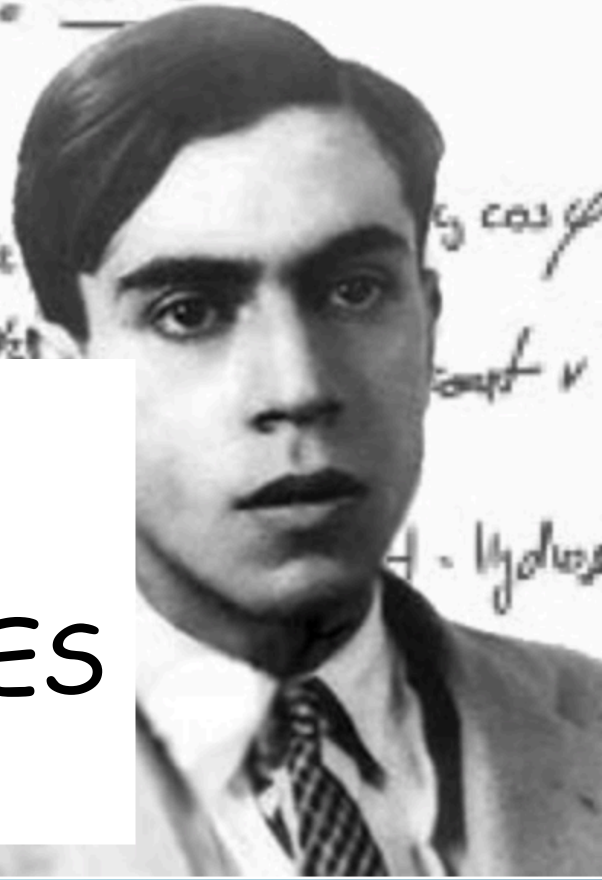
# XI series of Majorana Lectures



# CONSISTENT EFFECTIVE FIELD THEORIES

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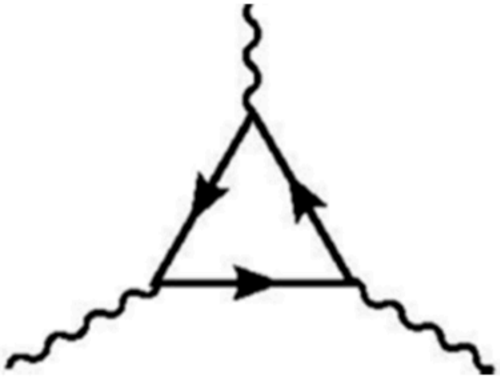
22, 23, 24 Febbraio 2022

# Part III

# gauge anomalies in EFTs

# Effective Field Theories and gauge anomalies

$$\begin{Bmatrix} u \\ d \end{Bmatrix} \begin{Bmatrix} c \\ s \end{Bmatrix} \begin{Bmatrix} t \\ b \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \nu_e \\ e \end{Bmatrix} \begin{Bmatrix} \nu_\mu \\ \mu \end{Bmatrix} \begin{Bmatrix} \nu_\tau \\ \tau \end{Bmatrix}$$



# gauge invariance and anomalies

- gauge invariance ↔ quantization of spin 1 particles

DILEMMA

Lorentz-covariant description $A_\mu$

unitarity

correct number of DOF
2 (massless case)

choosing $A_\mu$ we should eliminate extra DOFs

- massive case

$$\mathcal{L}_{MASSIVE} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

the theory is NOT gauge invariant

$$\partial^\nu \partial_\nu A^\mu - \partial^\mu \partial^\nu A_\nu + \mu^2 A^\mu = 0 \quad \Rightarrow \quad \mu^2 \partial_\mu A^\mu = 0$$

$$\partial^\nu \partial_\nu A^\mu + \mu^2 A^\mu = 0$$
$$\partial_\mu A^\mu = 0$$

Klein-Gordon equations plus a constraint

look for plane wave solutions $A_\mu \approx \epsilon_\mu e^{ikx}$

$$-k^2 + \mu^2 = 0 \quad \epsilon_\mu k^\mu = 0$$

when $k^\mu = (\omega, 0,0, k)$ $\quad \epsilon_\mu^1 = (0,1,0,0) \quad \epsilon_\mu^2 = (0,0,1,0)$

$$\epsilon_\mu^3 = (\frac{k}{\mu}, 0,0, \frac{k}{\mu})$$

3 independent polarizations

exercise: consider

$$\mathcal{L}_{MASSIVE} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{a}{2}\partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{2}\mu^2 A_\mu A^\mu$$

derive the equations of motion and plane wave solutions $A_\mu \approx \epsilon_\mu e^{ikx}$

distinguish two cases: $\epsilon_\mu k^\mu = 0$ and $\epsilon_\mu \propto k_\mu$

discuss the mass spectrum and the limit $a \to 1$

## massless case

$$\mathcal{L}_{MASSLESS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

not the trivial limit $\mu \to 0$:
- constraint $\mu^2\partial_\mu A^\mu = 0$ is lost
- 2 extra DOF

**the theory is GAUGE INVARIANT**

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)$$

this redundancy allows
to eliminate the extra DOF

$$\mathcal{L}_{MASSLESS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

$$\partial^\nu\partial_\nu A^\mu - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu A_\nu = 0 \quad \Rightarrow \quad (\partial^\nu\partial_\nu)\partial_\mu A^\mu = 0$$

$$\partial_\mu A^\mu = \left(\partial_\mu A^\mu\right)^+ + \left(\partial_\mu A^\mu\right)^- \quad \Rightarrow$$

