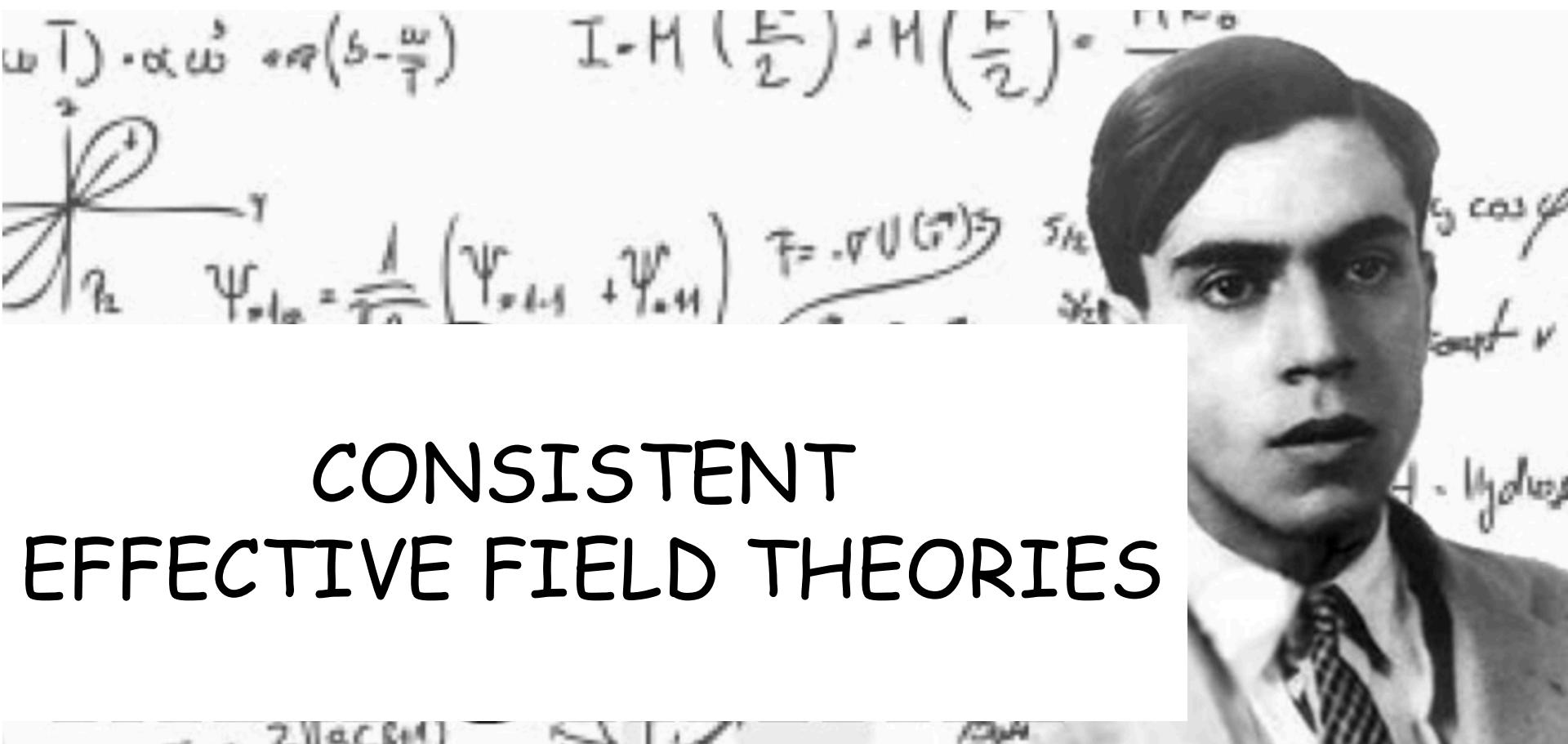


XI series of Majorana Lectures



CONSISTENT EFFECTIVE FIELD THEORIES

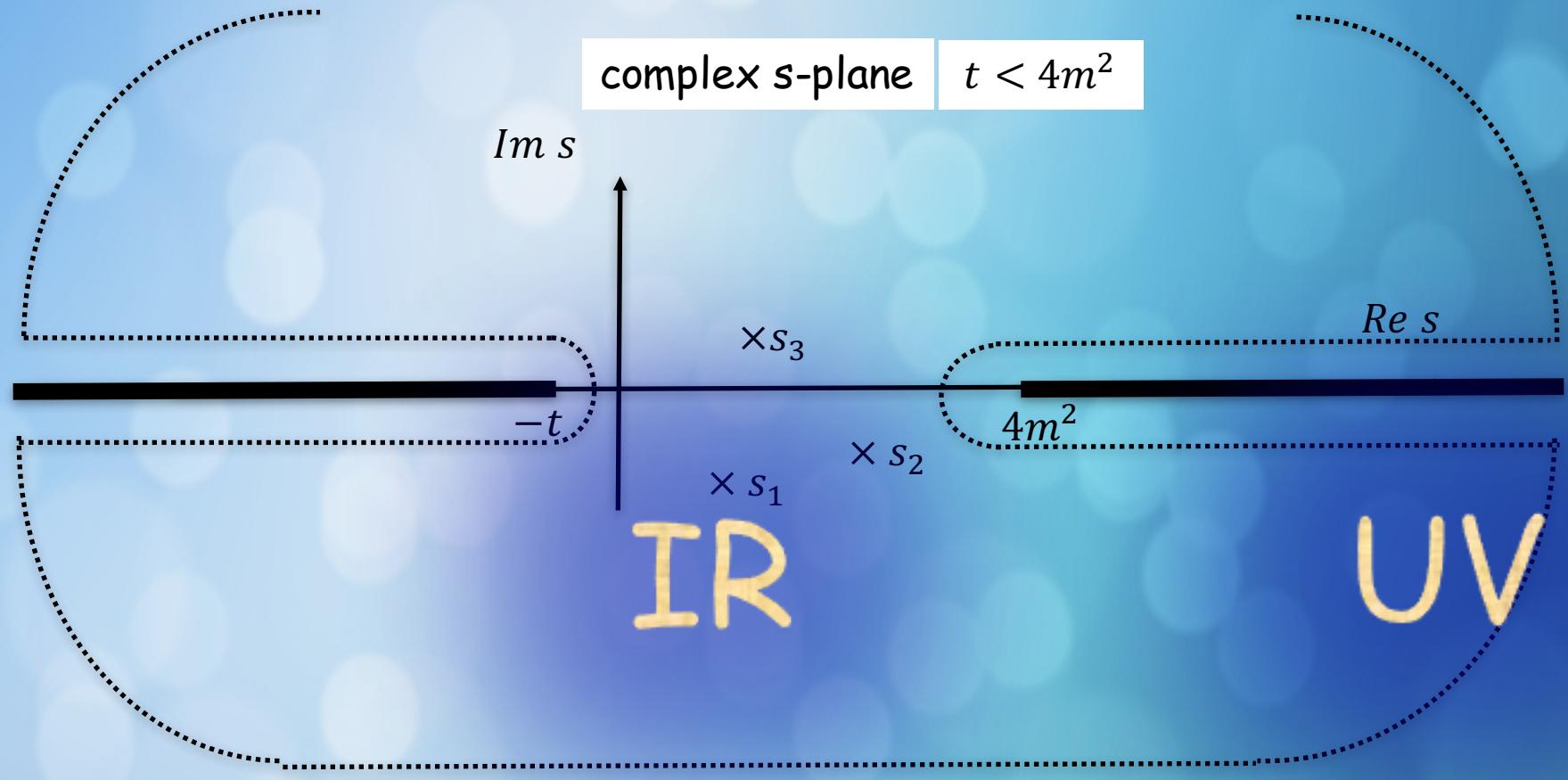
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22, 23, 24 Febbraio 2022

Part II

positivity bounds

dispersion relations



$$\frac{1}{2\pi i} \frac{T(s, t)}{(s - s_1)(s - s_2)(s - s_3)}$$

has 3 simple poles inside the contour

Cauchy formula

analyticity

$$\oint ds \frac{1}{2\pi i} \frac{T(s,t)}{(s-s_1)(s-s_2)(s-s_3)} = \frac{T(s_1,t)}{(s_2-s_1)(s_3-s_1)} + \frac{T(s_2,t)}{(s_1-s_2)(s_3-s_2)} + \frac{T(s_3,t)}{(s_1-s_3)(s_2-s_3)}$$

