Towards the determination of excited nucleon matrix elements with lattice QCD

Based on *[arXiv:2110.11908]* and *PRL* (prepared for submission)

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Introduction and Motivation

*

Nobel Prize

(ν_e) neutrino flux originated from the sun (pp-chain) and flux detected on earth are in discrepancy.

1930	1956	1957	1962	1960'	2000	2015
Prediction of an undetected particle $(\bar{\nu}_e)$ in β-decay	Discovery of ν_e *	Prediction of neutrino oscillations	Discovery of ν_μ *	**solar neutrino puzzle** *	"Discovery" of ν_τ	Clear observation of neutrino oscillations *

DONUT Experiment

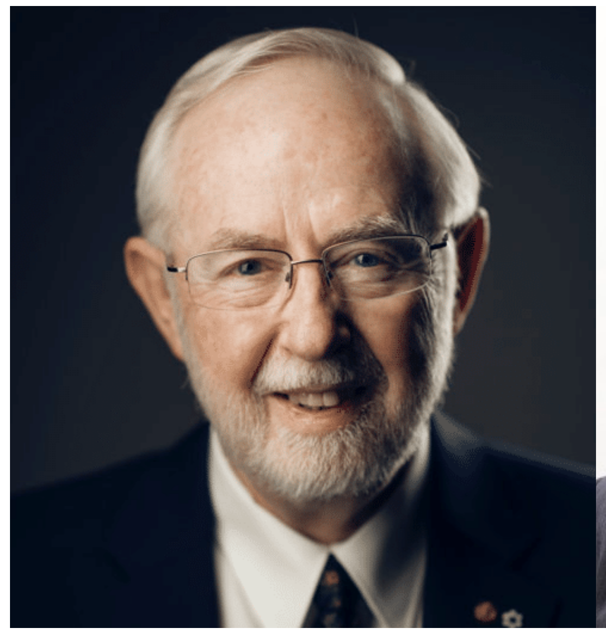
W. Pauli

F. Reines and C. Cowan *

B. Pontecorvo
(Z. Maki, M. Nakagawa, S. Sakata)

L. Lederman *

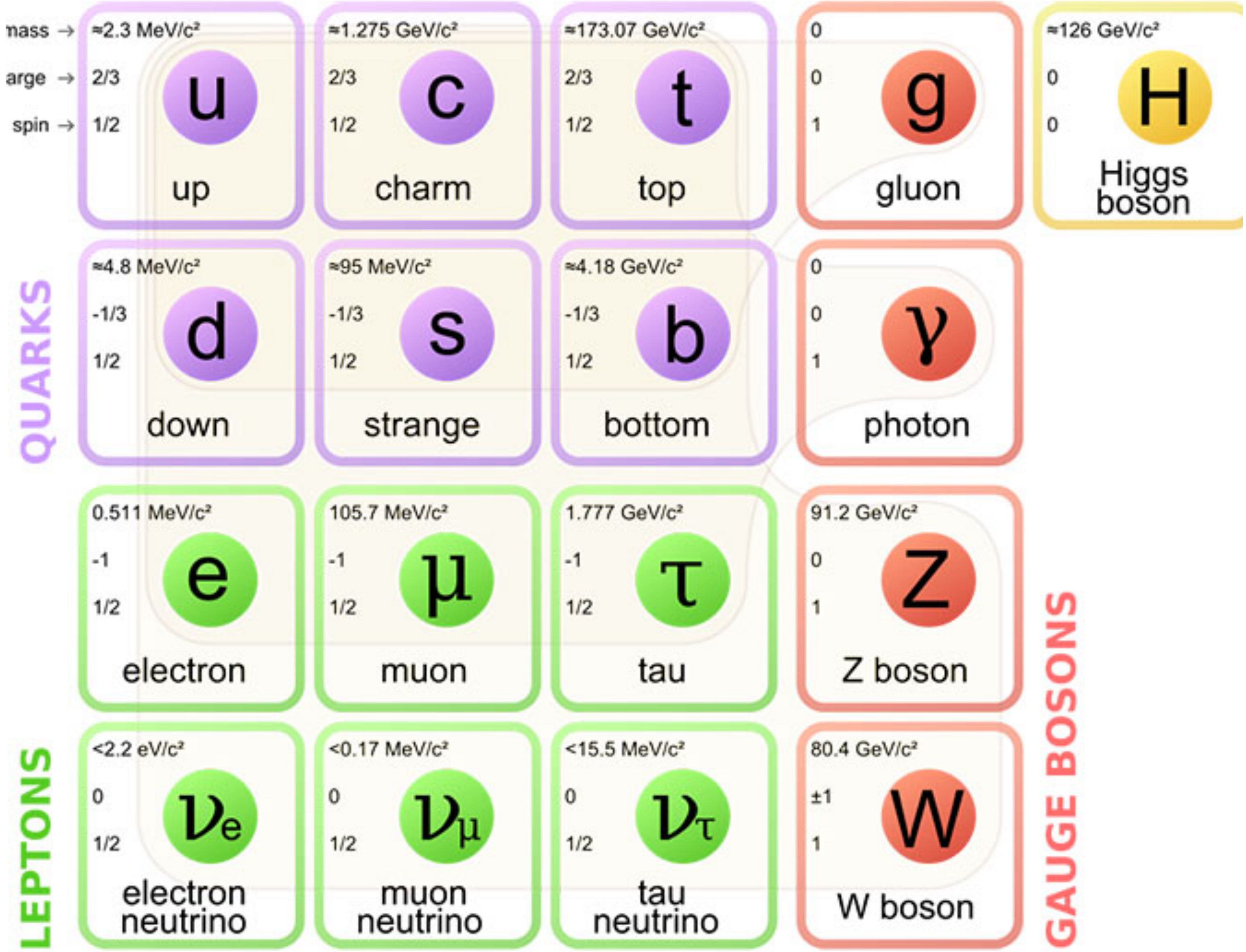
Homestake Experiment *



A. McDonald *
SNO Experiment

T. Kajita *
Super-Kamiokande

Introduction and Motivation



Neutrino oscillations (in vacuum)

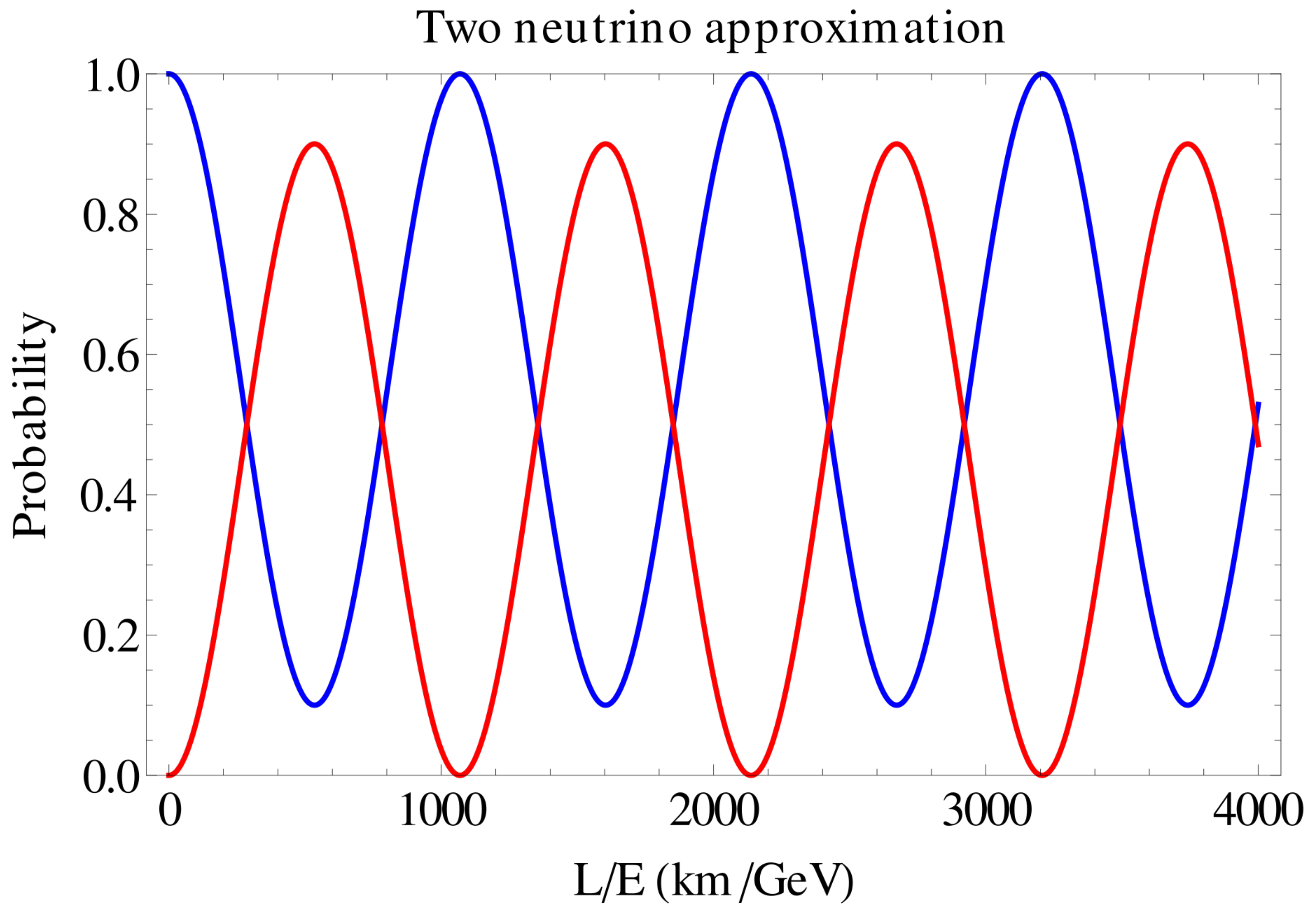
$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle$$

$$|\nu_i(t)\rangle = e^{-i(E_i t - \mathbf{p}_i \cdot \mathbf{x})} |\nu_i\rangle$$

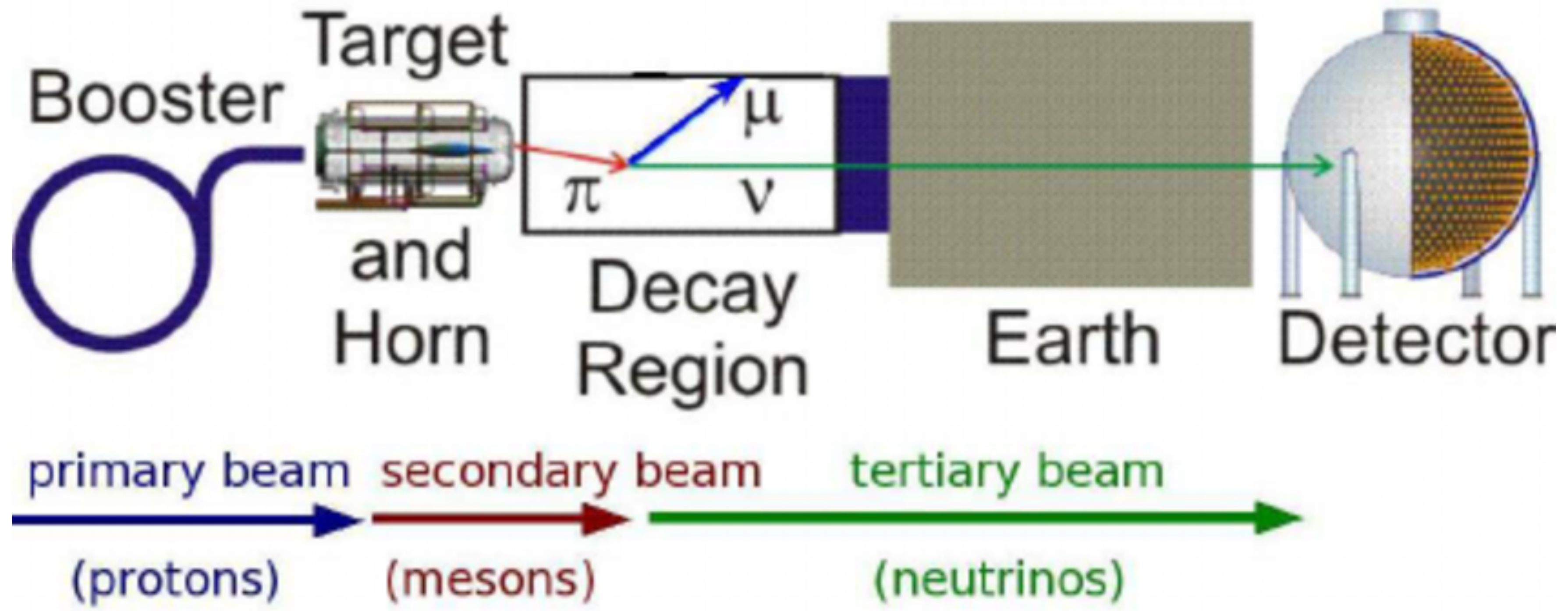
2-flavour case

Neutrinos cannot be massless!

$$P_{\alpha\to\beta} = |\langle \nu_\beta(L)|\nu_\alpha(0)\rangle|^2 = \sin^2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$