$\left(\partial_\mu A^\mu\right)^+$ |phys> 0

valid also in the interacting theory $A_\mu J^\mu$, provided $\partial_\mu J^\mu = 0$

**gauge invariance mandatory to define the Hilbert space and a unitary theory**

# anomalies

- classical, gauge invariant, action S

$$S = \int d^4x \;\mathcal{L}(\psi, \varphi, A)$$

$$\delta_\alpha S = 0$$

$\Rightarrow$ $\partial_\mu j_a^\mu = 0$

$$j_a^\mu = -\frac{1}{g}\frac{\delta S}{\delta A_{a\mu}}$$

- gauge invariance of effective, action W

$$e^{iW[\varphi,A]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{i\,S}$$

$$\delta_\alpha W = 0$$

$$\delta_\alpha W \neq 0$$

$\Leftrightarrow$ Anomaly

- $\delta_\alpha W$ efficiently studied through differential operators $L(x)$

$$\delta_\alpha W = \int d^4x\, \alpha_a(x) L_a(x) W[\varphi, A]$$

$$\delta_\alpha W = 0 \leftrightarrow L_a(x) W = 0$$

$$L(x) = \left[-\frac{1}{g}\partial_\mu \frac{\delta}{\delta A_\mu(x)} + i\varphi(x)\frac{\delta}{\delta\varphi(x)} - i\varphi^\dagger(x)\frac{\delta}{\delta\varphi^\dagger(x)}\right]$$

for instance:
abelian theory,
one charged scalar

# anomaly-free theories

PHYSICAL REVIEW D VOLUME 6, NUMBER 2 15 JULY 1972

**Gauge Theories Without Anomalies***

Howard Georgi† and Sheldon L. Glashow
*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138*
(Received 27 March 1972)

standard criterion about $\mathcal{L}_4$ :

$$tr\left(t^a\{t^b,t^c\}\right)=0$$

$t^a$ fermion generators of gauge group in a Weyl basis

1 well-known for renormalizable theories
valid also for EFT?

$$\mathcal{L}_{EFT}=\mathcal{L}_4+\frac{1}{\Lambda}\sum c_k^{(5)}\,O_k^{(5)}+\frac{1}{\Lambda^2}\sum c_k^{(6)}\,O_k^{(6)}+\cdots$$

2 assume now a classical gauge-invariant theory with $tr\left(t^a\{t^b,t^c\}\right)\neq 0$
can we give sense to this theory as a QFT?

# $tr\left(t^a\{t^b, t^c\}\right) = 0$ and EFTs

- regularization $W \rightarrow W_r$

$L(x)W_r \neq 0$ ⇒ is the theory anomalous ?

inspect the entire class $\{W\} = W_r + \int d^4y\, P(y)$

$P(y)$ local polynomial in the bosonic fields.

if $P_c(y)$ exists such that

$$L(x)\,[W_r + \int d^4y\, P_c(y)] = 0$$

$$W \equiv W_r + \int d^4y\, P_c(y)$$

$L(x)W = 0$

$P_c(y)$ defined up to a gauge invariant contribution

an anomaly is a non trivial equivalence class $\{L(x)W\}$

Physics Letters B
Volume 37, Issue 1, 1 November 1971, Pages 95-97

# Consequences of anomalous ward identities

J. Wess [a, b], B. Zumino

The anomalies of Ward identities are shown to satisfy consistency or integrability relations, which restrict their possible form. For the case of SU(3) × SU(3) we verify that

$$L_a(x)L_b(y)W_{\mathtt{r}} - L_b(y)L_a(x)W_{\mathtt{r}} = \delta^4(x-y)f_{ab}^{c}L_c(x)W_{\mathtt{r}}$$

- we can regard the class $\{L(x)W\}$ as the unknown in this equation
  the general solution is known from cohomology

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- we can regard the class $\{L(x)W\}$ as the unknown in this equation
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by quantizing $(\varphi, A_\mu)$ gauge invariance is replaced by BRST invariance

- $\delta_{BRST}\, W = 0 \qquad\qquad \delta_{BRST}^2 = 0$

WZ consistency condition reads

$$\delta_{BRST}\left(\delta_{BRST} W_r\right) = 0$$

Physics Letters B
Volume 37, Issue 1, 1 November 1971, Pages 95-97


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  the general solution is known from cohomology

Physics Reports
Volume 338, Issue 5, November 2000, Pages 439-569


# Local BRST cohomology in gauge theories

Glenn Barnich [a, b], Friedemann Brandt [c], Marc Henneaux [a, d]

We shall also consider “effective Yang–Mills theories” for which the Lagrangian contains all possible terms compatible with gauge invariance [118], [223] and thus involves derivatives of arbitrarily high order.

semisimple gauge group:
the anomaly is a polynomial of dimension 4 in the fields and derivatives

$$\{L_a(x)W\} = \frac{ie^2}{24\pi^2}\varepsilon^{\mu\nu\lambda\rho}\mathrm{tr}\, T^a \partial_\mu \left( A_\nu \partial_\lambda A_\rho + \frac{1}{2} A_\nu A_\lambda A_\rho \right)$$

