Entanglement entropy from non-equilibrium lattice simulations

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Overview

Motivation

Entanglement in QFT

Our algorithm

Validation: $2D$ Ising model

Some preliminary results for $3D$ Ising model

Conclusions and future prospects

1. Motivation
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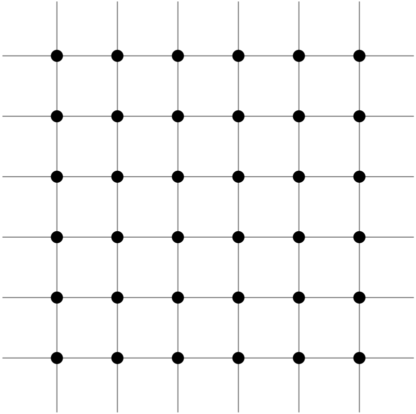
4. Validation: $2D$ Ising model
5. Some preliminary results for $3D$ Ising model
6. Conclusions and future prospects

Motivation

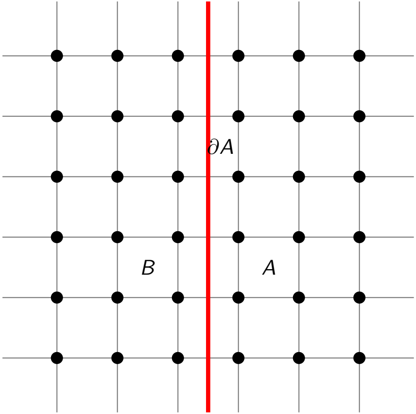
- The study of entanglement entropy is a fast-growing area of research, with applications in many different areas of physics, such as:
  - Quantum information
  - Condensed matter and CFT
  - AdS/CFT and quantum gravity
  - Gauge theories
- However analytical and numerical results are still limited to simple, highly symmetric systems.
- Non-equilibrium techniques can provide an efficient tool to calculate entanglement-related quantities [Alba 2016; D'Emidio 2019; Zhao *et al.* 2021].

Overview

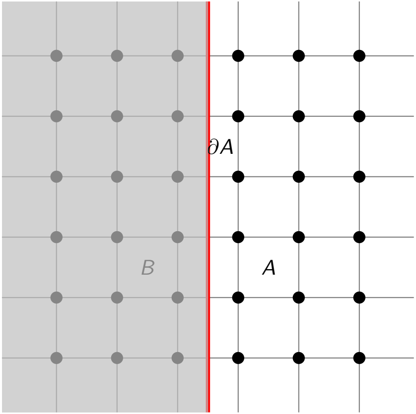
Entanglement in QFT



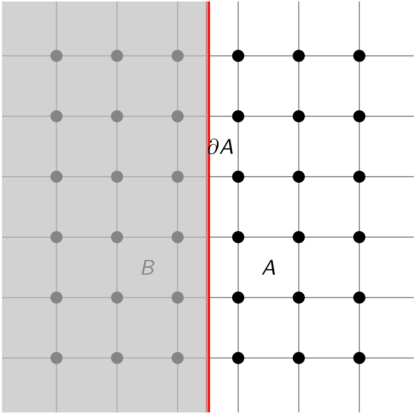
Entanglement in QFT



Entanglement in QFT



Entanglement in QFT



$$S(A) = -\operatorname{Tr}\{\rho_A \log \rho_A\}$$

Replica trick and entropic c-function

- The most efficient and general way to calculate the entanglement entropy is the replica trick [Calabrese, Cardy 2004].

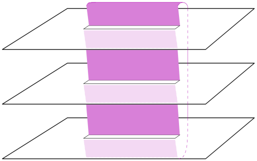


Image taken from [Cardy *et al.* 2007].

- In this geometry we can calculate $\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n}$ and extract the *Rényi entropies* and the *entropic c-functions*

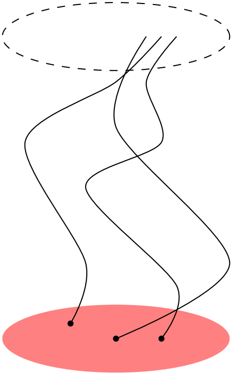
$$S_n(A) = -\frac{1}{n-1}\log\frac{Z_n}{Z^n} \qquad C_n(l) = \frac{l^{D-1}}{|\partial A|}\frac{\partial S_n(A)}{\partial l} \qquad n \in \mathbb{Z}_+$$

- Analytical results are available for two dimensional CFTs.
- Numerically the task is to calculate differences of free energy on the lattice.

