Finite temperature BKT phase transition in the planar ψ^4 model with a strongly modulating potential

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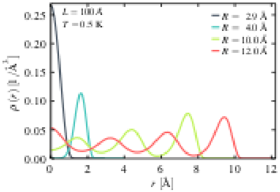
Overview

- $(1D \to 2D$ and $2D \to 1D)$ dimensional crossover in interacting bosonic systems
- Mapping between the modulated ψ^4 model and the modulated XY-model over a square lattice at finite T
- Renormalization group approach to the scaling of the superfluid fractions in the modulated XY model
- Comparison between analytical and numerical results
- Conclusions and further perspectives

1D (Luttinger liquid) physics in He^4 in nanopores

- Interaction with the wall of the pore $\Rightarrow$ concentric shells ($N_{\rm Shells}$ depends on the pore radius) [A. Del Maestro, I. Affleck, PRB82, 060515(R) (2010), A. Del Maestro, M. Boninsegni, and I. Affleck, PRL106, 105303 (2011)]



- A single shell hosts a Luttinger Liquid [F. D. M. Haldane, PRL47. 1840 (1981)]

$$H_{1S} = \frac{v}{2\pi}\int_0^L dx \left\{K^{-1}(\partial_x\phi(x))^2 + K(\partial_x\theta(x))^2\right\}$$

$$vK^{-1} = v_J = \frac{\pi\rho_0}{m} \quad , \quad vK = v_N = (\pi\rho_0^2\kappa)$$

LL calculation of physical observables

- Superfluid fraction [A. D.M., I. A. PRB82, 060515(R) (2010)]

$$\rho_s(T,L) = 1 - \frac{\pi v_J}{LT} \frac{\theta_3'(0|e^{-\frac{2\pi v_J}{LT}})}{\theta_3(0|e^{-\frac{2\pi v_J}{LT}})}$$

- Structure factor $S(q)$y R. Citro *et. al.*, New Journal of Physics 10, 045011 (2008), A. Nava *et. al.*, PRB105, 085402 (2022)]

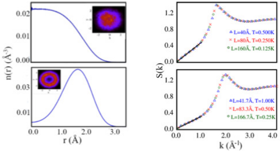
$$S(q) = \int_0^L dx \langle (n(x) - n_0)(n(0) - n_0) \rangle =$$

$$\int_0^L dx e^{iqx} \left\{ \frac{1}{2\pi^2 K} \partial_x^2 \ln \theta_1(\frac{\pi x}{L}|e^{-\frac{2\pi v_J}{LT}}) + A\cos(2\pi n_0 x) \left| \frac{\theta_1'(0|e^{-\frac{2\pi v_J}{LT}})}{\theta_1(\frac{\pi x}{L}|e^{-\frac{2\pi v_J}{LT}})} \right|^{\frac{2}{K}} \right\}$$

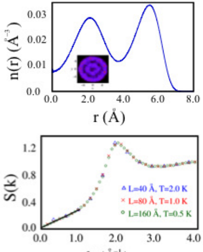
- Scaling functions of LT

Fingerprints of 1D physics from the scaling of $S(q)$

- Structure factor $S(q)$ [A. Nava *et al.*, PRB105, 085402 (2022)]



- When increasing the number of shells 1D physics is always recovered as $L \to \infty$ [W. Yang and I. Affleck, PRB102 295426 (2020), A. Nava *et al.*, PRB105, 085402 (2022)]



Fingerprints of 2D physics (I)

- Superfluid transition $\leftrightarrow$ BKT phase transition
- Paradigmatic model: XY-Hamiltonian over a square lattice

$$H_{\mathrm{XY}} = -2J \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos[\theta_{\vec{r}} - \theta_{\vec{r}'}]$$

- Vortex-induced finite-T transition: BKT RG equations

$$\frac{dD_2(T,\lambda)}{d\ln\lambda} = 2\pi^3 y^2(T,\lambda)$$
$$\frac{dy(T,\lambda)}{d\ln\lambda} = [2 - \pi/D_2(T,\lambda)]y(T,\lambda)$$

- $\lambda \rightarrow$ running scale parameter, $y(T,\lambda=1) \rightarrow$ single-vortex fugacity, $D_2(T,\lambda=1) = (k_B T/J) + \frac{1}{2}(k_B T/J)^2$ [J. V. José *et al.*, PRB16, 1217 (1977), T. Ohta *et al.*, PRB20, 139 (1979)]