Need for excited nucleon structure (in neutrino oscillation)



e.g. MiniBooNE

ν_μ flux is artificially produced and then detected after a travel length L

$$\nu_\mu\, n \xrightarrow{\mathcal{J}^-} \mu^- p \qquad \frac{d\sigma^{(\nu p)}}{dQ^2} \propto |\langle N | \mathcal{J}^- | N \rangle|^2$$

$\mathcal{J}^-$ is the weak CC

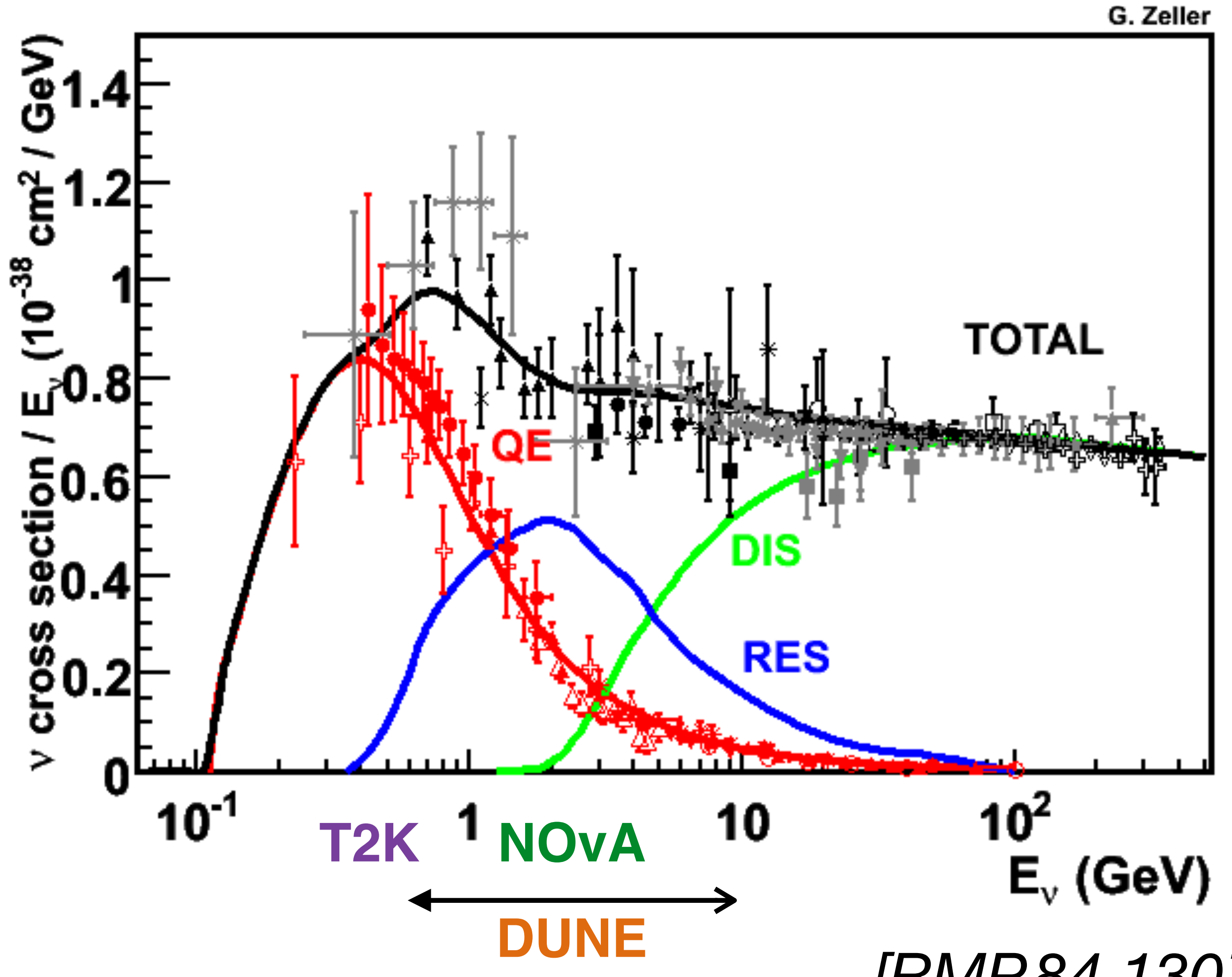
neutrino process	abbreviation	reaction	fraction (%)
CC quasielastic	CCQE	$\nu_\mu + n \to \mu^- + p$	39
NC elastic	NCE	$\nu_\mu + p(n) \to \nu_\mu + p(n)$	16
CC $1\pi^+$ production	CC$1\pi^+$	$\nu_\mu + p(n) \to \mu^- + \pi^+ + p(n)$	25
CC $1\pi^0$ production	CC$1\pi^0$	$\nu_\mu + n \to \mu^- + \pi^0 + p$	4
NC $1\pi^\pm$ production	NC$1\pi^\pm$	$\nu_\mu + p(n) \to \nu_\mu + \pi^+(\pi^-) + n(p)$	4
NC $1\pi^0$ production	NC$1\pi^0$	$\nu_\mu + p(n) \to \nu_\mu + \pi^0 + p(n)$	8
multi pion production, DIS, etc.	other	$\nu_\mu + p(n) \to \mu^- + N\pi^\pm + X$, etc.	4

[PRD.81.092005]

Main challenges

- Nuclear model to factorise neutrino-nuclei scattering into neutrino-nucleon scattering (known from EW theory);
- Q^2 in a range where excited nucleons are produced!

[arXiv:2203.09030]



[RMP.84.1307]

Required also knowledge of $\langle N^* | \mathcal{J}^- | N \rangle$, $\langle \Delta | \mathcal{J}^- | N \rangle$ and $\langle N\pi | \mathcal{J}^- | N \rangle$

We are the first to investigate $\langle N\pi | \mathcal{J}^- | N \rangle$ with LQCD

Accessing hadron structure through lattice QCD

1) Construct operator O_1 with $\mathcal{J}^P = \left(\frac{1}{2}\right)^+$ s.t. $\bar{\mathrm{O}}_1 |\Omega\rangle = c^N |N\rangle + c^{N^*} |N^*\rangle + c^{N\pi} |N\pi\rangle + \ \ldots$

2) Compute three-point functions (momentum $\mathbf{p}', \mathbf{p}, \mathbf{q} = \mathbf{p}' - \mathbf{p}$) and employ spectral decomposition

$(n, n' = N, N^*, N\pi, \ldots)$

$$\langle \mathrm{O}_1(\mathbf{p}', t)\ \mathcal{J}(\mathbf{q}, \tau)\ \bar{\mathrm{O}}_1(\mathbf{p},0)\rangle = \sum_{n',n} \frac{e^{-E_n'(t-\tau)} e^{-E_n \tau}}{2E_n 2E_{n'}} \langle \Omega | \mathrm{O}_1(\mathbf{p}') | n'\rangle\ \langle n' | \mathcal{J}(\mathbf{q}) | n\rangle\ \langle n | \bar{\mathrm{O}}_1(\mathbf{p}) | \Omega\rangle$$

3) Extract $\langle N(\mathbf{p}') | \mathcal{J}(\mathbf{q}) | N(\mathbf{p})\rangle$ at $t \gg \tau \gg 0$, where $\langle \mathrm{O}_1(\mathbf{p}', t)\ \mathcal{J}(\mathbf{q}, \tau)\ \bar{\mathrm{O}}_1(\mathbf{p},0)\rangle \propto \frac{e^{-E_N'(t-\tau)} e^{-E_N \tau}}{2E_N 2E_{N'}} \langle N(\mathbf{p}') | \mathcal{J}(\mathbf{q}) | N(\mathbf{p})\rangle$

Problem

large t, τ data is noisy w/ current statistics.

We use small t, τ data

Consequences

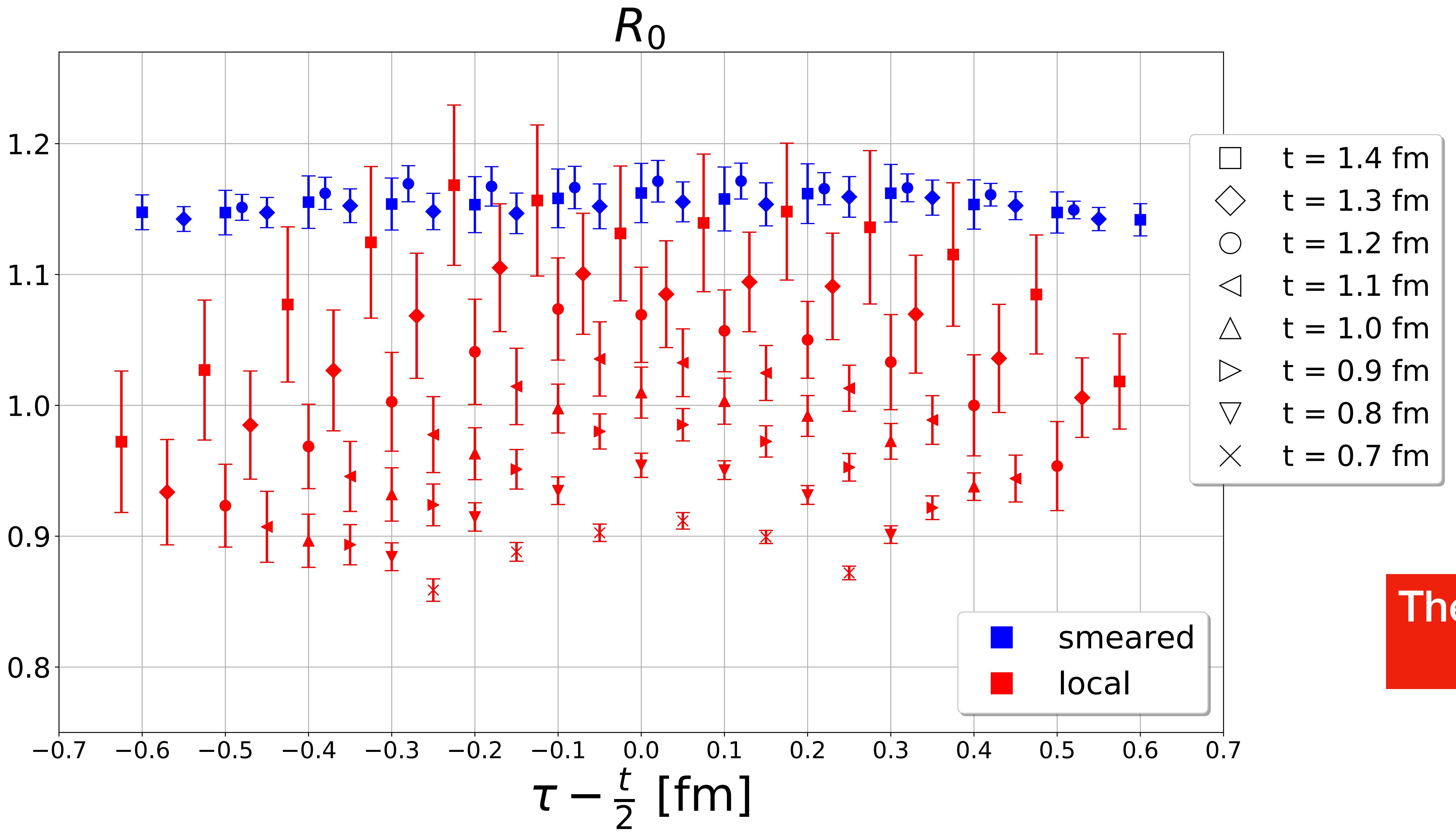
Contamination from excited nucleons $(N^*, \ldots)$

and multiparticle states $(N\pi, \ldots)$

Improvement through the construction of better operators

O_1 can be iteratively improved with smearing techniques (a technical and effective tool)

$$\langle N(\mathbf{0}) \,|\, \mathscr{A}_i(\mathbf{q}=\mathbf{0}) \,|\, N(\mathbf{0}) \rangle = g_A$$



Axial charge g_A

$$R_0 = \frac{\langle O_1(\mathbf{0},t)\ \mathscr{A}_i(\mathbf{0},\tau)\ \bar{O}_1(\mathbf{0},0)\rangle}{O_1(\mathbf{0},t)\ \bar{O}_1(\mathbf{0},0)} \propto g_A$$

$$\mathscr{A}_i = \bar{q}\ \gamma_i \gamma_5\ q \qquad \mathbf{p}' = \mathbf{q} = \mathbf{p} = \mathbf{0}$$