Overview

Jarzynski's equality

- Jarzynski's theorem [Jarzynski 1996] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.
- The theorem is valid both for real and Monte Carlo time evolution.
- Consider the one parameter evolution $H_{\lambda=0} \to H_{\lambda=1}$. Jarzynki's theorem states that



$$\left\langle \exp\left(-\int \beta \delta W\right)\right\rangle = \exp\{-\Delta(\beta F)\}$$

Out-of-equilibrium protocol

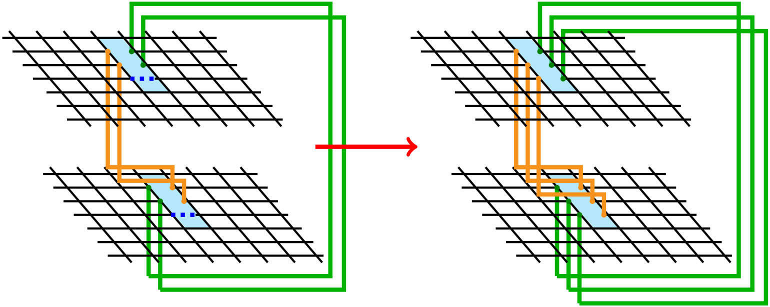
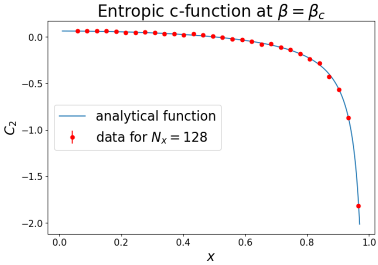


Figure adapted from [Alba 2016].

Overview

Scaling region

- The theoretical prediction for a CFT on a cylinder of spatial length N_x is $C_2(x) = \frac{c}{8}\frac{\pi x}{\tan(\pi x)}$, $x = \frac{l}{N_x}$.
- It can be compared to our data in the scaling region $N_x, l \gg 1$

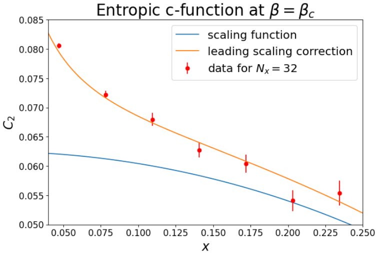


N_x	$\frac{\chi^2}{\nu}$, $\nu = 30$	$\frac{\chi^2}{\nu}$, $\nu = 28$
32	131.96	19.33
64	35.56	4.16
96	12.53	2.08
128	2.52	1.03

First scaling correction

- The general theory of unusual corrections to scaling of the entanglement entropy was developed in [Calabrese, Cardy 2010].
- In the case of the $2D$ Ising model one expects

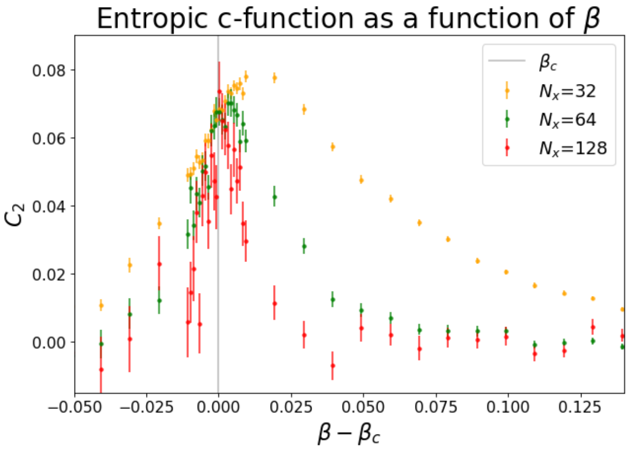
$$C_2(x, N_x) = C_2^{CFT}(x) + \frac{k}{N_x} \frac{x}{\sin(\pi x)\tan(\pi x)}$$



N_x	$\frac{\chi^2}{\nu}$, $\nu = 30$	k
32	2.10	0.260(4)
64	1.47	0.225(7)
96	1.70	0.209(12)
128	0.91	0.17(2)