The anomaly does not depend on $c_k$ in semisimple gauge theories [dependence on $c_k$ only in d > 4 contributions]

abelian theory, one charged scalar
non-trivial candidate anomalies $L(x)W$

$$\varphi^\dagger \varphi\, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu\, \partial_\rho A_\sigma \quad , \qquad i\, \varphi^\dagger \varphi\, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu (\varphi^\dagger D_\nu \varphi - D_\nu \varphi^\dagger \varphi)\, \partial_\rho A_\sigma$$

these are solutions of

$$L_a(x) L_b(y) W_{\mathtt{r}} - L_b(y) L_a(x) W_{\mathtt{r}} = \delta^4(x-y) f_{ab}^c L_c(x) W_{\mathtt{r}}$$

dependence on $c_k$ not forbidden by BRST cohomology if $G_{\text{gauge}}$ is not semisimple

PHYSICAL REVIEW D VOLUME 35, NUMBER 6 15 MARCH 1987

# Spinor loop anomalies with very general local fermion Lagrangians

Jooha Minn, Jewan Kim, and Choonkyu Lee
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(Received 27 October 1986)

fundamental level.] An effective local Lagrangian will typically include renormalizable couplings[33] which play a dominant dynamical role, and nonrenormalizable couplings[34] (involving operators of dimension larger than four in the case of four spacetime dimensions) which may be less important dynamically but still crucial for some processes. There is a question which arises naturally in an effective low-energy (chiral or nonchiral) gauge theory with spinor fields. Will there not be certain restrictions to the structure of allowed higher dimensional local spinor couplings (besides naive gauge invariance of the forms) because of possible gauge anomaly problems? We can now give a definite answer to that—as long as the spinor field contents are such that the usual gauge anomaly cancellation condition[2] is satisfied, the effective gauge theory Lagrangian may include any gauge-invariant, renormalizable or nonrenormalizable, local spinor couplings without encountering gauge inconsistency by spinor loop effects.

# $tr\left(t^a\{t^b,t^c\}\right) \neq 0$ and EFT

- assume now a classical gauge-invariant theory with $tr\left(t^a\{t^b,t^c\}\right) \neq 0$

  can we give sense to this theory as a QFT?

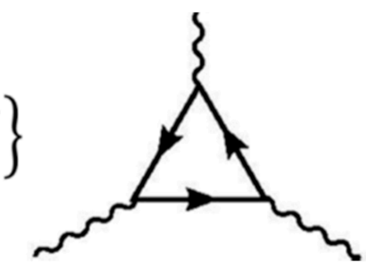
  Yes!

  consider a world coinciding with ours but for the heaviness of $(t,b)$

  $m_t, m_b \gg v$ (ew scale)

  gauge theory $G = SU(3)\times SU(2)\times U(1)$

  $$\begin{Bmatrix} u \\ d \end{Bmatrix} \begin{Bmatrix} c \\ s \end{Bmatrix} \begin{Bmatrix} t \\ b \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \nu_e \\ e \end{Bmatrix} \begin{Bmatrix} \nu_\mu \\ \mu \end{Bmatrix} \begin{Bmatrix} \nu_\tau \\ \tau \end{Bmatrix}$$

  $\mathcal{L}_{SM}(\Psi)$ equipped with Higgs mechanism

  gauge anomalies cancelled within each generation

  $\Psi = (\Psi_{light}, t, b)$

  consider the $\mathcal{L}_{EFT}\left(\Psi_{light}\right)$ at $E \ll m_t, m_b$

  this is a consistent gauge theory with anomalous fermion content

  [contribution to $tr\left(t^a\{t^b,t^c\}\right)$ from 3rd generation do not cancel any more]

# a closer look to $\mathcal{L}_{EFT}(\Psi_{light})$

toy model

| | $l_L$ | $l_R$ | $q_L$ | $q_R$ | $\varphi$ |
|---|---|---|---|---|---|
| $Q$ | $-1$ | $0$ | $+1$ | $0$ | $-1$ |
| $B$ | $0$ | $0$ | $+1$ | $+1$ | $0$ |
| $L$ | $+1$ | $+1$ | $0$ | $0$ | $0$ |

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&+ i\bar{l}_L\gamma^\mu(\partial_\mu - igA_\mu)l_L + i\bar{l}_R\gamma^\mu\partial_\mu l_R \\
&+ i\bar{q}_L\gamma^\mu(\partial_\mu + igA_\mu)q_L + i\bar{q}_R\gamma^\mu\partial_\mu q_R \\
&+ D_\mu\varphi^\dagger D^\mu\varphi - V(\varphi^\dagger\varphi) \\
&- y_l(\bar{l}_L\varphi l_R + h.c.) - y_q(\bar{q}_L\varphi q_R + h.c.)
\end{aligned}
$$

$$tr(Q^3) = 0$$

anomaly free
UV theory

$$V(\varphi^\dagger\varphi) = \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$$

$$<\varphi>= \frac{v}{\sqrt{2}}$$

$$v^2 = -\frac{\mu^2}{\lambda}$$

SB phase

$$\varphi = \frac{(\sigma + v)}{\sqrt{2}} e^{i\frac{\xi}{v}}$$

gauge transformation

$$\xi' = \xi + \alpha(x) v$$

$$\sigma' = \sigma$$

| | $A_\mu$ | $\sigma$ | $l$ | $q$ |
|---|---|---|---|---|
| $mass$ | $gv$ | $\sqrt{2\lambda}v$ | $y_l v/\sqrt{2}$ | $y_q v/\sqrt{2}$ |

consider now $y_q \gg y_l, g, \lambda$

define an EFT valid at $E < m_q$

$$e^{iW_{EFT}[A,\varphi,l]} \equiv \int \mathcal{D}q \; e^{iS[A,\varphi,l,q]}$$

the theory described by $W_{EFT}[A,\varphi,l]$ is anomaly-free, despite $tr(Q_l^3) \neq 0$

one-line proof

$$e^{iW[A,\varphi]} = \int \mathcal{D}l \; e^{iW_{EFT}[A,\varphi,l]} = \int \mathcal{D}l \, \mathcal{D}q \; e^{iS[A,\varphi,l,q]}$$