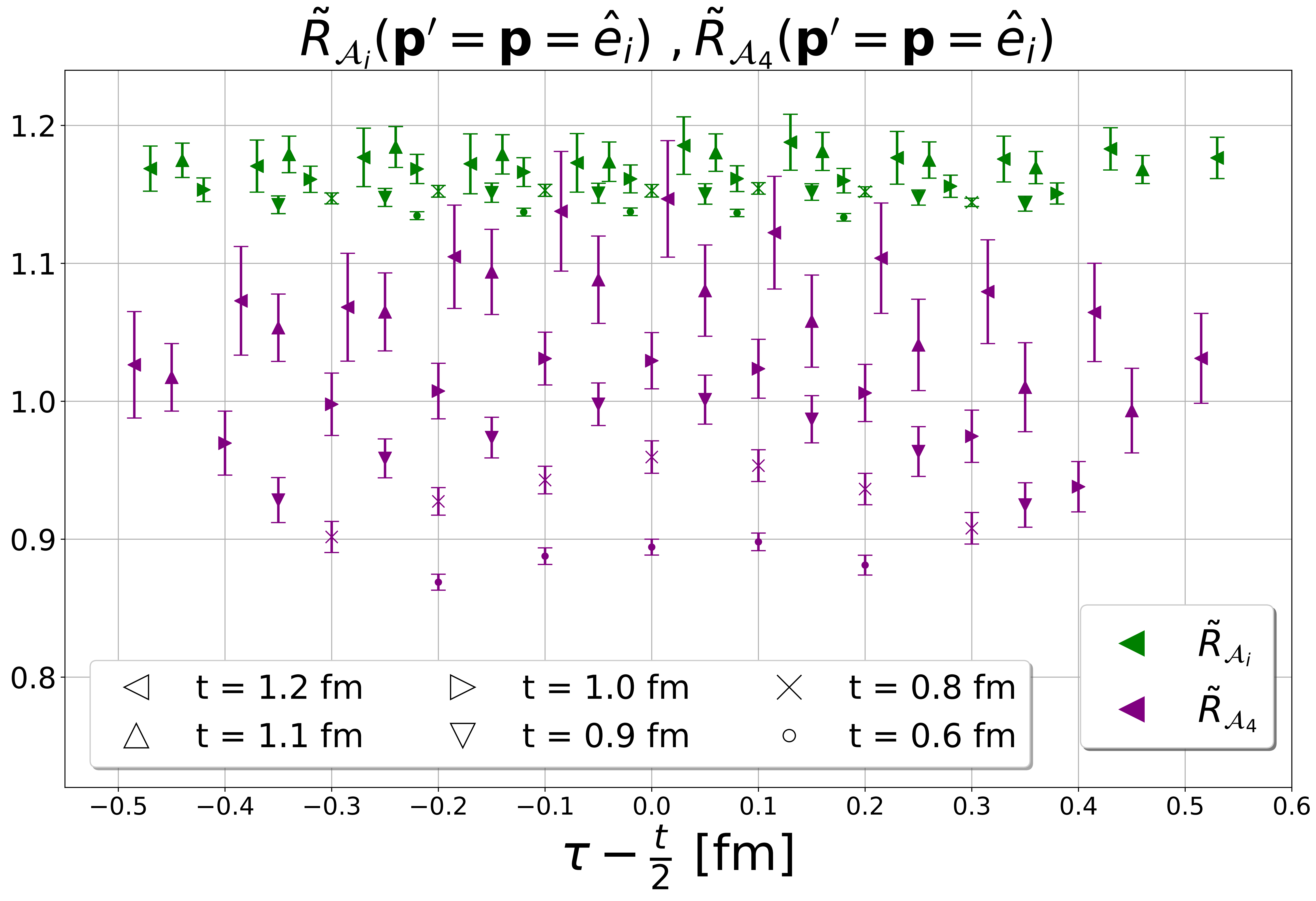
There is a clear sign of excited state contamination with local nucleon operators

The effect of the smearing is evident

$g_A = 1.16 \pm 0.07$ at $m_\pi \approx 426\ \mathrm{MeV},\ a \approx 0.098\ \mathrm{fm},\ L = 24a, T = 2L$

The axial charge g_A from $\langle N(\mathbf{p}) | \mathscr{A}_i(\mathbf{q}=\mathbf{0}) | N(\mathbf{p}) \rangle$ and $\langle N(\mathbf{p}) | \mathscr{A}_4(\mathbf{q}=\mathbf{0}) | N(\mathbf{p}) \rangle$

g_A can be extracted from $\mathscr{A}_i = \bar{q}\gamma_i\gamma_5 q$ and $\mathscr{A}_4 = \bar{q}\gamma_4\gamma_5 q$ with $\mathbf{p}' = \mathbf{p} = \hat{e}_i = \dfrac{2\pi}{L}\hat{n}_i$



$$\tilde{R}_{A_i} = \frac{\langle \mathrm{O}_1(\mathbf{p},t)\ \mathscr{A}_i(\mathbf{q}=\mathbf{0},\tau)\ \bar{\mathrm{O}}_1(\mathbf{p},0)\rangle}{\mathrm{O}_1(\mathbf{p},t)\ \bar{\mathrm{O}}_1(\mathbf{p},0)} = g_A + \ldots$$

$$\tilde{R}_{A_4} = \frac{\langle \mathrm{O}_1(\mathbf{p},t)\ \mathscr{A}_4(\mathbf{q}=\mathbf{0},\tau)\ \bar{\mathrm{O}}_1(\mathbf{p},0)\rangle}{\mathrm{O}_1(\mathbf{p},t)\ \bar{\mathrm{O}}_1(\mathbf{p},0)} \left(\frac{E}{p_i}\right) = g_A + \ldots$$

Results with $\mathscr{J} = \mathscr{A}_i$ are consistent with rest frame

Results with $\mathscr{J} = \mathscr{A}_4$ show 5%-20% discrepancy

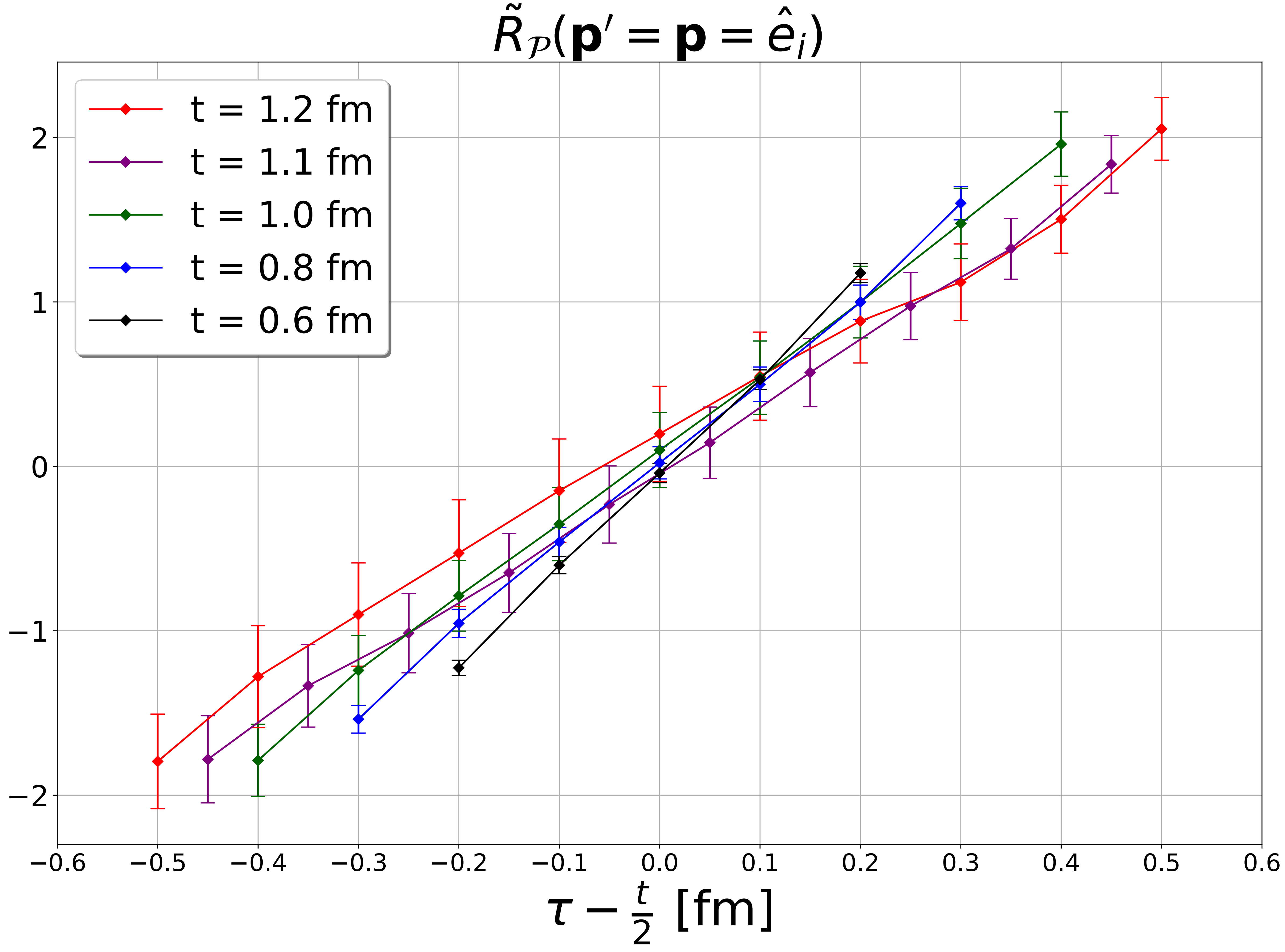
Observed also by χPT collaboration [arXiv:1612.04388]

$m_\pi \approx 426$ MeV, $a \approx 0.098$ fm, $L = 24a, T = 2L$

Excited state effects in the pseudoscalar channel $(\mathbf{q} = \mathbf{0})$

We investigate, for the first time, channels with $\mathcal{J} = \mathcal{P}$ and $\mathbf{p}' = \mathbf{p} = \hat{e}_i = \dfrac{2\pi}{L}$

$$\bar{\mathrm{O}}_1 |\Omega\rangle = c_N |N> + c_{N\pi} |N\pi>$$



$$\tilde{R}_P = \frac{\langle \mathrm{O}_1(\mathbf{p},t)\ \mathcal{P}(\mathbf{q}=\mathbf{0},\tau)\ \bar{\mathrm{O}}_1(\mathbf{p},0)\rangle}{\mathrm{O}_1(\mathbf{p},t)\ \bar{\mathrm{O}}_1(\mathbf{p},0)} \frac{E}{p_i} = 0 + \ldots$$

The signal is purely from excited states and in particular $N\pi$

$m_\pi \approx 426$ MeV, $a \approx 0.098$ fm, $L = 24a, T = 2L$

This channel is the clearest case of $N\pi$ state contamination

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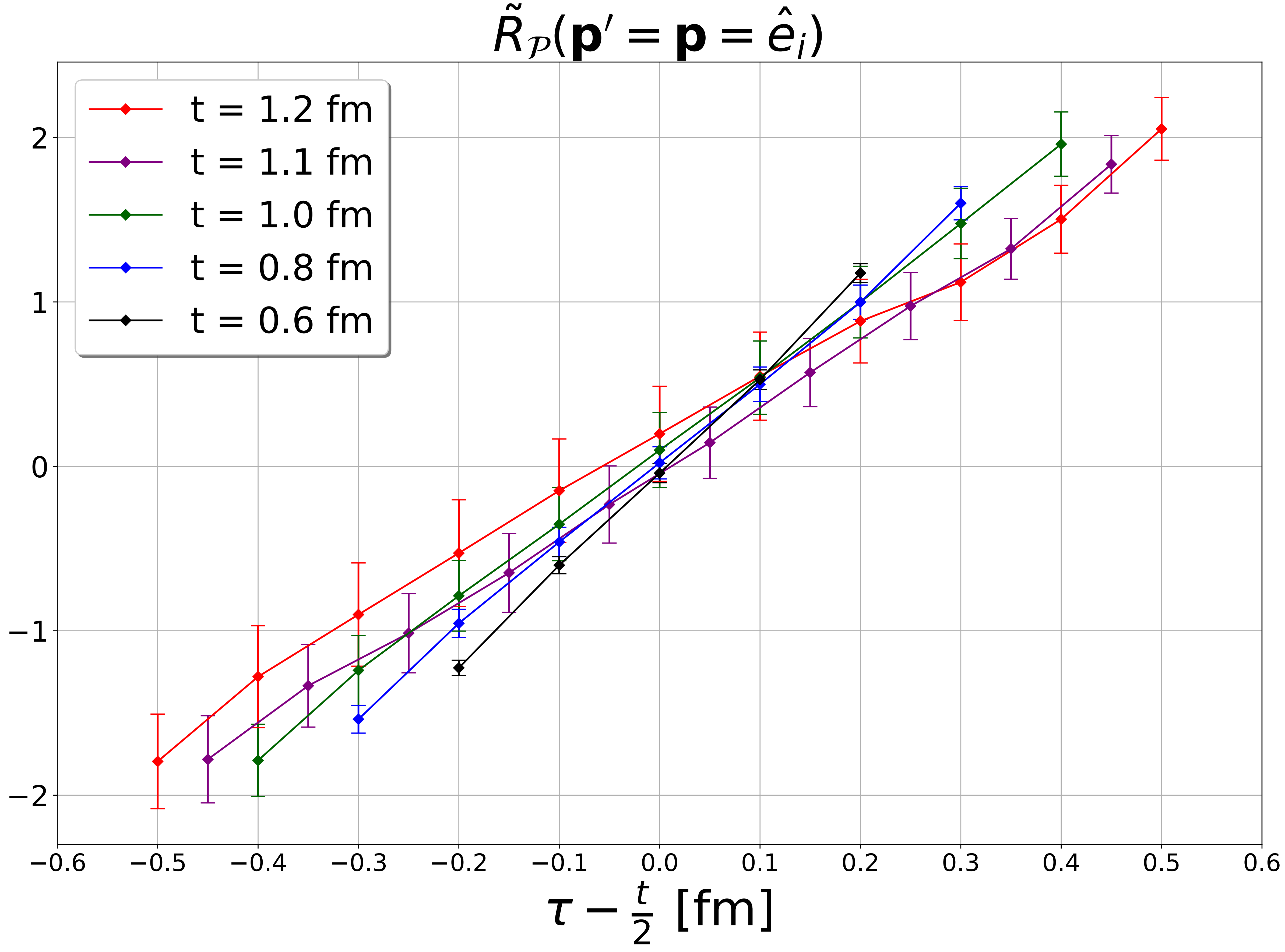

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The signal is purely from excited states and in particular $N\pi$

ChPT predicts that terms $\propto \langle N\pi|\mathcal{P}|N\rangle$ and $\langle N|\mathcal{P}|N\pi\rangle$ are large.