Fingerprint of 2D physics (II)

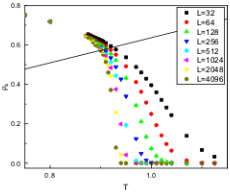
- Critical scaling of the superfluid fraction: the "universal jump" at the BKT phase transition

$$\rho_s(T, L) = k_B T/[J D_2(T, L)]$$

$$\rho_s(T, L) \text{ smooth function of } T \quad (L \text{ finite})$$

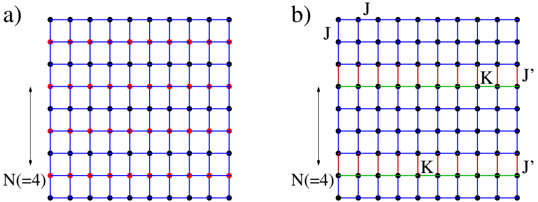
$$\lim_{T \to T_c^-} \rho_s(T, L \to \infty) = \frac{2T_c}{\pi}, \quad \lim_{T \to T_c^+} \rho_s(T, L \to \infty) = 0$$

- Monte Carlo verification P. H. Nguyen and M. Boninsegni, Appl. Sci.11, 4931 (2021)]



1D → 2D crossover, and vice versa

- 1D → 2D: *Is it possible to induce a dimensional crossover in, e.g., mustishell nanopores filled with ^{4}He by increasing the number of shells?*
  If the number of shells is finite and the pore il long enough it will always exhibit 1D physics [A. Nava *et. al.*, PRB105, 085402 (2022)]
- 2D → 1D: *Is it possible to trade a two-dimensional bosonic system for an array of effectively decoupled one-dimensional channels by means of a strong enough potential modulation?*
- Site- and bond-modulation



- ψ^4 model $\rightarrow$ prototypical (classical) lattice model for interacting bosons

$$H_{\mathrm{BH}} = -t \sum_{\langle \vec{r}, \vec{r}' \rangle} \{\Psi^\dagger_{\vec{r}} \Psi_{\vec{r}'} + \mathrm{h.c.}\} + \sum_{\vec{r}} \left\{ \frac{U}{2} (\Psi^\dagger_{\vec{r}} \Psi_{\vec{r}})^2 - (\mu + v_y) \Psi^\dagger_{\vec{r}} \Psi_{\vec{r}} \right\}$$

$$v_y = c_1 \cos(2\pi y / N)$$

- Polar decomposition of $\Psi_{\vec{r}}$ and saddle point solution

$$\Psi_{\vec{r}} \equiv \sqrt{\rho_{\vec{r}}} e^{i\theta_{\vec{r}}}$$

- $\rho_{\vec{r}}(\tau) \rightarrow n_{\vec{r}} + \delta\rho_{\vec{r}}$, with $n_{\vec{r}}$ the saddle-point solution and

$$2t \sum_{\vec{r}'} \sqrt{n_{\vec{r}'}} - \sqrt{n_{\vec{r}}} \{U n_{\vec{r}} - v_{\vec{r}} - \mu\} = 0$$

$$n_{\vec{r}} \geq 0$$

Saddle-point solution in the large-U limit

- $\theta_{\vec{r}} \leftrightarrow$ critical phase fluctuations $\Rightarrow$ modulated XY model

$$H_{\mathrm{XY}} = -2 \sum_{\langle \vec{r}, \vec{r}' \rangle} J_{\vec{r},\vec{r}'} \cos[\theta_{\vec{r}} - \theta_{\vec{r}'}]$$

$$J_{\vec{r},\vec{r}'} = t \langle \sqrt{[n_{\vec{r}} + \delta\rho_{\vec{r}}][n_{\vec{r}'} + \delta\rho_{\vec{r}'}]} \rangle$$