# general structure of $W_{EFT}[A, \varphi, l]$

gauge transformation on $W[A, \varphi]$

$$A_\alpha = A + \partial\alpha$$

$$\varphi_\alpha = e^{i\alpha}\varphi$$

$$l_\alpha = \begin{cases} e^{i\alpha} l_L \\ l_R \end{cases}$$

$$e^{iW[A_\alpha,\varphi_\alpha]} = \int \mathcal{D}l \; e^{iW_{EFT}[A_\alpha,\varphi_\alpha,l]}$$

change of variables $l \to l_\alpha$

$$\mathcal{D}l_\alpha = \mathcal{D}l \; e^{i\frac{g^2}{48\pi^2}\int d^4x\alpha\, F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

$$= \int \mathcal{D}l \; e^{i\frac{g^2}{48\pi^2}\int d^4x\alpha F_{\mu\nu}\tilde{F}^{\mu\nu}} \times e^{iW_{EFT}[A_\alpha,\varphi_\alpha,l_\alpha]}$$

$$= e^{iW[A,\varphi]} = \int \mathcal{D}l \; e^{iW_{EFT}[A,\varphi,l]}$$

$$W_{EFT}[A_\alpha, \varphi_\alpha, l_\alpha] + \frac{g^2}{48\pi^2}\int d^4x\; \alpha F_{\mu\nu}\tilde{F}^{\mu\nu} = W_{EFT}[A, \varphi, l]$$

loop expansion of $W_{EFT}[A,\varphi,l]$

$$W_{EFT}[A,\varphi,l] = S[A,\varphi,l,0] + W_{EFT}^{\geq 1}[A,\varphi,l]$$

$$W_{EFT}[A_\alpha,\varphi_\alpha,l_\alpha] + \frac{g^2}{48\pi^2}\int d^4x\alpha F_{\mu\nu}\tilde{F}^{\mu\nu} = W_{EFT}[A,\varphi,l]$$

satisfied by $W_{EFT}^{\geq 1}$, since $S[A,\varphi,l,0]$ is gauge invariant

general solution

$$W_{EFT}[A,\varphi,l] = S[A,\varphi,l,0] - \frac{g^2}{48\pi^2 v}\int d^4x\,\xi F_{\mu\nu}\tilde{F}^{\mu\nu} + \Delta W_{EFT}^{\geq 1}[A,\varphi,l]$$

gauge-invariant

# Lorentz-invariant, unitary theory despite anomalous fermion content

cut-off estimate

$$\frac{g^2}{48\pi^2 v}\int d^4x\,\xi\,F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \frac{1}{M}\int d^4x\,\xi\,F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$\Lambda \leq 4\pi M = \frac{192\pi^3 v}{g^2}$$

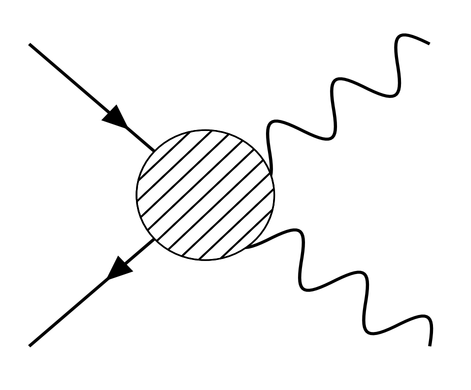
$$\frac{m_A}{\Lambda} \geq \frac{g^3}{192\pi^3}$$

- massive spin-one particle in the spectrum
  [$m_A \to 0$ limit not allowed]
- nonrenormalizable EFT
  [$\Lambda \to \infty$ limit not possible]

# Exercise: independence on gauge fixing

show the independence on the gauge-fixing parameter of the scattering amplitude

$$ll \to AA$$



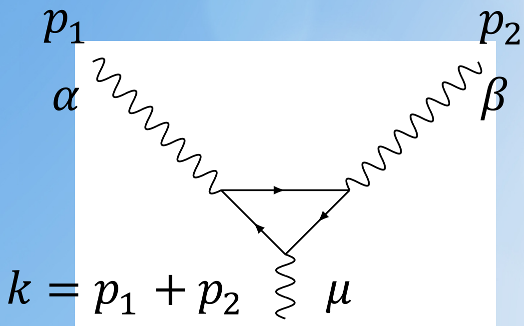
$$\mathcal{L}_{EFT} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{l}_L\gamma^\mu(\partial_\mu - igA_\mu)l_L$$

$$+i\,\bar{l}_R\gamma^\mu\partial_\mu l_R + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\left(1+\frac{\sigma}{v}\right)^2\left(\partial_\mu\xi - gvA_\mu\right)^2$$