[PRD.100.054507] *[PRD.99.054506]*

With LO-ChPT (EFT), the correction to the 3pt at tree-level is

$$\boxed{\delta^{\mathcal{P}}_{\chi PT} = A\frac{E'}{E_\pi}e^{-(E'-m_\pi/2)t}\sinh\left(m_\pi(\tau-t/2)\right)}$$

[JHEP05(2020)126]

where $A \propto g_A, \mathbf{p}$

This channel is the clearest case of $N\pi$ state contamination

More general approach: Variational Method

Variational method

Construct a basis $\mathbb{B}_n = \{\mathrm{O}_1, \mathrm{O}_2, \ldots, \mathrm{O}_n\}$ of operators with same quantum numbers $J^P = \left(\frac{1}{2}\right)^+$

Construct a matrix $C(t)_{ij} = \langle \mathrm{O}_i(t)\ \bar{\mathrm{O}}_j(0)\rangle$ where $O_k \in \mathbb{B}_n$

$\mathrm{O}_1 \propto (qqq)$ $\mathrm{O}_2 \propto (qqq)(\bar{q}q)$

Suppose we find $n = 2$ operators that overlap with the physical states $|N\rangle$ and $|N\pi\rangle$:

$$\bar{\mathrm{O}}_1|\Omega\rangle = c_1^N|N\rangle + c_1^{N\pi}|N\pi\rangle$$

$$\bar{\mathrm{O}}_2|\Omega\rangle = c_2^N|N\rangle + c_2^{N\pi}|N\pi\rangle$$

$$C(t) = \begin{pmatrix} \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_2(0)\rangle \\ \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_2(0)\rangle \end{pmatrix}$$

solve $C(t)v^\alpha(t,t_0) = C(t_0)\ \lambda^\alpha(t,t_0)v^\alpha(t,t_0)$ **GEVP**

$v^\alpha(t_0)$, $\lambda^\alpha(t_0)$ are Generalised Eigenvectors and Eigenvalues

(Amazing) Properties

$$\lambda^\alpha(t_0) = d^\alpha(t_0)\ e^{-E_\alpha(t-t_0)}$$

$$\sum_i v_i^\alpha(t_0)\ v_j^\beta(t_0) \propto \delta^{\alpha\beta}$$

$$\bar{\mathrm{O}}_\alpha = \sum_i v_i^\alpha(t_0)\ \bar{\mathrm{O}}_i \quad \text{s.t.} \quad \bar{\mathrm{O}}_\alpha|\Omega\rangle = c_\alpha|\alpha\rangle$$

System is diagonalised! e.g. $\bar{\mathrm{O}}_N|\Omega\rangle = c_N|N\rangle$

GEVP results with $\mathbf{p} = (2\pi/L)\ \hat{n}_z$

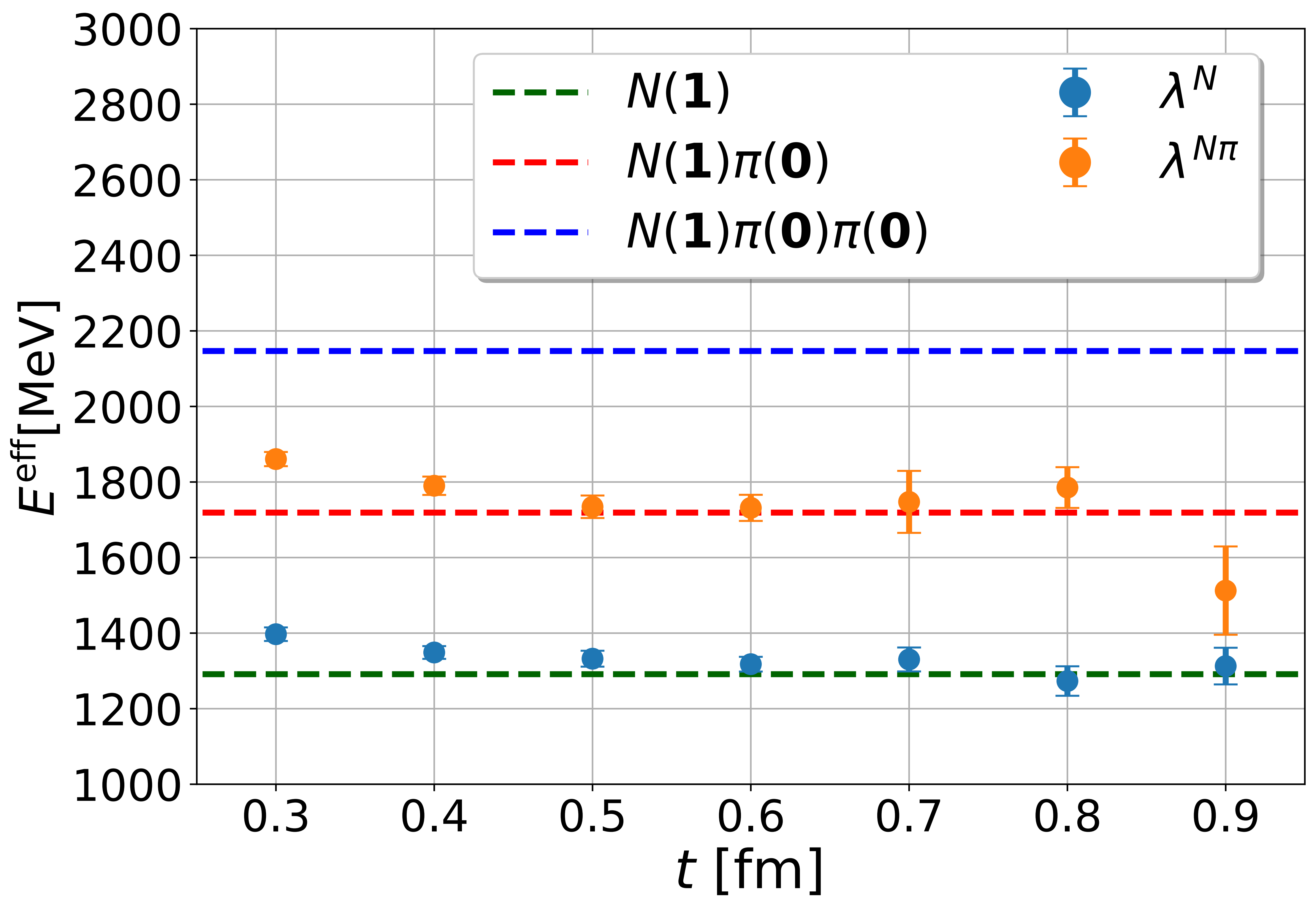
$$C(t) = \begin{pmatrix} \langle O_1(t)\ \bar{O}_1(0)\rangle & \langle O_1(t)\ \bar{O}_2(0)\rangle \\ \langle O_2(t)\ \bar{O}_1(0)\rangle & \langle O_2(t)\ \bar{O}_2(0)\rangle \end{pmatrix}$$

$$C(t) v^\alpha(t,t_0) = C(t_0)\ \lambda^\alpha(t,t_0) v^\alpha(t,t_0)$$

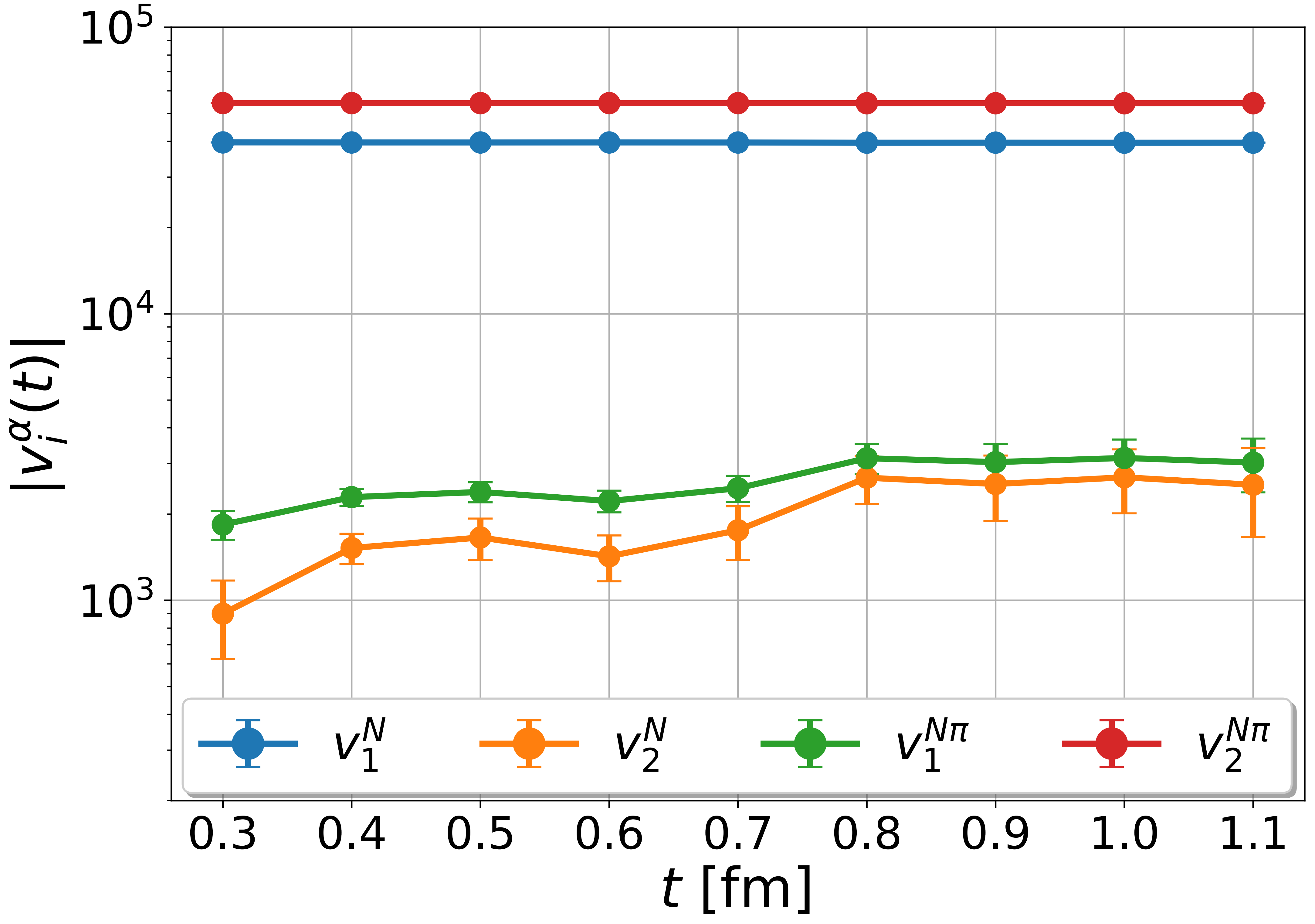
$$\lambda^1 \propto e^{-E_N(t-t_0)} \equiv \lambda^N$$

$$\lambda^2 \propto e^{-E_{N\pi}(t-t_0)} \equiv \lambda^{N\pi}$$

We extract the (effective) energies from the eigenvalues: $E^{\text{eff}}_\alpha = \log\left(\lambda^\alpha(t-a)\ /\ \lambda^\alpha(t)\right)$



(Dashed lines are non-interacting energy levels)