- Modulated exchange strengths in the XY model

$$J_{\vec{r},\vec{r}+\hat{x}} = J^x(y) \; , \quad J_{\vec{r},\vec{r}+\hat{y}} = J^y(y)$$

$$H_{\mathrm{XY}} = -\sum_{x,y} \{ J^x(y) \cos[\theta_{(x,y)} - \theta_{(x+1,y)}] + J^y(y) \cos[\theta_{(x,y)} - \theta_{(x,y+1)}]$$

$$J^{x,y}(y+N) = J^{x,y}(y)$$

Superfluid fraction(s) and dual Hamiltonian

- "Twisting" of the θ-fields in both directions

$$\begin{aligned} &\theta_{\vec{r}} = \theta'_{\vec{r}} + \Phi_{\vec{r}} \ , \ \theta'_{\vec{r}+\hat{x}L} = \theta'_{\vec{r}+\hat{y}L} = \theta'_{\vec{r}} \\ &\Phi_{\vec{r}+\hat{x}L} = \Phi_{\vec{r}} + \mathcal{Q}_x \ , \ \Phi_{\vec{r}+\hat{y}L} = \Phi_{\vec{r}} + \mathcal{Q}_y \end{aligned} \tag{1}$$

- Modulation $\Rightarrow$ sum over the *reduced* Brillouin zone

$$\Phi_{\vec{r}} = \frac{1}{V} \sum_{k_x \in [-\pi,\pi]} \sum_{k_y \in \left[-\frac{\pi}{N}, \frac{\pi}{N}\right]} e^{i\vec{k}\cdot\vec{r}} \sum_{\nu=0}^{N-1} e^{\frac{2\pi i \nu}{N}} \Phi_{\vec{k},\nu}$$

- Integrate over the $\theta'_{\vec{r}}$-field $\rightarrow$ effective Hamiltonian $H[\{\Phi\}]$

$$\beta H[\{\Phi\}] = \frac{1}{2V} \sum_{\vec{k}\in\mathcal{B}_N} \sum_{\nu,\nu'=0}^{N-1} \Delta_{2;(\nu,\nu')}(\vec{k}) \Phi^*_{\vec{k},\nu} \Phi_{\vec{k},\nu'} \ , \ \Delta_{2,(0,0)}(\vec{k}) \rightarrow_{k\rightarrow 0} \frac{k_x^2 k_y^2}{k_x^2 \mathcal{D}_{2,y}(T) + k_y^2 \mathcal{D}_{2,x}(T)}$$

- Superfluid fractions

$$\rho_{s,(x,y)}(T) = \lim_{k_{(y,x)}\rightarrow 0} \{ \lim_{k_{(x,y)}\rightarrow 0} [\Delta_{2;(0,0)}(\vec{k};T)/(k^2_{(x,y)})] \}$$

Vortex-induced renormalization of the spin stifness(es)

- Coupling to vortices $\Rightarrow$ RG flow of the $\mathcal{D}_{2,(x,y)}(T)$

$$\frac{d\mathcal{D}_{2,x}(T;\lambda)}{d\ln\lambda} = 2\pi^3 y^2(T;\lambda)\sqrt{\mathcal{D}_{2,x}(T;\lambda)/\mathcal{D}_{2,y}(T;\lambda)}$$
$$\frac{d\mathcal{D}_{2,y}(T;\lambda)}{d\ln\lambda} = 2\pi^3 y^2(T;\lambda)\sqrt{\mathcal{D}_{2,y}(T;\lambda)/\mathcal{D}_{2,x}(T;\lambda)}$$

- First step: an "extra" RG invariant

$$\frac{d}{d\ln\lambda}\left(\frac{\mathcal{D}_{2,x}(T;\lambda)}{\mathcal{D}_{2,y}(T;\lambda)}\right) = \frac{d\mathcal{K}(T;\lambda)}{d\ln\lambda} = 0$$

- Second step: mapping onto the "isotropic" BKT equations

$$\mathcal{D}_2(T;\lambda) = \sqrt{\mathcal{D}_{2,x}(T;\lambda)\mathcal{D}_{2,y}(T;\lambda)}$$
$$\frac{d\mathcal{D}_2(T;\lambda)}{d\ln\lambda} = 2\pi^3 y^2(T;\lambda)$$
$$\frac{dy(T;\lambda)}{d\ln\lambda} = \left[2 - \frac{\pi}{\mathcal{D}_2(T;\lambda)}\right] y(T;\lambda)$$

Critical temperature and *critical jump*

- Reduced variables and second invariant

$$\mathcal{Y}_\lambda(T) = 2\pi y(T;\lambda) \ , \ \Theta_\lambda(T) = 2 - \pi/D_2(T;\lambda)$$

$$\frac{d\mathcal{Y}_\lambda^2(T)}{d\ln\lambda} \approx 2\Theta_\lambda(T)\mathcal{Y}_\lambda^2(T) \ , \ \frac{d\Theta_\lambda(T)}{d\ln\lambda} \approx 2\Theta_\lambda(T)\mathcal{Y}_\lambda^2(T) \ , \ \frac{d}{d\ln\lambda}\{\mathcal{Y}_\lambda^2(T) - \Theta_\lambda^2(T)\} = 0$$