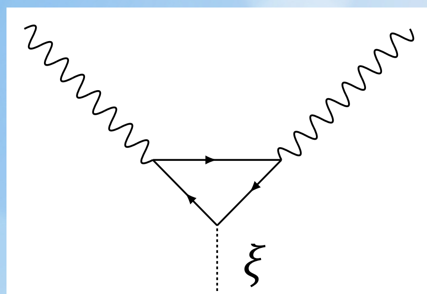
$$\boxed{-\frac{1}{2\lambda}\left(\partial_\mu A^\mu + gv\xi\right)^2} - \frac{g^2}{48\pi^2 v}\xi F_{\mu\nu}\tilde{F}^{\mu\nu} + \cdots$$

1. show that at the tree-level there is no $\lambda$-dependence

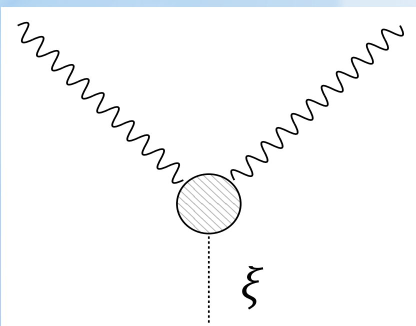
2. list the one-loop contributions depending on $\lambda$




$$= \Gamma_A^{\mu\alpha\beta}\,(p_1, p_2)$$



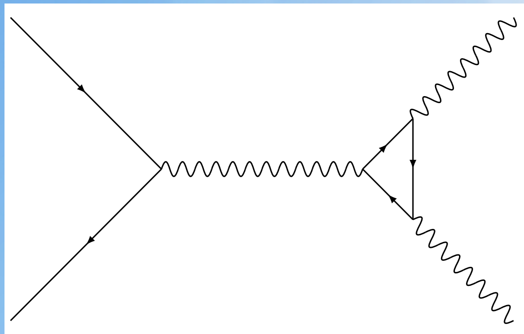

$$= \Gamma_\xi^{\alpha\beta}\,(p_1, p_2)$$




$$= \Gamma_{WZ}^{\alpha\beta}\,(p_1, p_2) = -i\,\frac{g^2}{12\pi^2 v}\epsilon^{\mu\nu\alpha\beta} p_{1\mu} p_{2\nu}$$

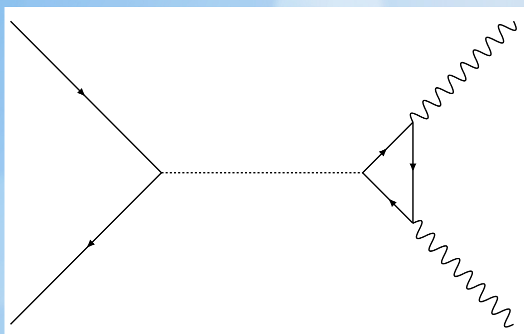
**3.** derive the Ward Identity

$$i\, k_\mu \Gamma_A^{\mu\alpha\beta}(p_1, p_2) = m_A \Gamma_\xi^{\alpha\beta}(p_1, p_2) + m_A \Gamma_{WZ}^{\alpha\beta}(p_1, p_2)$$

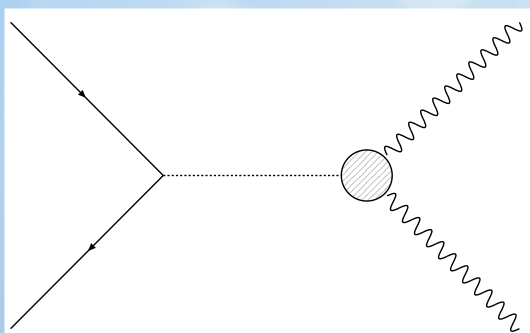


$$= g\,\bar{v}\,\gamma_\mu \frac{(1-\gamma_5)}{2} u \,\frac{1}{k^2-m_A^2}\,\Gamma_A^{\mu\alpha\beta}(p_1,p_2) +$$

$$-g\,m_l\,\overline{v}\,\gamma_5 u \frac{1}{k^2-\lambda m_A^2}(1-\lambda)\frac{1}{k^2-m_A^2}\,k_\mu\,\Gamma_A^{\mu\alpha\beta}(p_1,p_2)$$



$$\frac{m_l}{v}\,\overline{v}\,\gamma_5 u \frac{i}{k^2-\lambda m_A^2}\Gamma_\xi^{\alpha\beta}(p_1,p_2)$$



$$\frac{m_l}{v}\,\overline{v}\,\gamma_5 u \frac{i}{k^2-\lambda m_A^2}\Gamma_{WZ}^{\alpha\beta}(p_1,p_2)$$

$$= g\,\bar{v}\,\gamma_\mu \frac{(1-\gamma_5)}{2} u \,\frac{1}{k^2-m_A^2}\,\Gamma_A^{\mu\alpha\beta}(p_1,p_2) +$$

$$\frac{m_l}{v}\,\overline{v}\,\gamma_5 u \frac{i}{k^2-m_A^2}\left(\Gamma_\xi^{\alpha\beta}(p_1,p_2)+\Gamma_{WZ}^{\alpha\beta}(p_1,p_2)\right)$$

# $tr\left(t^a\{t^b, t^c\}\right) \neq 0$ and EFT

so far: we started from an anomaly-free UV gauge theory and built $\mathcal{L}_{EFT}(\Psi_{light})$ by integrating out an heavy chiral fermion

here: we start from a classical gauge theory $\mathcal{L}(\psi)$ with anomalous fermion content and check under which conditions it gives rise to a consistent EFT