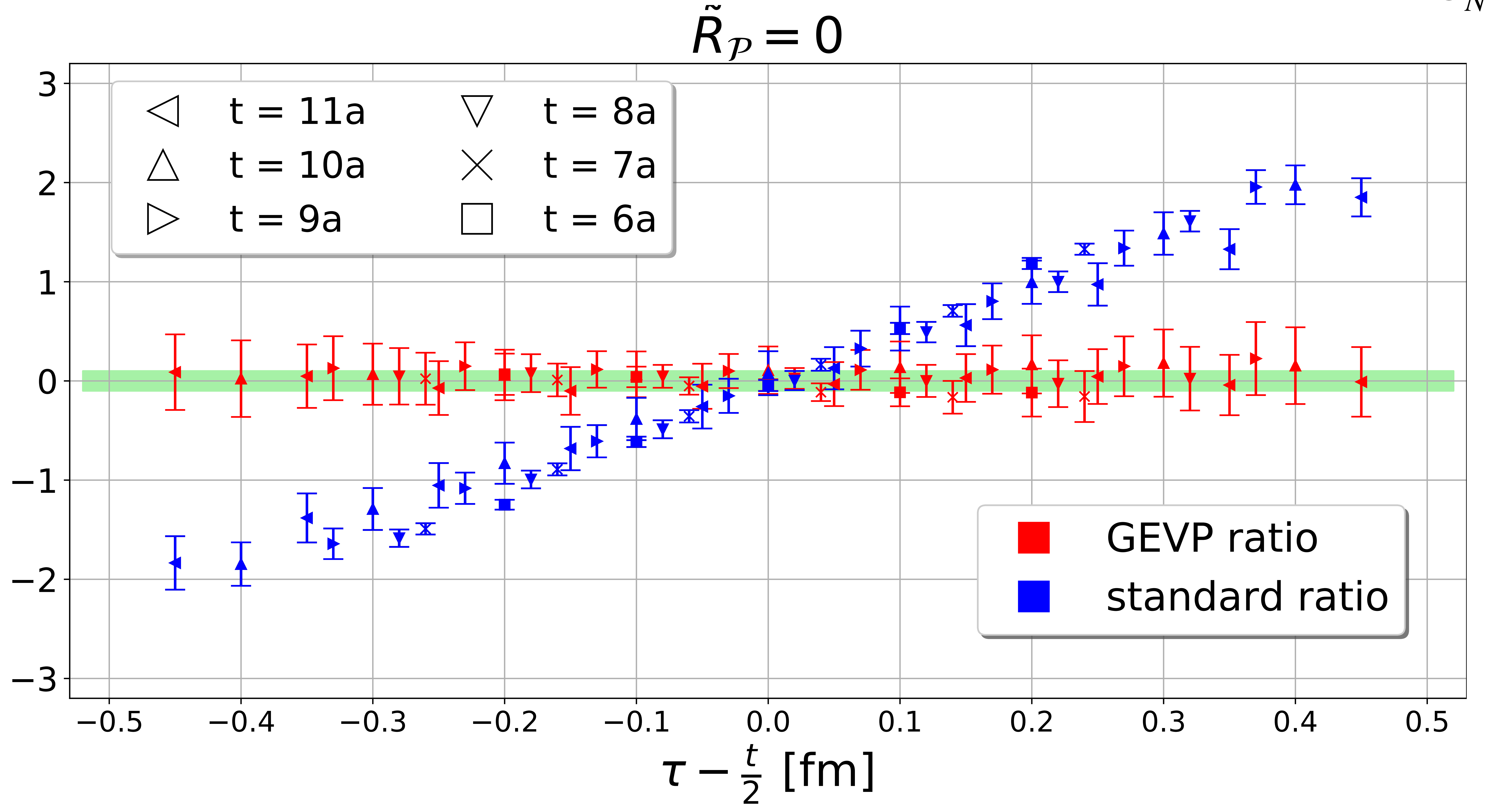
$v^\alpha(t,t_0)$ normalised s.t. $(v^\alpha(t,t_0), C(t_0)v^\beta(t,t_0)) = \delta^{\alpha\beta}$

GEVP ratio in the pseudoscalar channel $(\mathbf{q}=\mathbf{0})$

$$\tilde{R}_{\mathscr{P}} = \frac{\langle \mathrm{O}_N(\mathbf{p}', t)\ \mathscr{P}(\mathbf{q}=\mathbf{0}, \tau)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0)\rangle}{\mathrm{O}_N(\mathbf{p}, t)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0)} \frac{E}{p_i} = 0 + \ldots$$

we replace O_1 with O_N to get the GEVP ratio

$$\mathrm{O}_N = \sum_i v_i^N(t_0)\ \mathrm{O}_i$$

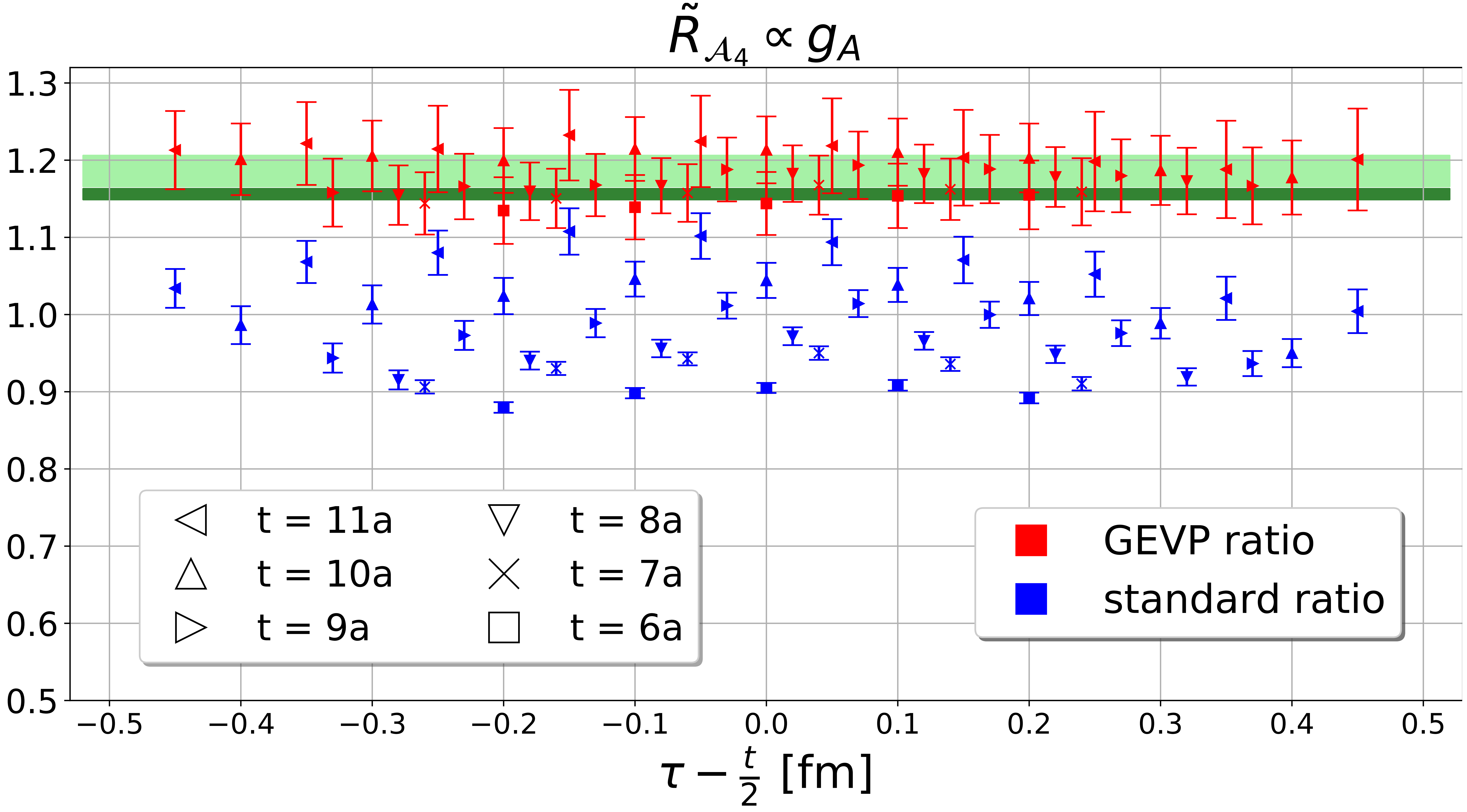


GEVP ratio in the axial temporal channel $(\mathbf{q} = \mathbf{0})$

$$\tilde{R}_{\mathscr{A}_4} = \frac{\langle \mathrm{O}_N(\mathbf{p}', t)\ \mathscr{A}_4(\mathbf{q}=\mathbf{0}, \tau)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0)\rangle}{\mathrm{O}_N(\mathbf{p}, t)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0)}\ \frac{-E}{p_i} = g_A + \ \ldots$$

we replace O_1 with O_N to get the GEVP ratio

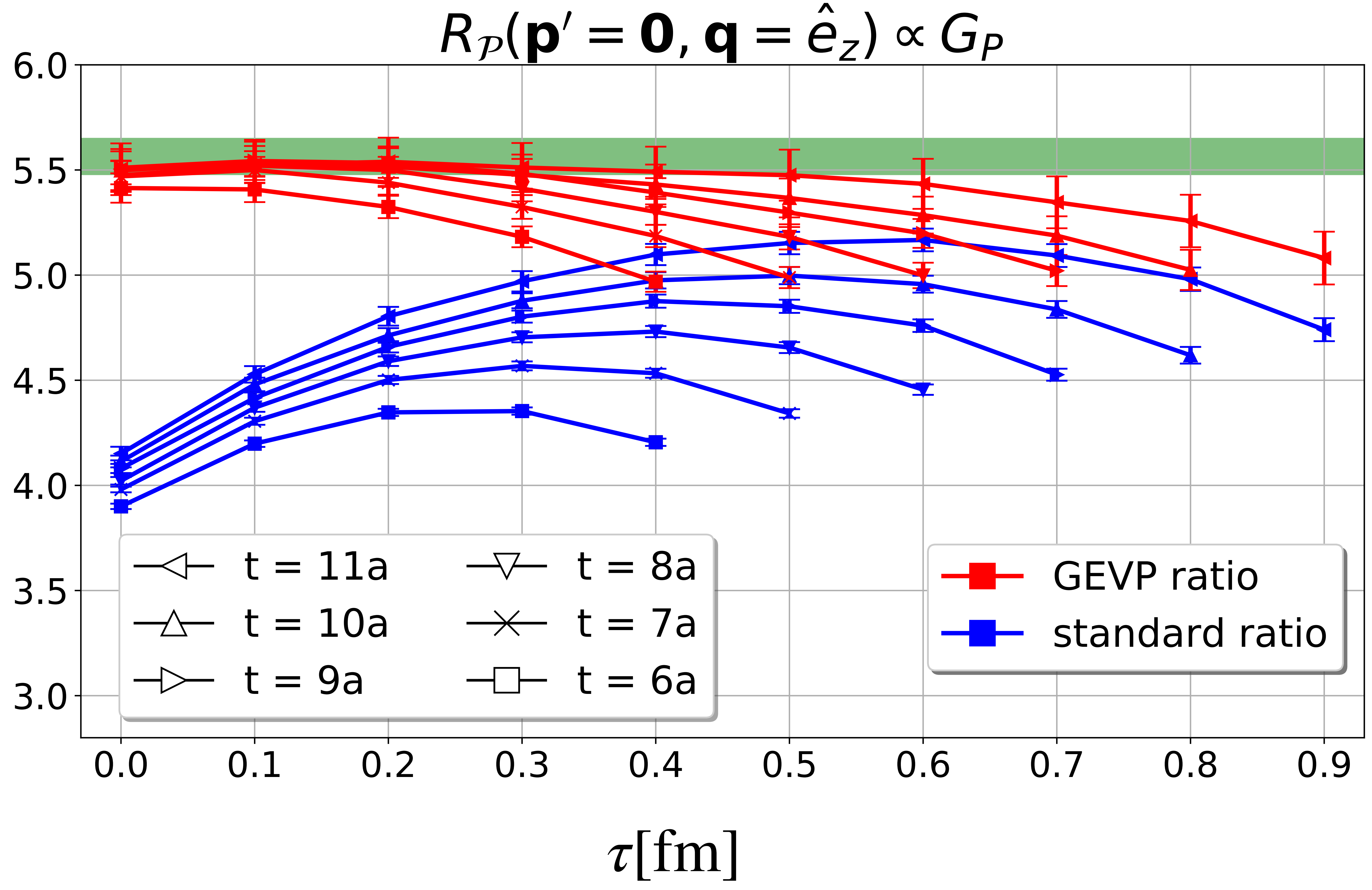
$$\mathrm{O}_N = \sum_i v_i^N(t_0)\ \mathrm{O}_i$$



GEVP ratio at $Q^2 = 0.297\ \mathrm{GeV}^2$ in the pseudoscalar channel

Phenomenologically more interesting are nucleon form factors $G_A, G_P, G_{\tilde{P}}$ at $Q^2 \neq 0$.
Unfortunately, a traditional fit to lattice data gives unreliable FF.

ChPT studies* show that $N\pi$ contribution can be quite large!