- Data for $N_x = 128$ have been collected in a relatively small amount of time: 8 simulations of ~ 2 hours each on Marconi100 machine (CINECA)

Variation of C_2 with β

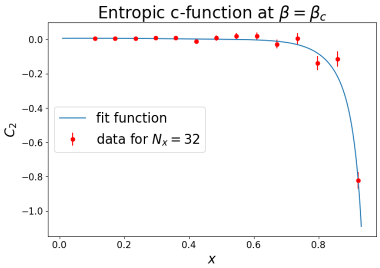


Overview

Scaling region

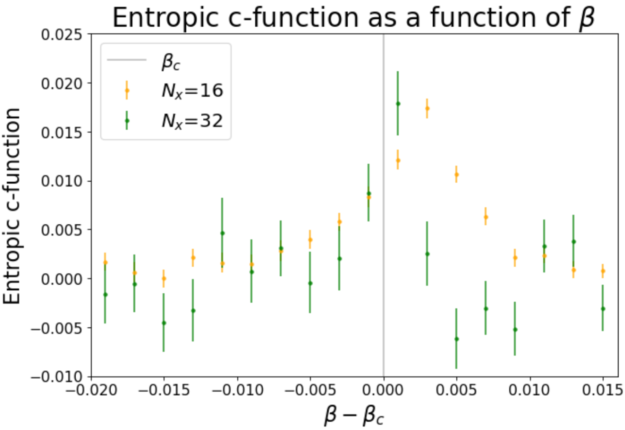
- The fit function we used for the three dimensional case is proportional to the exact c-function describing the quantum Lifshitz universality class [Inglis *et al.* 2013; Chen *et al.* 2014]

$$f(x;\tilde{c}) = \frac{\tilde{c}}{2}x^2\frac{\mathrm{d}J_2}{\mathrm{d}x}(x) \quad J_n(x) = \frac{n}{1-n}\log\left\{\frac{\eta(i)^2}{\theta_3(2i)\theta_3(\frac{i}{2})}\frac{\theta_3(2ix)\theta_3(2i(1-x))}{\eta(2ix)\eta(2i(1-x))}\right\}$$



N_x	$\frac{\chi^2}{\nu}$, $\nu = 13$	$\tilde{c}$
16	2.90	0.0543(16)
32	1.19	0.043(2)

Variation of C_2 with β



Overview

Conclusions

- Our data for the $2D$ Ising model are in perfect agreement with the CFT prediction.
- Results are promising also for the $3D$ Ising model.
- Thanks to the efficiency of our algorithm we have access to lattices that are already in the scaling limit $N_x, l \gg 1$.

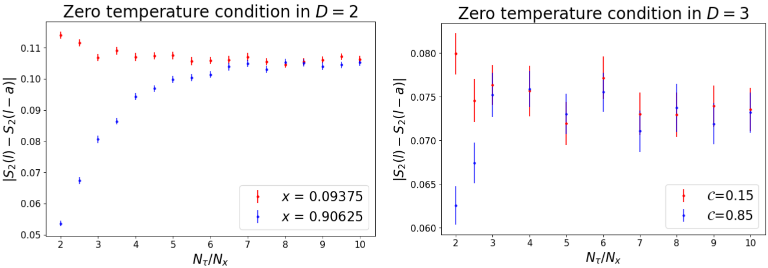
Future work:

- Exploit the duality properties of the $3D$ Ising model to study the entanglement content of the $\mathbb{Z}_2$ gauge theory.
- $SU(N)$ gauge theories [Buividovich, Polikarpov 2008; Rabenstein *et al.* 2018; Rindlisbacher *et al.* 2022].

Appendix

The code

- We used an already existing code [Komura, Okabe 2014] that we modified to implement the replica space and the Jarzynski's theorem.
- The code is written in CUDA C to achieve high parallelization.
- We simulated lattices with $N_\tau \gg N_x$ so that $T = 0$; the entropic c-function itself provides a way to check if this condition is satisfied since at $T = 0$ $\frac{\partial S_n}{\partial l}(l) = -\frac{\partial S_n}{\partial l}(N_x - l)$



Consistency of Jarzynski's algorithm

- Theoretically if we invert initial and final Hamiltonian the result does not change.
- Numerically, since the initial state is an equilibrium state while during the evolution the system is driven more and more far from the equilibrium, comparing direct and reverse realizations of the Jarzynski's algorithm is a non-trivial check.