explicit integration

contribution at infinity vanishes

Froissart bound

$$\begin{aligned} \oint ds \frac{1}{2\pi i} \frac{T(s,t)}{(s-s_1)(s-s_2)(s-s_3)} &= \int_{4m^2}^{+\infty} ds \frac{1}{2\pi i} \frac{T(s+i\epsilon,t) - T(s-i\epsilon,t)}{(s-s_1)(s-s_2)(s-s_3)} \\ &\quad + \int_{-\infty}^{-t} ds \frac{1}{2\pi i} \frac{T(s + i\epsilon, t) - T(s - i\epsilon, t)}{(s - s_1)(s - s_2)(s - s_3)} \end{aligned}$$

change of variables in 2nd integral

$$s \rightarrow 4m^2 - t - s$$

crossing symmetry

$$T(s - i\epsilon, t) = T(4m^2 - t - (s + i\epsilon), t)$$

$$\int_{-\infty}^{-t} ds \frac{1}{2\pi i} \frac{T(s + i\epsilon, t) - T(s - i\epsilon, t)}{(s - s_1)(s - s_2)(s - s_3)} = \int_{4m^2}^{+\infty} ds \frac{1}{2\pi i} \frac{T(s + i\epsilon, t) - T(s - i\epsilon, t)}{(s - u_1)(s - u_2)(s - u_3)}$$

$$u_i \equiv 4m^2 - t - s_i$$

define

$$Disc T(s, t) \equiv \frac{T(s + i\epsilon, t) - T(s - i\epsilon, t)}{2i}$$

we get the dispersion relation

$$\frac{T(s_1, t)}{(s_2 - s_1)(s_3 - s_1)} + \frac{T(s_2, t)}{(s_1 - s_2)(s_3 - s_2)} + \frac{T(s_3, t)}{(s_1 - s_3)(s_2 - s_3)} =$$
$$\int_{4m^2}^{+\infty} \frac{ds'}{\pi} Disc T(s', t) \left[\frac{1}{(s' - s_1)(s' - s_2)(s' - s_3)} + \frac{1}{(s' - u_1)(s' - u_2)(s' - u_3)} \right]$$

IR = UV

application

choose $t = 0$ $\text{Disc } T(s, 0) = \text{Im } T(s, 0)$

take the limit $S_{1,2,3} \rightarrow S$

go to the symmetric point $s = 2m^2$ here $u = s$

$$\left. \frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \right|_{s=2m^2} = 2 \int_{4m^2}^{+\infty} \frac{ds'}{\pi} \text{Im}T(s', 0) \frac{1}{(s' - 2m^2)^3}$$

positivity bound

$$= 4 \int_{4m^2}^{+\infty} \frac{ds'}{\pi} \sqrt{s'(s' - 4m^2)} \sigma(s') \frac{1}{(s' - 2m^2)^3} > 0$$

for theories where m is very small or zero

$$\left. \frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \right|_{s=0} = 4 \int_0^{+\infty} \frac{ds'}{\pi} \sigma(s') \frac{1}{(s')^2} > 0$$

constraints on EFT

Back to our counterexample

$$\mathcal{L}_{IR} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + g (\partial^\mu \varphi \partial_\mu \varphi)^2 + \dots$$

symmetries: Lorentz and $\varphi(x) \rightarrow \varphi(x) + c$

field content and symmetries allow for (real) arbitrary g

EFT validity extends up to $E \approx 1/\sqrt[4]{|g|}$

add a small mass term $-\frac{1}{2}m^2\varphi^2$ to the theory
it breaks the shift symmetry. Eventually $m \rightarrow 0$

$$m \ll 1/\sqrt[4]{|g|}$$

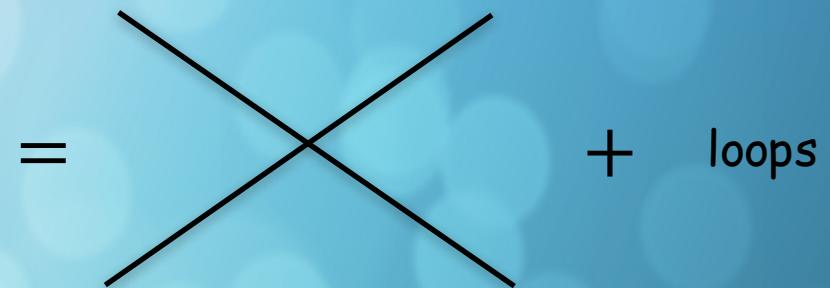
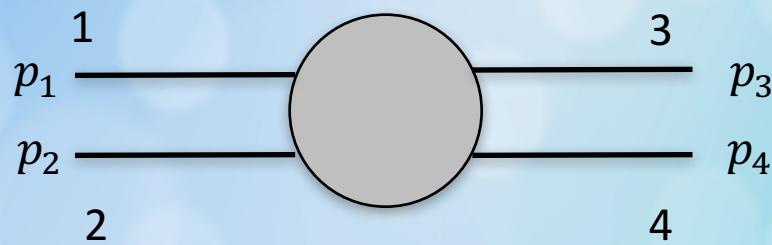
apply the dispersion relation:

$$\left. \frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \right|_{s=0} = 4 \int_0^{+\infty} \frac{ds'}{\pi} \sigma(s') \frac{1}{(s')^2} > 0$$

$$\frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \Big|_{s=2m^2}$$

can be evaluated within IR theory

exercise: compute $T(s, t)$ at the tree level from \mathcal{L}_{IR}



$$= 2 i g (s^2 + t^2 + u^2) + \dots$$

$$t = 0$$

$$u = -s$$

$$T(s, 0) = 4 g s^2 + \dots$$

plug this in dispersion relation
and take the limit $m \rightarrow 0$

$$g + \dots = \int_0^{+\infty} \frac{ds'}{\pi} \frac{\sigma(s')}{s'^2} > 0$$

$$\frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \Big|_{s=0}$$

IR

$$\int_0^{+\infty} \frac{ds'}{\pi} \frac{\sigma(s')}{s'^2} > 0$$

UV

0 $2m^2$ $4m^2$

IR = UV

$$\mathcal{L}_{IR} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + g (\partial^\mu \varphi \partial_\mu \varphi)^2 + \dots$$

$$g + [loops] > 0$$

unless

$g + [loops]$ cannot be computed in IR theory. Inequality right but useless. e.g. strong-coupling regime and PT not applicable.
["ineffective" EFT where tree approximation is usually assumed to be a good one]

theory cannot be embedded into a UV complete but causality and/or unitarity and/or crossing are violated

exercise

show that \mathcal{L}_{IR} can be derived from

$$\mathcal{L}_{UV} = \partial^\mu \chi^+ \partial_\mu \chi - V(\chi^+ \chi)$$

$$V(\chi^+ \chi) = \lambda \left(\chi^+ \chi - \frac{v^2}{2} \right)^2 \quad \lambda > 0$$

Symmetries:

Lorentz and global U(1)

$$\chi(x) \rightarrow e^{-i\alpha} \chi(x)$$

$$\chi(x) = \frac{(\sigma(x) + v)}{\sqrt{2}} e^{-i\varphi(x)}$$

$$\mathcal{L}_{UV} = \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma + \frac{1}{2} \left(1 + \frac{\sigma}{v} \right)^2 \partial^\mu \varphi \partial_\mu \varphi - \frac{\lambda}{4} (\sigma^2 + 2 v \sigma)^2$$

absolute minimum of $V(\sigma)$ at $\sigma = 0$ $V(\sigma) = \frac{1}{2} (2\lambda v^2) \sigma^2 + \dots$

$$m_\sigma^2 = 2\lambda v^2$$

$$m_\varphi = 0$$

below $E \approx m_\sigma$ we can consider an EFT
to describe φ and its interactions

static equations of motion of σ (tree-level)

$$\sigma^2 + 2 v \sigma = \frac{2}{m_\sigma^2} \partial^\mu \varphi \partial_\mu \varphi$$

plugging this into \mathcal{L}_{UV}

$$\mathcal{L}_{IR} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \frac{1}{2m_\sigma^2 v^2} (\partial^\mu \varphi \partial_\mu \varphi)^2 + O(\partial \varphi)^6$$



$$g = \frac{1}{2m_\sigma^2 v^2} > 0$$

we got a bound on

$$\left. \frac{1}{2!} \frac{d^2 T(s, 0)}{ds^2} \right|_{s=0}$$

what can be said on

$$\left. \frac{1}{n!} \frac{d^n T(s, 0)}{ds^n} \right|_{s=0} ?$$

in this simple model we can compute $T(s, t)$ in the UV theory

for example, at the tree-level [exercise]

$$T(s, t) = -\frac{1}{v^2} \left(\frac{s^2}{s - m_\sigma^2} + \frac{t^2}{t - m_\sigma^2} + \frac{u^2}{u - m_\sigma^2} \right)$$