- Critical line $\mathcal{Y}_{\lambda=1}(T = T_c) + \Theta_{\lambda=1}(T = T_c) = 0$
- Superfluid fractions and $\mathcal{D}_2(T;L)$

$$\rho_{s,x}(T;L) = \frac{k_B T}{\bar{J}_x\sqrt{\mathcal{K}(T)}\mathcal{D}_2(T;L)} \ , \ \rho_{s,y}(T;L) = \frac{k_B T\sqrt{\mathcal{K}(T)}}{\bar{J}_x\mathcal{D}_2(T;L)}$$

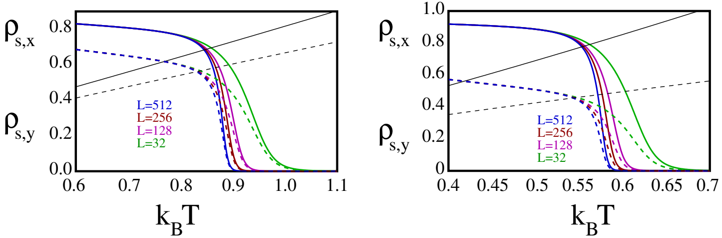
- "Critical jump" in the superfluid fractions

$$\lim_{T\to T_c^+}\{\lim_{L\to\infty}\rho_{s,x}(T;L)\} = 0 \ , \ \lim_{T\to T_c^-}\{\lim_{L\to\infty}\rho_{s,x}(T;L)\} = \frac{2k_B T_c}{\pi\bar{J}_x\sqrt{\mathcal{K}(T_c)}}$$

$$\lim_{T\to T_c^+}\{\lim_{L\to\infty}\rho_{s,y}(T;L)\} = 0 \ , \ \lim_{T\to T_c^-}\{\lim_{L\to\infty}\rho_{s,y}(T;L)\} = \frac{2k_B T_c\sqrt{\mathcal{K}(T_c)}}{\pi\bar{J}_x}$$

Analytical results for $N = 2$

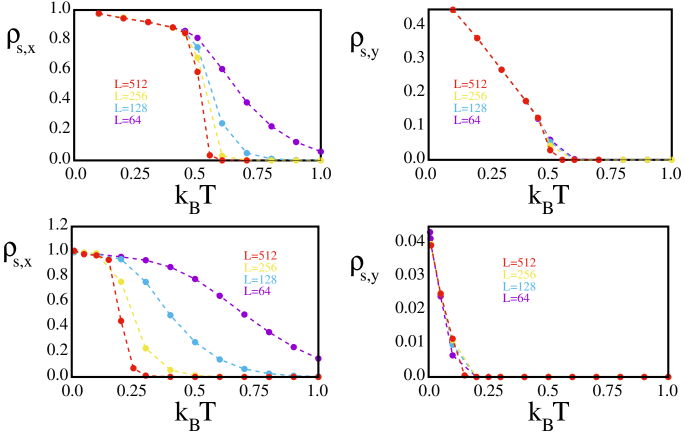
- All the machinery applied to the ψ^4 with $U = c_0 = 40$, $t = 1$, $c_1 = 40$ and $c_1 = 60$ D. Giuliano, P. H. Nguyen, A. Nava, and M. Boninsegni, in preparation]



- T_c is the same for both $\rho_{s,x}$ and $\rho_{s,y}$, as it must be; increasing c_1 increases the effective anisotropy, with a resulting backshift of T_c toward smaller values
- Yet, the two-dimensional nature of the system, witnessed by the BKT phase transition, is never lost, even at "extreme" values of the modulation strength

Numerical results for $N = 8$

- Monte Carlo data from the ψ^4 model



- Fit to T_c $\Rightarrow$ input data for the vortex fugacities in the modulated XY model

Conclusions

- ψ^4 lattice model with a modulating potential $\rightarrow$ modulated XY model
- Agreement between Monte Carlo results for the ψ^4 and analytical results for the XY model
- The relevant physics is 2D
- Effects of the modulation over a single impurity (what happens to, for instance, Kondo physics?)
- Effects of disorder
- ... and so on