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{\psi}_L\gamma^\mu(\partial_\mu + igQA_\mu)\psi_L + i\,\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R$$

classically invariant under local transformations

$$A_\mu \to A_\mu + \partial_\mu\alpha \qquad \psi_L \to e^{-iQ\alpha}\psi_L \qquad \psi_R \to \psi_R$$

$Q \neq 0$ ⇒ $tr(Q^3) \neq 0$ anomalous fermion content

cannot describe a massless spin-1 particle: Hilbert space not defined

does it describe a massive $A_\mu$ ?

$$e^{iW[A]} \equiv \int \mathcal{D}\psi \, e^{iS[A,\psi]}$$

by proceeding as before

$$\mathcal{D}\psi_\alpha = \mathcal{D}\psi \, e^{-i\frac{g^2 Q^3}{48\pi^2}\int d^4x\alpha \, F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

$$W[A_\alpha] = W[A] - \frac{g^2 Q^3}{48\pi^2}\int d^4x\alpha \, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

anomaly

1. introduce the dimensionless field $\vartheta$

$$\vartheta \to \vartheta + \alpha$$

2. "repair" the anomaly by adding the new term

$$\delta\mathcal{L}[\vartheta] = +\frac{g^2 Q^3}{48\pi^2}\vartheta \, F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \mathcal{L}_\vartheta = \mathcal{L} + \delta\mathcal{L}[\vartheta]$$

$$e^{iW_\vartheta[A,\vartheta]} \equiv \int \mathcal{D}\psi \, e^{iS_\vartheta[A,\vartheta,\psi]}$$

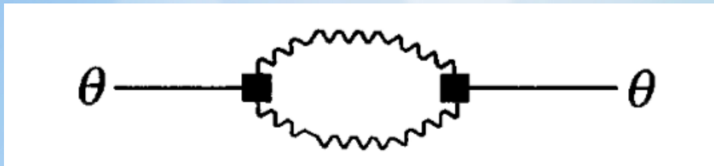
anomaly free

we have NOT modified the theory: $\mathcal{L}$ is simply $\mathcal{L}_\vartheta = \mathcal{L} + \delta\mathcal{L}[\vartheta]$ in the gauge $\vartheta = 0$

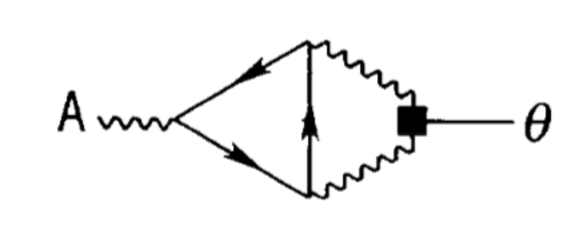
inclusion of $\vartheta$ as a device to make perturbative expansion easier

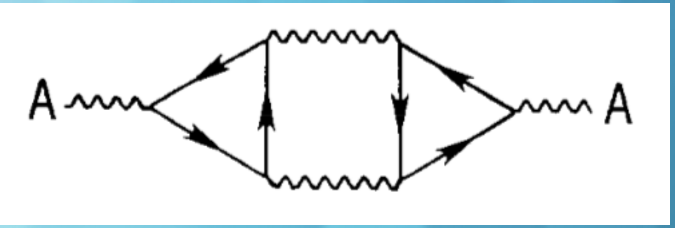
3. compute quantum corrections to the classical theory $\mathcal{L}_\vartheta = \mathcal{L} + \delta\mathcal{L}[\vartheta]$

Feynman rules plus UV cutoff $\Lambda$ as regulator
[plus finite counterterms to maintain the induced gauge invariance]

a) 


$$\approx \frac{1}{16\pi^2}\left(\frac{g^2Q^3}{16\pi^2}\right)^2 \Lambda^2\, \partial_\mu\vartheta\partial^\mu\vartheta$$

b) 


c) 


$$a) + b) + c) = \frac{v^2}{2}\left(\partial_\mu\vartheta - g\,A_\mu\right)^2$$

$$v \approx \frac{g^2Q^3}{64\pi^3}\Lambda$$

$$= \frac{1}{2}\left(\partial_\mu\xi - g\,vA_\mu\right)^2$$

after rescaling $\vartheta = \frac{\xi}{v}$

gauge-invariant kinetic term for $\xi$

back to the unitary gauge $\vartheta = 0$

$$m_A = gv$$

massive
spin-1
particle

can we remove the cutoff?

$$\mathcal{L}_\vartheta = \mathcal{L} + \frac{g^2 Q^3}{48\pi^2}\frac{\xi}{v} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\left(\partial_\mu \xi - g\, v A_\mu\right)^2 \qquad \Lambda \approx \frac{64\pi^3}{g^2 Q^3} v$$

$$\frac{m_A}{\Lambda} \approx \frac{g^3 Q^3}{64\pi^3}$$

- massive spin-one particle in the spectrum
  [$m_A \to 0$ limit not allowed]
- nonrenormalizable EFT
  [$\Lambda \to \infty$ limit not possible]

the anomalous classical gauge theory $\mathcal{L}(\psi)$ describe a massive spin-1 particle and has a finite energy domain of validity

# non-abelian gauge group $G$

$\psi \to \Omega^{-1}\psi$ $\qquad$ $\Omega \equiv e^{i\alpha_a t_R^a}$

$A_\mu \to \Omega^{-1}A_\mu\,\Omega + \Omega^{-1}\partial_\mu\,\Omega$ $\qquad$ $A_\mu \equiv igt_R^a A_{a\mu}$

assume $R$ irreducible
and
$tr\left(t^a\{t^b,t^c\}\right) \neq 0$

$$\mathcal{L} = -\frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi$$

introduce the dimensionless field $\mathcal{U}$, "repair" the anomaly by adding a new term
compute quantum corrections to the classical theory