$R_{\mathscr{P}}$ is constructed with $\mathscr{J} = \mathscr{P}$

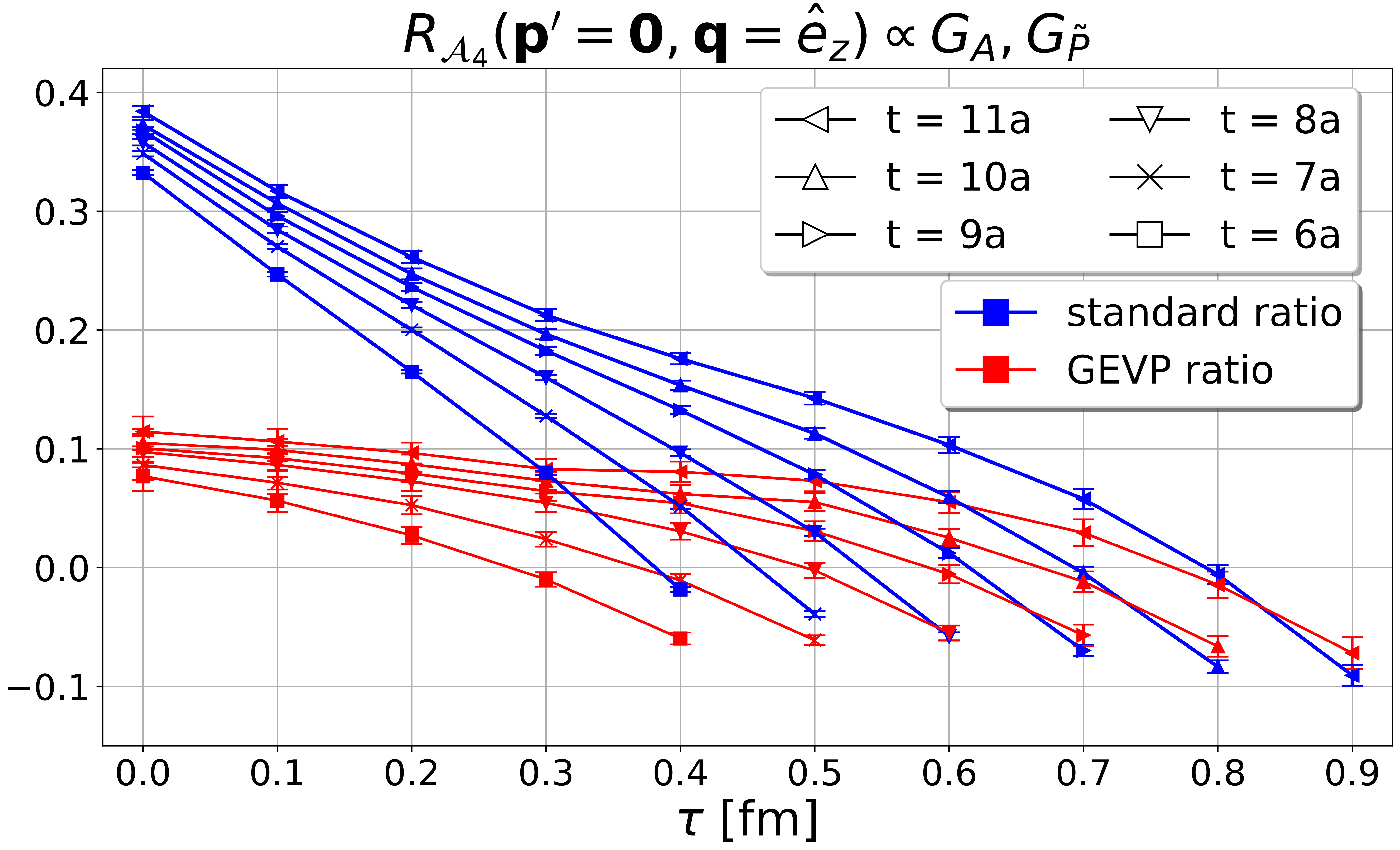
The GEVP improves significantly the ratios, as they approach the green band (nucleon ground state)

There is still a trace of contamination left at the sink $\tau = t$ (rightmost part)

*[PRD.100.054507, PRD.99.054506]

GEVP ratio at $Q^2 = 0.297\ \mathrm{GeV}^2$ in the axial temporal channel

The most dramatic channel is with $\mathcal{J} = \mathcal{A}_4$. Excited states at source and sink have different signs



$R_{\mathcal{A}_4}$ **is constructed with** $\mathcal{J} = \mathcal{A}_4$

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$G_A, G_P, G_{\tilde{P}}$ **satisfy PCAC with a simple fit.**

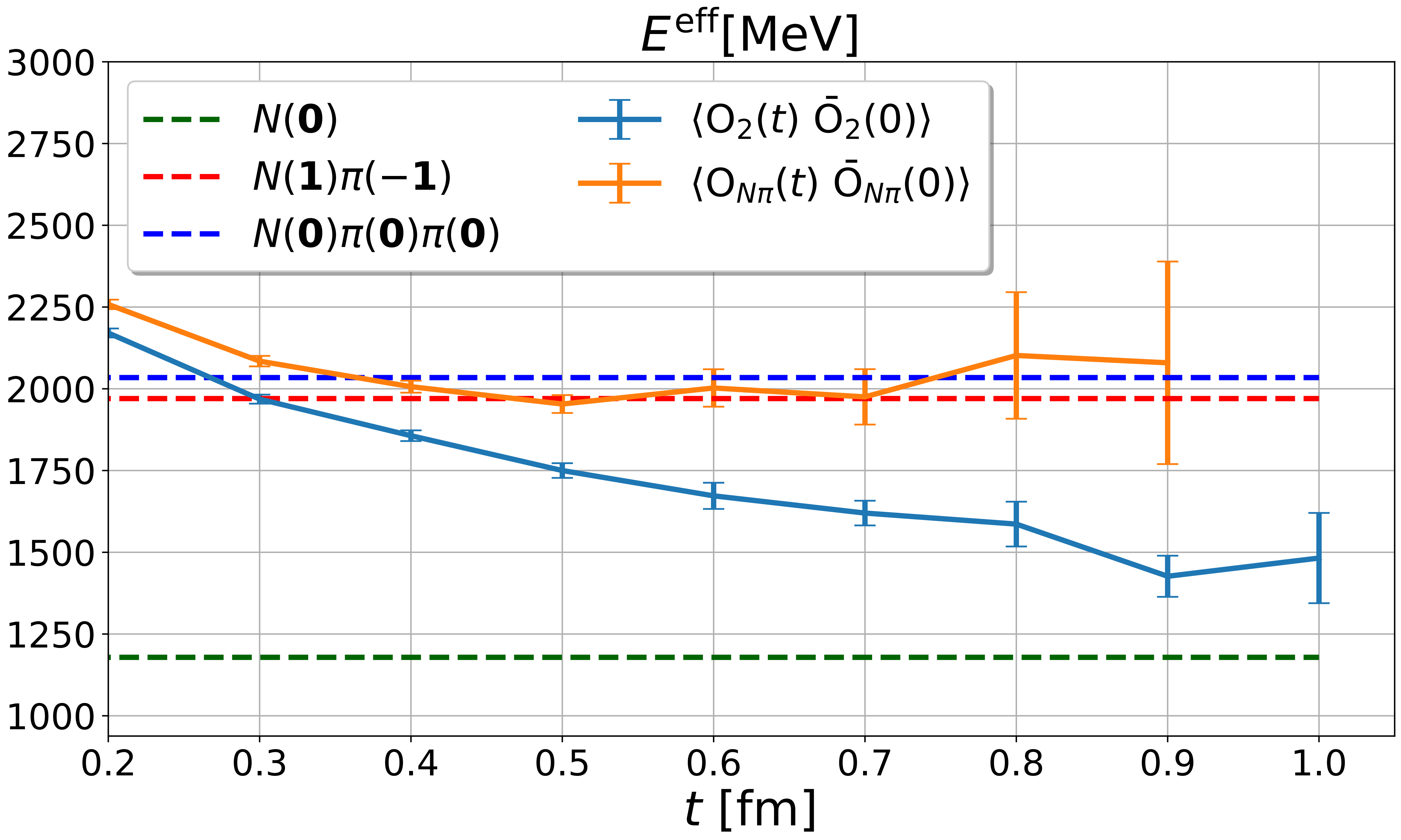
$$m_N G_A(Q^2) = m_\ell G_P(Q^2) + \frac{Q^2}{4m_N} G_{\tilde{P}}(Q^2)$$

GEVP-projected operators ($\mathbf{p} = 0$)

We use eigenvectors to project operators: $\langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_2(0) \rangle \approx c_2^{N\pi} e^{-E_{N\pi}t} + c_2^{N} e^{-E_N t}$ at $t \gg 0$ the dominant term is the nucleon

After GEVP-projection: $\mathrm{O}_{N\pi} = \sum_i v_i^{N\pi}\ \mathrm{O}_i = v_1^{N\pi}\ \mathrm{O}_N + v_2^{N\pi}\ \mathrm{O}_{N\pi}$

$$\langle \mathrm{O}_{N\pi}(t)\ \bar{\mathrm{O}}_{N\pi}(0) \rangle \approx c_{N\pi} e^{-E_{N\pi}t}$$



(Dashed lines are non-interacting energy levels)

$$E^{\mathrm{eff}} = \log \left(\frac{\langle \mathrm{O}(t-a)\ \bar{\mathrm{O}}(0) \rangle}{\langle \mathrm{O}(t)\ \bar{\mathrm{O}}(0) \rangle} \right)$$

The correlation functions with O_2 don't exhibit a plateau here because of the mixing with N states

New step will be the computation of

$$\langle (N\pi)(\mathbf{p}') |\ \mathcal{J}(\mathbf{q})\ | N(\mathbf{p}) \rangle$$

through

$$\langle \mathrm{O}_{N\pi}(\mathbf{p}', t)\ \mathcal{J}(\mathbf{q}, \tau)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0) \rangle$$

Conclusions

Hadron structure

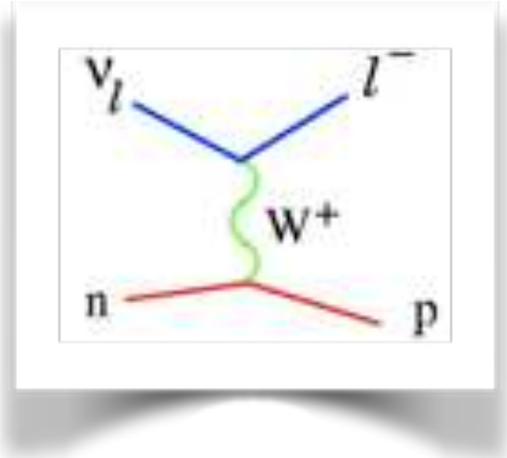
- Structure of the nucleons and excited nucleons is relevant for neutrino oscillation experiments
- Variational method gives promising results for the nucleon ground state matrix elements
- Studies of $\langle N\pi \,|\, \mathcal{J} \,|\, N \rangle$ are undergoing *[PRD.92.074509] (M. Hansen & R. Briceño)*
- Need to clarify the remaining contamination: $N\pi\pi$ in S-wave?
- First step needed in order to study $\langle N^* \,|\, \mathcal{J} \,|\, N \rangle$ and $\langle \Delta^+ \,|\, \mathcal{J} \,|\, N \rangle$

Thank you!

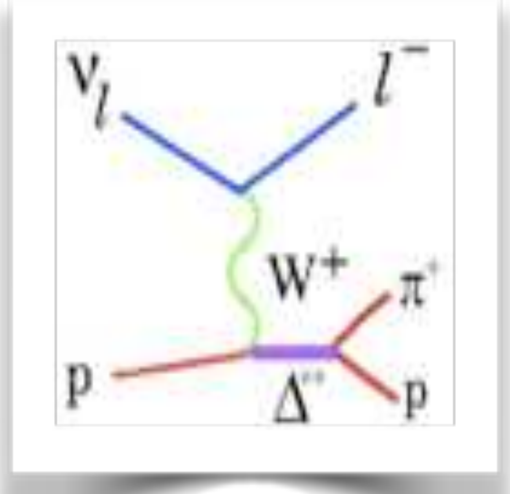
This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 813942

Backup Slides

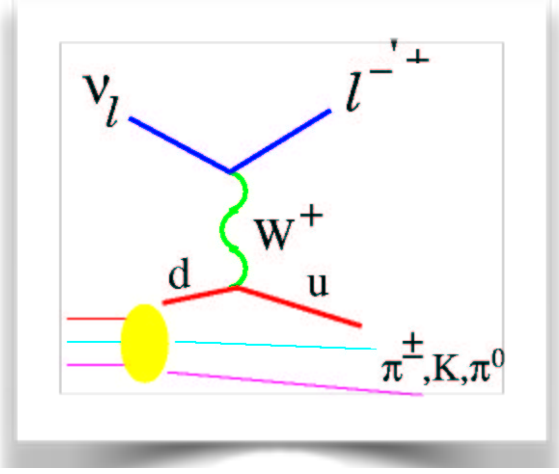
Quasi-elastic scattering (QE)

