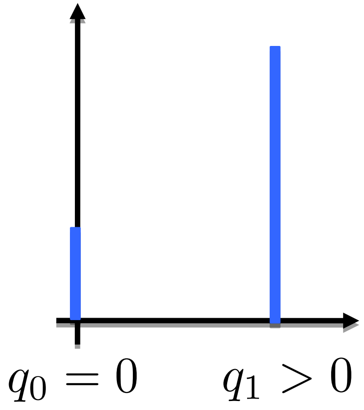


Ergodic T > T_K

Typically confs are different



T = T_K



Glass T < T_K

Typically confs are similar

Ergodicity Breaking: Parisi’s order parameter

RANDOM FIRST-ORDER TRANSITION

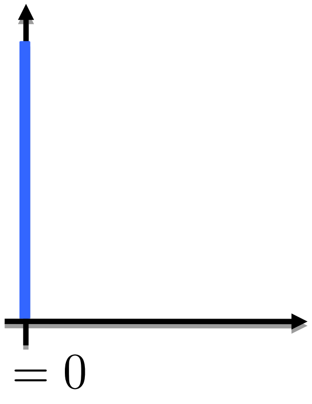
- NO LATENT HEAT AT THE CRITICAL TEMPERATURE T_K

- AVERAGE VALUE OF ORDER PARAMETER CONTINUOUS AT THE TRANSITION

$$\int dq \; P(q) \; q = (1-m) \; q_1$$

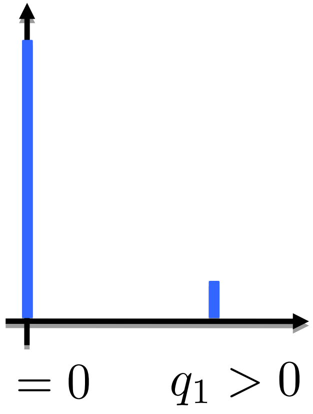
$P(q)$

‘First-order like’ behaviour

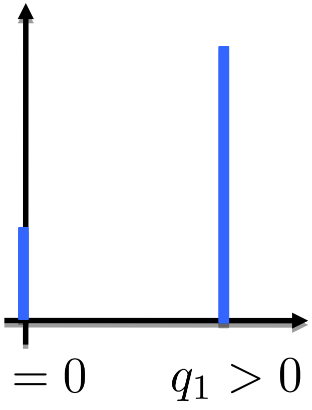


Ergodic T > T_K

Typically confs are different



T = T_K



Glass T < T_K

Typically confs are similar

What about localization in Glasses?

P-spin model on Complete Graph

#-interactions = N^4

$$\mathcal{H} = -\sum_{ijkl} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$$

$$\sum_{i=1}^{N} \sigma_i^2 = N$$

You have all possible (independent) interacting quadruplets !

Partition function is dominated by homogeneous solutions (replica theory)

$$q^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{\alpha}\sigma_i^{\beta}$$

α, β **replica indices**
Replicas : independent equilibrium configurations samples with identical disorder

RANDOM LASER : a possible benchmark for glass+localization transition

1) **Modes of electromagnetic field in a disordered cavity** $A_k(t) = |A_k(t)|\ e^{i\varphi_k(t)}$

2) **What we study**: Stationary probability distribution. Numerical sampling

$$P[A_1, \ldots, A_N] = e^{-\beta \mathcal{H}[A_1,\ldots,A_N]}\ \delta\Big(\epsilon N - \sum_{i=1}^{N} |A_k|^2\Big)$$

Non-linearity

$$\mathcal{H}[\mathbf{A}] = - \sum_{\langle ijkl \rangle_{\mathrm{FMC}}} J_{ijkl}\ |A_i||A_j||A_k||A_l| \cos(\varphi_i - \varphi_j + \varphi_k - \varphi_l)$$

Disorder: J_{ijkl} are Gaussian random variables

3) Selection rule for interacting modes

typical of random lasers

Frequency Matching Condition

$$|i - j + k - l| = 0$$

DILUTION : not all the quadruples are interacting