$\mathcal{U} \to \Omega^{-1}\mathcal{U}$ $\qquad$ $\mathcal{U} \equiv e^{i\frac{\xi_a}{f}t_R^a}$ $\qquad$ unitary gauge: $\mathcal{U} = 1$

a mass term for the spin-1 particles is generated

$$-\frac{m_A^2}{2g^2}tr\left(A_\mu + \partial_\mu\mathcal{U}\cdot\mathcal{U}^{-1}\right)^2$$

it contains a self-interaction for the scalar fields $\xi_a(x)$

$$-\frac{m_A^2}{2g^2} tr\left(\partial_\mu \mathcal{U} \cdot \mathcal{U}^{-1}\right)^2 + \cdots$$

$$= \frac{m_A^2}{2g^2 f^2} \partial_\mu \, \xi_a \partial^\mu \xi_a + \frac{m_A^2}{2g^2 f^4} (\partial\xi \cdot \xi \; \partial\xi \cdot \xi) + \cdots$$

$$\frac{m_A^2}{g^2 f^2} = 1$$

$$\frac{g^2}{2m_A^2}$$

$$\Lambda \leq 4\pi \frac{m_A}{g}$$

this term describes, via the equivalence theorem,
the interaction of longitudinally polarized spin-1 particles,

$$\frac{m_A}{\Lambda} \geq \frac{g}{4\pi}$$

same conclusion as in abelian case

ratio is different since in the abelian case there is no self-interaction

# RGE flow

## "Irreversibility" of the flux of the renormalization group in a 2D field theory

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There exists a function $c(g)$ of the coupling constant $g$ in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where $c$ is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

Some of the information on the ultraviolet behavior of the field theory is lost under renormalization transformations with $t > 0$, since in the field theory it is not legitimate to examine correlations at scales smaller than the cutoff. We would therefore expect that a motion of the space $Q$ under the influence of the renormalization group would become an "irreversible" process, similar to the time evolution of dissipative systems.

# RGE flow in 2d

d=2

a function $c(g)$ of the coupling constants $g$ exists such that

$c(g)$ decreases along the RG flow

[Zamolodchicov 1986]

$c_{UV} > c_{IR}$

the RG flow is irreversible

at fixed points of RG flow the theory is scale invariant and $c(g) = c$

$$[L_m, L_n] = (m-n)L_{m+n} + c\frac{1}{12}(m^3 - m)\delta_{m+n}$$

(central charge of CFT)

for a CFT in flat space

$T^\mu_\mu = 0$

trace of energy - momentum tensor

$-\frac{c}{12}R$ is the anomaly of scale transformations when the CFT is in curved space

scalar curvature

d=4

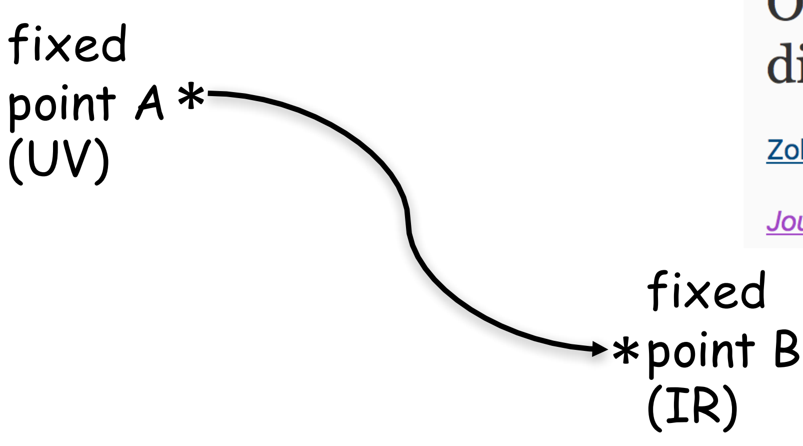
A CFT in curved space has an anomaly depending on (c,a)

$$T_\mu^\mu = c\,W^2 - a\,E$$

$$W^2 = R_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$E = R_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

a-theorem:




On renormalization group flows in four dimensions

Zohar Komargodski & Adam Schwimmer

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in A and B the theory is CFT

then: $a_{UV} > a_{IR}$

1. to induce the RG flow from A to B we perturb the UV theory by adding a “mass term”

$$S_{UV} = S_{UV}^{CFT} + \frac{1}{2}\int d^4x\; m^2\,\varphi^2$$

[flat space classical theory]

breaks scale invariance and push the theory to IR

2. scale invariance in flat space $\leftrightarrow$ rigid Weyl invariance in curved space

$$x^\mu \to e^\sigma x^\mu \qquad g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$$

$$\varphi \to e^{-\sigma}\varphi \qquad \varphi \to e^{-\sigma}\varphi$$

$$\tilde{S}_{UV} = \tilde{S}_{UV}^{CFT} + \frac{1}{2}\int d^4x\,\sqrt{-g}\,m^2\,\varphi^2$$

[curved space classical theory]

3. Weyl invariance recovered by adding a dilaton $\tau$

$$\tau \to \tau + \sigma$$

$$\tilde{S}_{UV} = \tilde{S}_{UV}^{CFT} + \frac{1}{2}\int d^4x\,\sqrt{-g}\,m^2\,e^{-2\tau}\varphi^2$$