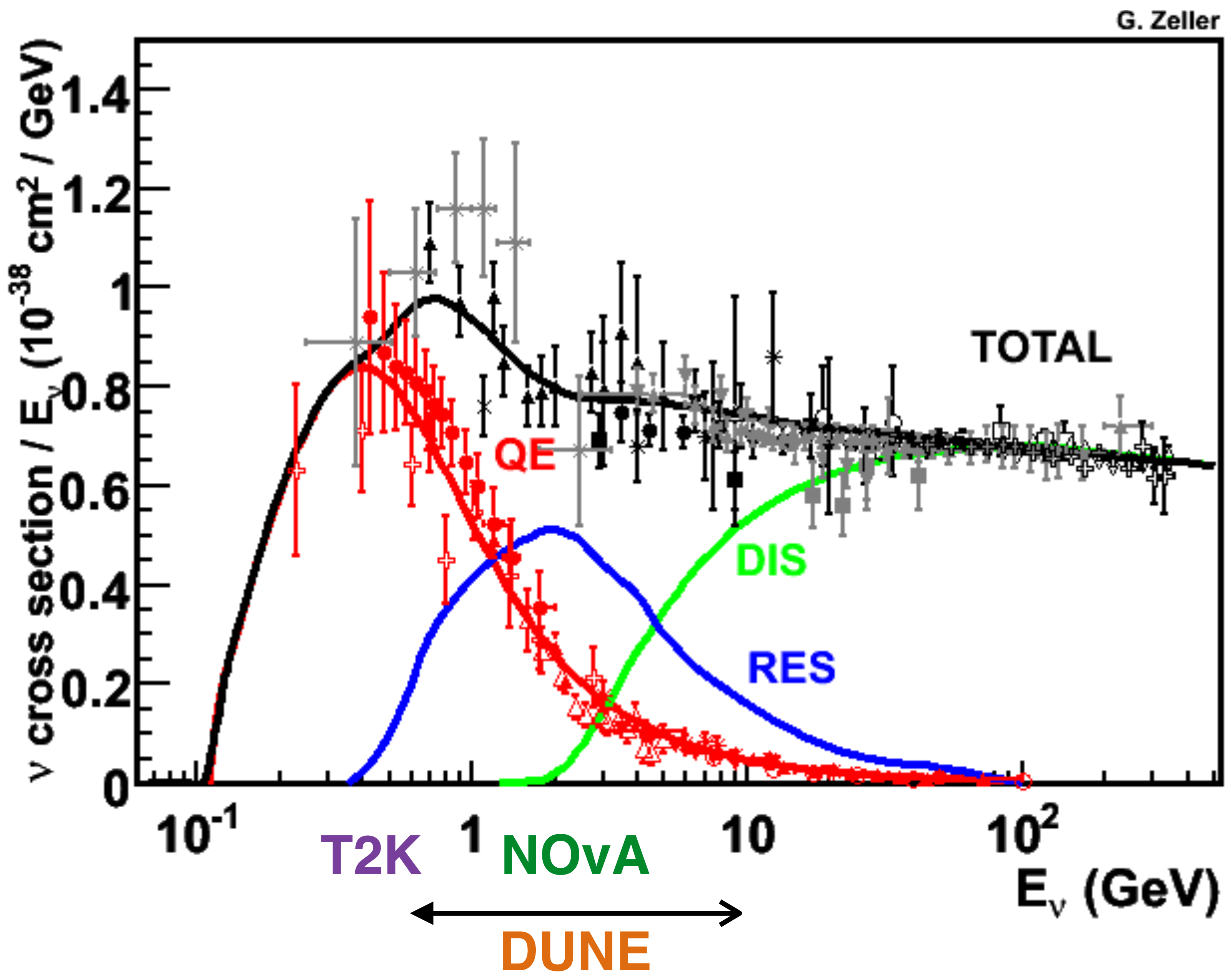


Resonance production (RES)



Deep Inelastic scattering (DIS)





J.A. Formaggio, G. Zeller, Reviews of Modern Physics, 84 (2012)

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multi pion production, DIS, etc.	other	$\nu_\mu + p(n) \to \mu^- + N\pi^\pm + X$, etc.	4

[PRD.81.092005]

[RMP.84.1307] [arXiv:2203.09030]

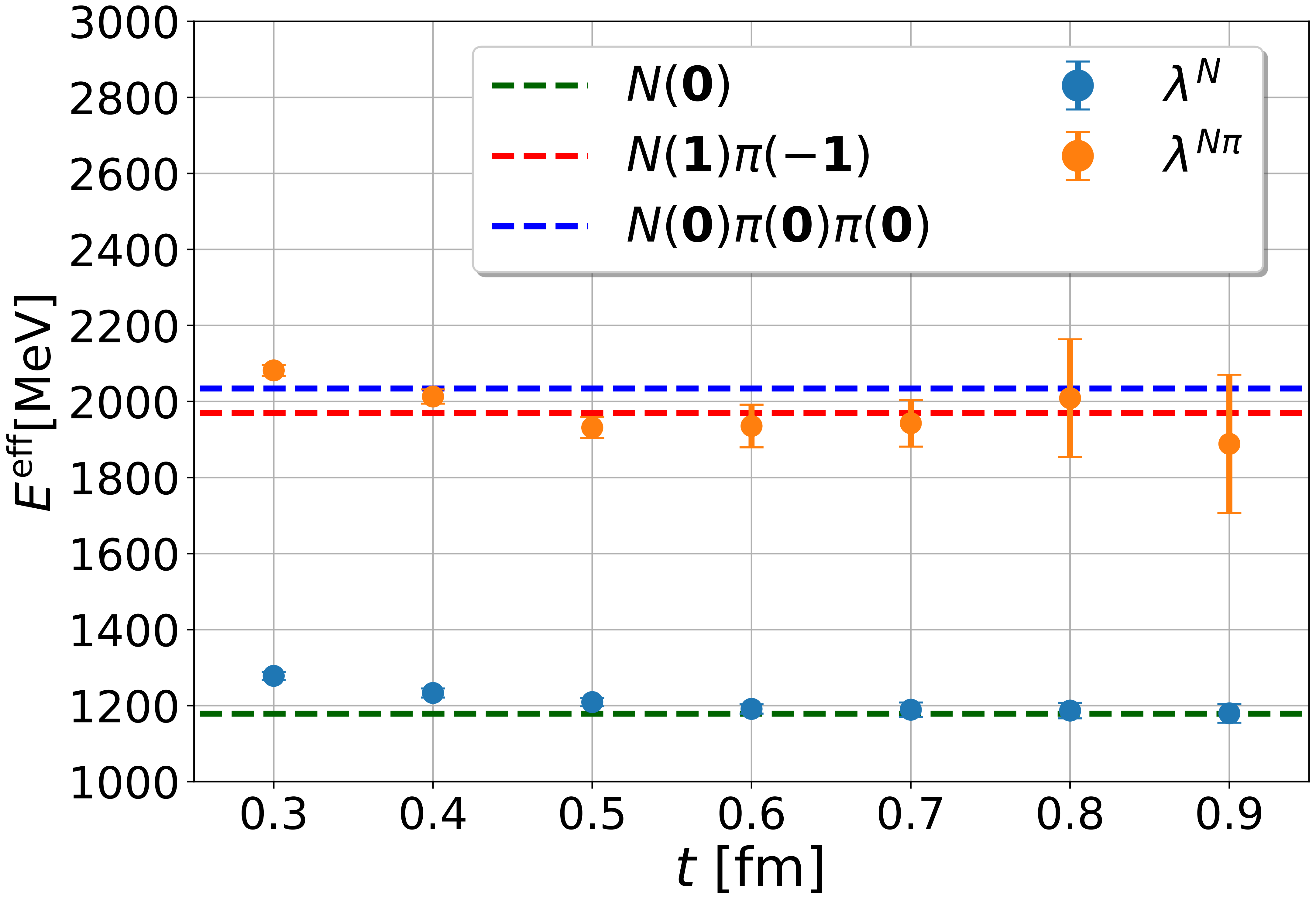
GEVP results with $\mathbf{p} = \mathbf{0}$

$$C(t) = \begin{pmatrix} \langle O_1(t)\ \bar{O}_1(0)\rangle & \langle O_1(t)\ \bar{O}_2(0)\rangle \\ \langle O_2(t)\ \bar{O}_1(0)\rangle & \langle O_2(t)\ \bar{O}_2(0)\rangle \end{pmatrix}$$

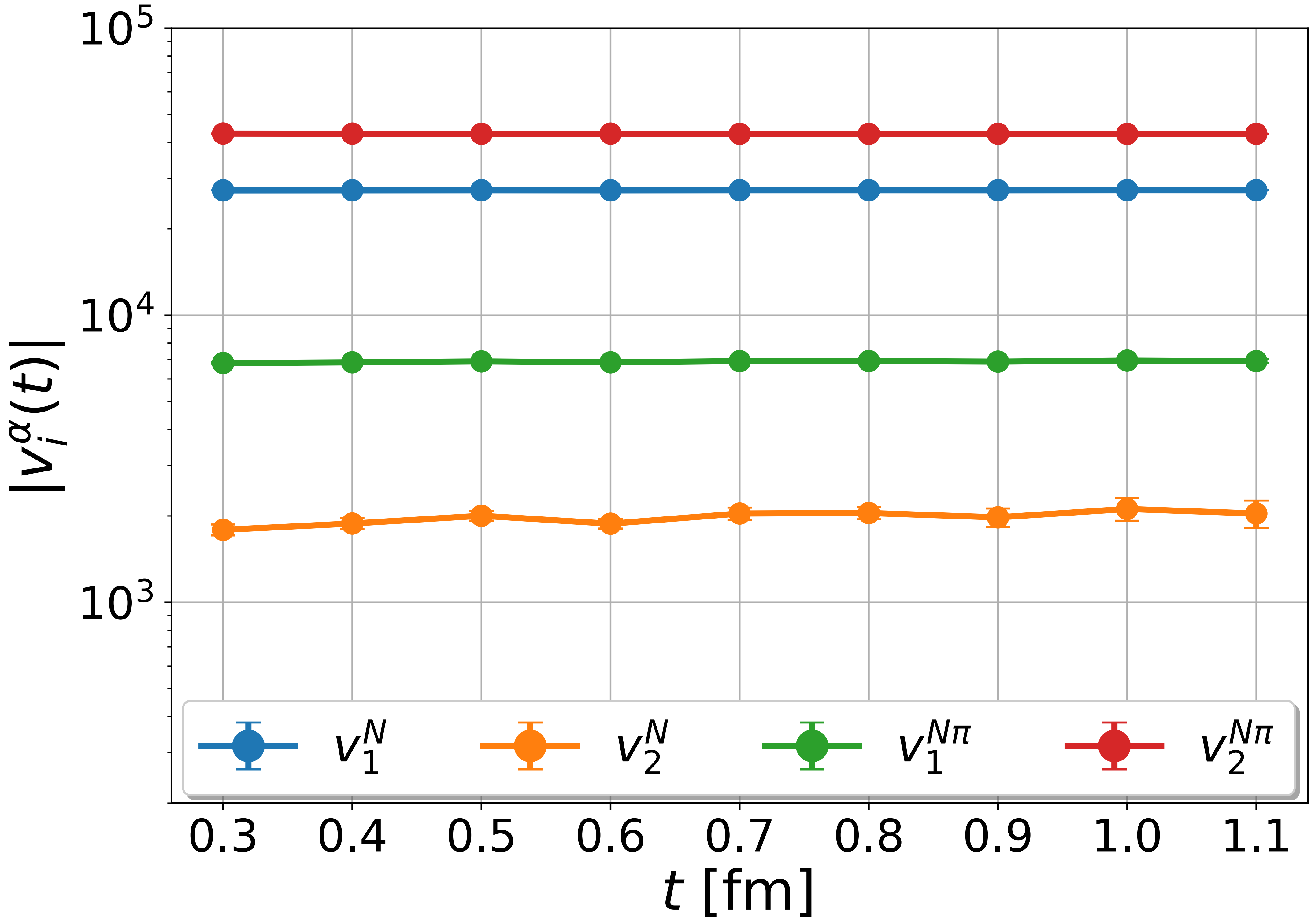
$$C(t)v^\alpha(t,t_0) = C(t_0)\ \lambda^\alpha(t,t_0)v^\alpha(t,t_0) \longrightarrow \begin{array}{l} \lambda^1 \propto e^{-E_N(t-t_0)} \equiv \lambda^N \\ \lambda^2 \propto e^{-E_{N\pi}(t-t_0)} \equiv \lambda^{N\pi} \end{array}$$

We extract the (effective) energies from the eigenvalues: $E_\alpha^{\text{eff}} = \log\left(\lambda^\alpha(t-a)\ /\ \lambda^\alpha(t)\right)$

$v^1 \equiv v^N,\ v^2 \equiv v^{N\pi}$



(Dashed lines are non-interacting energy levels)



$v^\alpha(t,t_0)$ normalised s.t. $(v^\alpha(t,t_0), C(t_0)v^\beta(t,t_0)) = \delta^{\alpha\beta}$

GEVP results with $\mathbf{p} = (2\pi/L)\ \hat{e}_i$

$$C(t) = \begin{pmatrix} \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_2(0)\rangle \\ \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_2(0)\rangle \end{pmatrix}$$