[curved space classical theory]

we require

$\tau$ very weakly coupled, not to modify the RGE flow

$\tau$ massless such that it survives till the IR

# low-energy EFT

at the classical level the IR theory is Weyl-invariant

$$\tilde{S}_{IR} = \tilde{S}_{IR}^{CFT} + \Delta\,\tilde{S}_{IR}(g_{\mu\nu},\tau)$$

$$\hat{g}_{\mu\nu} \equiv e^{-2\tau} g_{\mu\nu}$$

$$\Delta\,\tilde{S}_{IR}(g_{\mu\nu},\tau) = f^2 \int d^4x\,\sqrt{-\hat{g}}\,\frac{\hat{R}}{6} + \alpha \int d^4x\,\sqrt{-\hat{g}}\hat{R}^2 + \cdots$$

in flat space

$$\Delta\,\tilde{S}_{IR}(g_{\mu\nu},\tau) \to \Delta S_{IR}(\tau) = f^2 \int d^4x\, e^{-2\tau}(\partial\tau)^2 + \cdots$$

dilaton decay constant

terms vanishing along EOM

$$\partial^2\tau - (\partial\tau)^2 = 0$$

$$\tau \to \frac{\tau}{f}$$

$$\Delta S_{IR}(\tau) = \int d^4x\, e^{-2\frac{\tau}{f}}(\partial\tau)^2 + \cdots$$

$$\tilde{S}_{UV} = \tilde{S}_{UV}^{CFT} + \frac{1}{2}\int d^4x \,\sqrt{-g}\, m^2\, e^{-2\tau}\varphi^2$$

[curved space classical theory]

$$\tilde{S}_{IR} = \tilde{S}_{IR}^{CFT} + \Delta\,\tilde{S}_{IR}(g_{\mu\nu},\tau)$$

both are classically Weyl-invariant

by including quantum corrections, the Weyl transformation has an anomaly

$$\delta_\sigma \tilde{S} = \int d^4x \sqrt{-g}\,\sigma(c\,W^2 - a\,E\,)$$

we require that the overall anomaly is the same in UV and IR

already seen for gauge theories

absence of anomalies in UV $\Rightarrow$ absence of anomalies in IR

also true in rigid symmetries, e.g. $\pi^0 \to \gamma\gamma$ decay determined by matching UV and IR anomalies of $U(1)_{3A}$ subgroup of chiral symmetry anomaly matching requires a Wess- Zumino term

$$\tilde{S}_{IR} = \tilde{S}_{IR}^{CFT} + \Delta\,\tilde{S}_{IR}(g_{\mu\nu},\tau) + \tilde{S}_{WZ}$$

$$\delta_\sigma \tilde{S}_{WZ} = \delta_\sigma \tilde{S}_{UV}^{CFT} - \delta_\sigma \tilde{S}_{IR}^{CFT}$$

$$= \int d^4x\,\sqrt{-g}\,\sigma[(c_{UV} - c_{IR})\,W^2 - (a_{UV} - a_{IR})\,E]$$

solution:

A. Schwimmer (Weizmann Inst.), S. Theisen (Potsdam, Max Planck Inst.) (Nov, 2010)

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$$\tilde{S}_{WZ} = (c_{UV} - c_{IR}) \int d^4x\,\sqrt{-g}\,\tau\,W^2$$

$$-(a_{UV} - a_{IR}) \int d^4x\,\sqrt{-g}\,\left[\tau\,E + 4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau\right]$$

$$-(a_{UV} - a_{IR}) \int d^4x\,\sqrt{-g}\,[-4(\partial\tau)^2\partial^2\tau + 2(\partial\tau)^4]$$

in the flat limit:

$$\tilde{S}_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \, (\partial\tau)^4 + \cdots$$

causality,
unitarity,
crossing

$\Rightarrow$

$$a_{UV} > a_{IR}$$

RG flow is irreversible in d=4

in a free theory

$$c = n_s + 6n_f + 12n_v, \qquad a = \frac{1}{3}\left(n_s + 11n_f + 62n_v\right).$$

a counts the DOF of the theory. As we move down in energy, more znd more DOF are removed.


thanks to Yu Nakayama, Francesco Riva, for very helpful clarifications

additional Weyl-invariant terms can be added to the theory

$$\Delta\,\tilde{S}_{UV}(g_{\mu\nu},\tau) = f^2 \int d^4x\,\sqrt{-\hat{g}}\,\frac{\hat{R}}{6} + \alpha \int d^4x\,\sqrt{-\hat{g}}\hat{R}^2 + \cdots$$

$$\hat{g}_{\mu\nu} \equiv e^{-2\tau} g_{\mu\nu}$$

[curved space classical theory]

in flat space

$$\Delta\,\tilde{S}_{UV}(g_{\mu\nu},\tau) \to \Delta S_{UV}(\tau) = f^2 \int d^4x\, e^{-2\tau}(\partial\tau)^2 + \cdots$$

dilaton decay constant

terms vanishing along EOM

$$\partial^2\tau - (\partial\tau)^2 = 0$$

$$\tau \to \frac{\tau}{f}$$

$$\Delta S_{UV}(\tau) = \int d^4x\, e^{-2\frac{\tau}{f}}(\partial\tau)^2 + \cdots$$

very large $f$ $\to$ very weak coupling