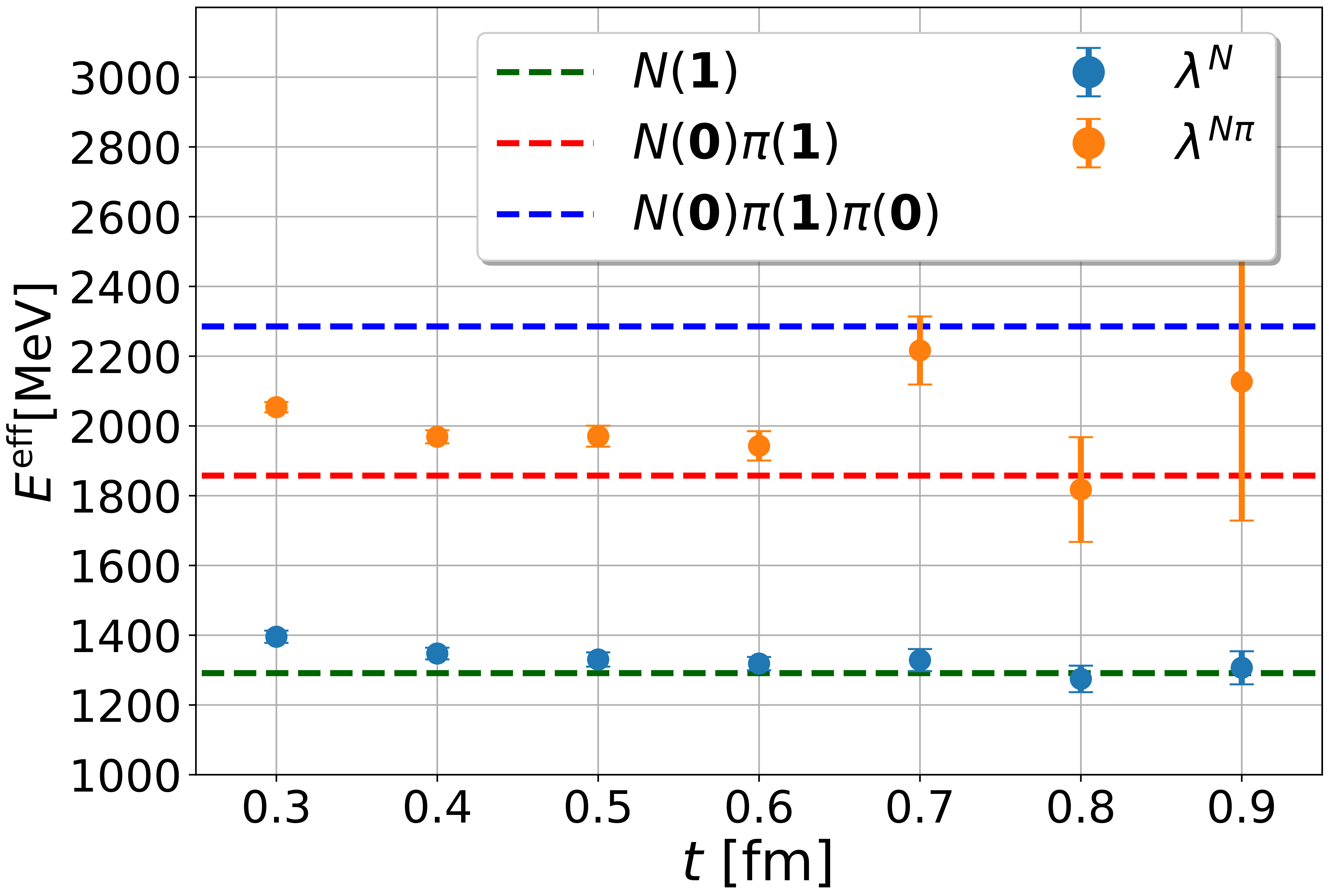
$$C(t)v^\alpha(t,t_0) = C(t_0)\ \lambda^\alpha(t,t_0)v^\alpha(t,t_0)$$

$$\lambda^1 \propto e^{-E_N(t-t_0)} \equiv \lambda^N$$

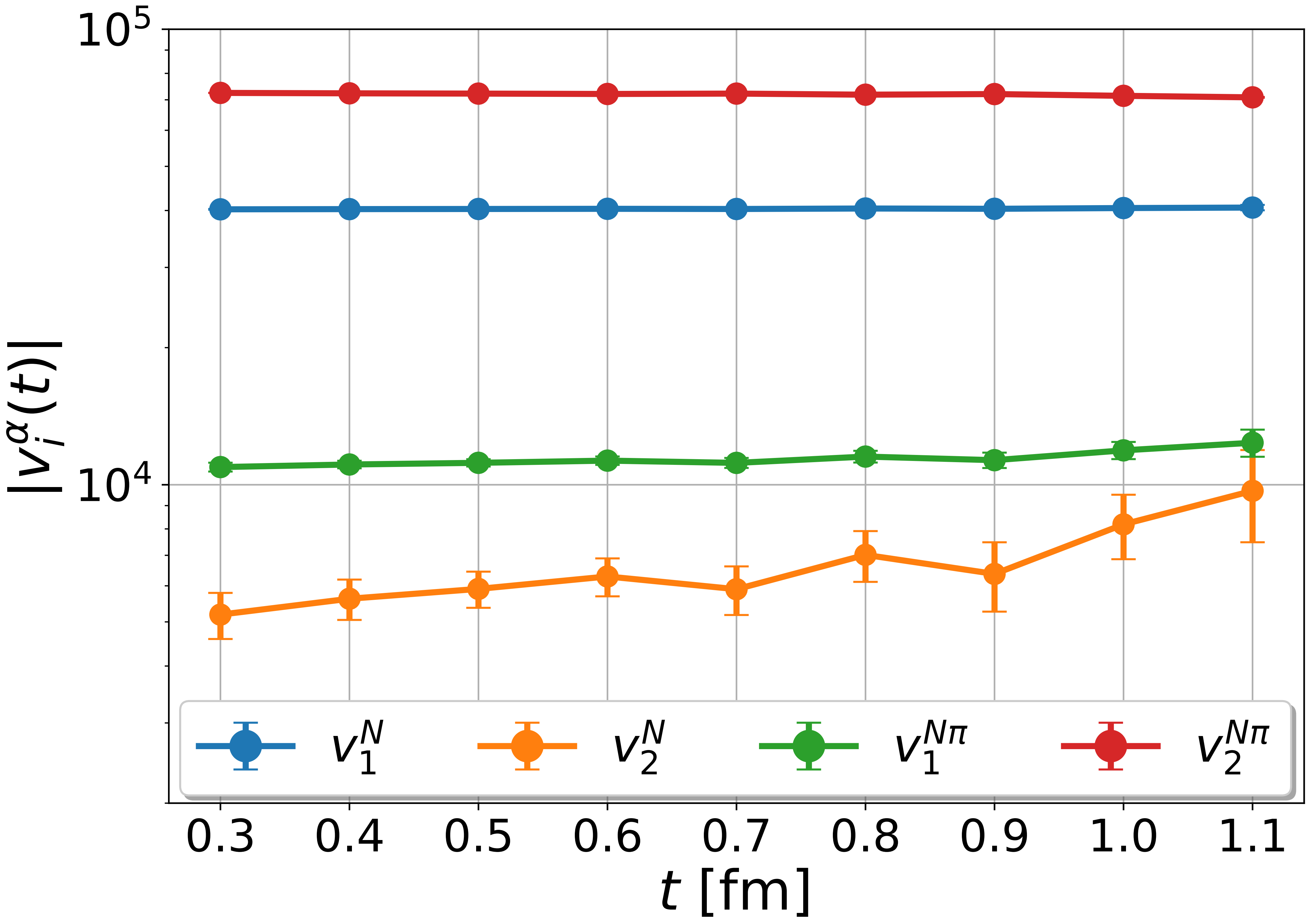
$$\lambda^2 \propto e^{-E_{N\pi}(t-t_0)} \equiv \lambda^{N\pi}$$

We extract the (effective) energies from the eigenvalues: $E^{\rm eff}_\alpha = \log\left(\lambda^\alpha(t-a)\ /\ \lambda^\alpha(t)\right)$

$$v^1 \equiv v^N,\ v^2 \equiv v^{N\pi}$$



(Dashed lines are non-interacting energy levels)



$v^\alpha(t,t_0)$ normalised s.t. $(v^\alpha(t,t_0), C(t_0)v^\beta(t,t_0)) = \delta^{\alpha\beta}$

Extraction of form factors

$$C_{2pt}(\mathbf{p}, t) = \langle \mathrm{O}_N(\mathbf{p}, t)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0) \rangle \qquad C^{\mathcal{J}}_{3pt}(\mathbf{p}', t; \mathbf{q}, \tau) = \langle \mathrm{O}_N(\mathbf{p}', t)\ \mathcal{J}(\mathbf{q}, \tau)\ \bar{\mathrm{O}}_N(\mathbf{p}, 0) \rangle$$

$$R_{\mathcal{J}}(\mathbf{p}', t; \mathbf{q}, \tau) = \frac{C^{\mathcal{J}}_{3pt}(\mathbf{p}', t; \mathbf{q}, \tau)}{C_{2pt}(\mathbf{p}', t)} \sqrt{\frac{C_{2pt}(\mathbf{p}', \tau)\ C_{2pt}(\mathbf{p}', t)\ C_{2pt}(\mathbf{p}, t - \tau)}{C_{2pt}(\mathbf{p}, \tau)\ C_{2pt}(\mathbf{p}, t)\ C_{2pt}(\mathbf{p}', t - \tau)}}$$

$$\langle N(\mathbf{p}') \,|\, \mathcal{J}(\mathbf{q}) \,|\, N(\mathbf{p}) \rangle = u_{\mathbf{p}'} FF[\mathcal{J}] u_{\mathbf{p}}$$

$$\propto\ \mathrm{tr}\left[\mathbb{P}\ (-i\gamma_\mu p'_\mu + m_N)\ FF[\mathcal{J}]\ (-i\gamma_\mu p_\mu + m_N)\right]$$

$$\langle N(\mathbf{p}') \,|\, \mathcal{A}_\mu(\mathbf{q}) \,|\, N(\mathbf{p}) \rangle = u_{\mathbf{p}'} \left[\gamma_\mu \gamma_5 G_A(Q^2) + \frac{q_\mu}{2m_N} \gamma_5 G_{\tilde{P}}(Q^2) \right] u_{\mathbf{p}}$$

$$\langle N(\mathbf{p}') \,|\, \mathcal{P}(\mathbf{q}) \,|\, N(\mathbf{p}) \rangle = u_{\mathbf{p}'}\ \gamma_5 G_P(Q^2)\ u_{\mathbf{p}}$$

$$\langle N(\mathbf{p}') \,|\, \mathcal{V}_\mu(\mathbf{q}) \,|\, N(\mathbf{p}) \rangle = u_{\mathbf{p}'} \left[\gamma_\mu F_1(Q^2) + i \frac{\sigma_{\mu\nu} q_\mu}{2m_N} F_2(Q^2) \right] u_{\mathbf{p}}$$

Operators with $J^P = (1/2)^+$ and $I = 1/2,\ I_z = -1/2$ (neutron channel)

$$C(t) = \begin{pmatrix} \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_1(t)\ \bar{\mathrm{O}}_2(0)\rangle \\ \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_1(0)\rangle & \langle \mathrm{O}_2(t)\ \bar{\mathrm{O}}_2(0)\rangle \end{pmatrix}$$

$$\mathrm{O}_1(x) = \epsilon^{abc}\left(d^a_\alpha(x)\ C\gamma_5\ u^b_\beta(x)\right) d^c_\gamma(x)$$

$$\mathrm{O}_2(x,y) = \left(q(x)q(x)q(x)\right)\left(\bar{q}(y)q(y)\right)$$

O_2 must be projected to represent $J^P = (1/2)^+$ and $I_z = -1/2$

Isospin projection with Clebsch-Gordan

$$\mathrm{O}_2(x,y) = \frac{1}{\sqrt{3}}\mathrm{O}_p(x)\ \mathrm{O}_{\pi^-}(y) - \frac{2}{\sqrt{3}}\mathrm{O}_n(x)\ \mathrm{O}_{\pi^0}(y)$$

Helicity projection with (Lattice) Group Theory

$$\mathrm{O}_{2,\uparrow}(\mathbf{P}=\mathbf{0}) = \mathrm{O}_{N\downarrow}(-\hat{e}_x)\mathrm{O}_\pi(\hat{e}_x) - \mathrm{O}_{N\downarrow}(\hat{e}_x)\mathrm{O}_\pi(-\hat{e}_x) - i\mathrm{O}_{N\downarrow}(-\hat{e}_y)\mathrm{O}_\pi(\hat{e}_y) + i\mathrm{O}_{N\downarrow}(\hat{e}_y)\mathrm{O}_\pi(-\hat{e}_y) + \mathrm{O}_{N\uparrow}(-\hat{e}_z)\mathrm{O}_\pi(\hat{e}_z) - \mathrm{O}_{N\uparrow}(\hat{e}_z)\mathrm{O}_\pi(-\hat{e}_z)$$

$$\mathrm{O}_2^{(1)}(\mathbf{P}=\hat{e}_i) = \mathrm{O}_N(\mathbf{0})\mathrm{O}_\pi(\hat{e}_i) \qquad \mathrm{O}_2^{(2)}(\mathbf{P}=\hat{e}_i) = \mathrm{O}_N(\hat{e}_i)\mathrm{O}_\pi(\mathbf{0})$$

$$\hat{e}_i = \frac{2\pi}{L}\hat{n}_i$$