by expanding $T(s, 0)$ in inverse powers of m_σ^2

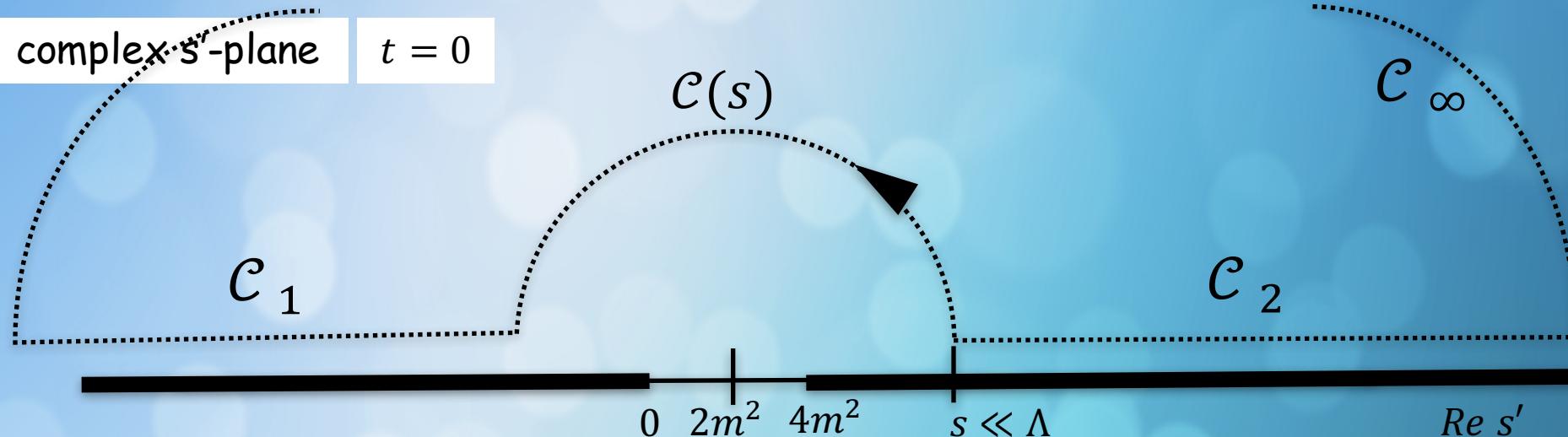
$$T(s, 0) = \frac{2m_\sigma^2}{v^2} \left(\frac{s^2}{m_\sigma^4} + \frac{s^4}{m_\sigma^8} + \frac{s^6}{m_\sigma^{12}} + \frac{s^8}{m_\sigma^{16}} + \dots \right)$$

we see that, in the TL approximation

$$\frac{1}{n!} \left. \frac{d^n T(s, 0)}{ds^n} \right|_{s=0} > 0 \quad [n \text{ even}]$$

good approximation if $\frac{2m_\sigma^2}{v^2} = 4\lambda \ll 1$

the moment problem



define

$$a_n(s) = \frac{1}{\pi i} \int_{\mathcal{C}(s)} ds' \frac{T(s', 0)}{(s' - 2m^2)^{2n+3}} \quad n = 0, 1, 2, \dots$$

exploit

$$\int_{\mathcal{C}(s)} + \int_{\mathcal{C}_1} + \int_{\mathcal{C}_\infty} + \int_{\mathcal{C}_2} = 0 \quad \rightarrow \quad \int_{\mathcal{C}(s)} = \int_{-\mathcal{C}_1} + \int_{-\mathcal{C}_2}$$

we find [exercise]

$$a_n(s) = \frac{2}{\pi} \int_s^{+\infty} ds' \frac{\text{Im } T(s', 0)}{(s' - 2m^2)^{2n+3}} > 0 \quad n = 0, 1, 2, \dots$$

relation to Wilson coefficients

$$T(s, t) = T(t, s) = T(u, t)$$

depends on the crossing-symmetric variables

$$\begin{aligned} s + t + u &= 4m^2 \\ st + tu + us &\leftrightarrow s^2 + t^2 + u^2 \\ stu \end{aligned}$$

$$T(s, 0) = f((s - 2m^2)^2)$$

under crossing

$$s - 2m^2 \rightarrow 2m^2 - s$$

at low energies, $s \ll \Lambda$, we expand $T(s, 0)$:

$$T(s, 0) = c_0 + c_2(s - 2m^2)^2 + c_4(s - 2m^2)^4 + \dots$$

for instance, in our previous example ($m = 0$), at the tree-level:

$$c_0 = 0$$

$$c_{2n+2} = \frac{2m_\sigma^2}{v^2} \frac{1}{(m_\sigma^2)^{2n+2}}$$

plugging

$$T(s, 0) = c_0 + c_2(s - 2m^2)^2 + c_4(s - 2m^2)^4 + \dots$$

into the definition

$$a_n(s) = \frac{1}{\pi i} \int_{\mathcal{C}(s)} ds' \frac{T(s', 0)}{(s' - 2m^2)^{2n+3}} \quad n = 0, 1, 2, \dots$$

we get [exercise]

$$c_{2n+2} = a_n(s) > 0 \quad (n = 0, 1, 2, \dots)$$

[s -dependence trivial at the tree-level]

plugging

$$T(s, 0) = c_0 + c_2(s - 2m^2)^2 + c_4(s - 2m^2)^4 + \dots$$

into the definition

$$a_n(s) = \frac{1}{\pi i} \int_{\mathcal{C}(s)} ds' \frac{T(s', 0)}{(s' - 2m^2)^{2n+3}} \quad n = 0, 1, 2, \dots$$

we get [exercise]

$$c_{2n+2} = a_n(s) > 0 \quad (n = 0, 1, 2, \dots)$$

[s -dependence trivial at the tree-level]

however, there is more than that

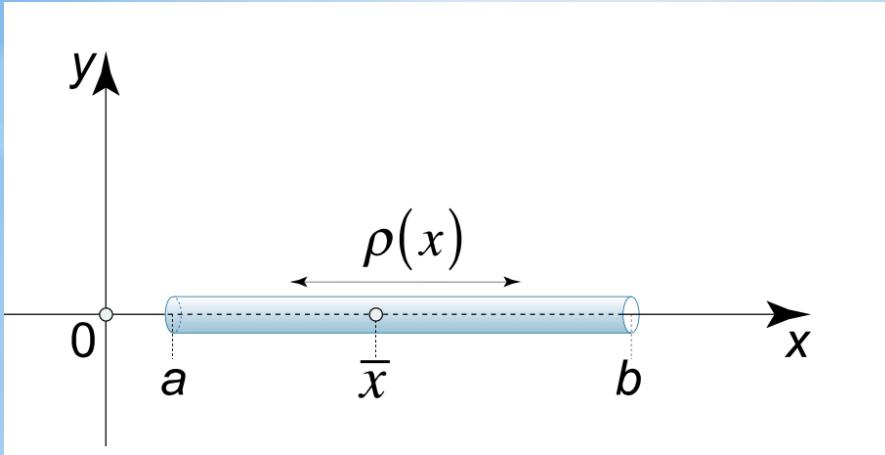
change of variable

$$\left(\frac{s - 2m^2}{s' - 2m^2} \right)^2 = x \quad \text{in} \quad a_n(s) = \frac{2}{\pi} \int_s^{+\infty} ds' \frac{\text{Im } T(s', 0)}{(s' - 2m^2)^{2n+3}}$$

$$b_n \equiv (s - 2m^2)^{2n+2} a_n(s) = \int_0^1 x^n \underbrace{\frac{1}{\pi} \text{Im } \tilde{T}(s, x)}_{>0} dx > 0$$

sequence of (dimensionless) moments

mass density distribution



$$m_n = \int_0^1 x^n \rho(x) dx > 0$$

m_0 mass

m_1 (center of mass) \times mass

m_2 moment of inertia

multiple expansion of electrostatic potential $V(r)$ due to a charge distribution $\rho(r')$ at $r \gg R$

moment problem: given $\{m_n\}$ how to establish whether it is a sequence of moments and can we reconstruct the density $\rho(x)$ from $\{m_n\}$?

take an **arbitrary positive** polynomial in $[0,1]$

$$P_N(x) = \sum_{k=1}^N \alpha_k x^k > 0$$

$$\sum_{k=1}^N \alpha_k b_n = \int_0^1 P_N(x) \rho(x) dx > 0$$

$$\rho(x) = \frac{1}{\pi} \operatorname{Im} \tilde{T}(s, x) > 0$$

we have an infinite set of constraints related to
the infinite choices of $P_N(x)$

a basis of positive polynomial in $[0,1]$ is

$$P_{n+k}(x) = x^n (1-x)^k$$

Bernstein polynomials

we have

$$= \int_0^1 x^n (1-x)^k \frac{1}{\pi} \operatorname{Im} \tilde{T}(s, x) dx > 0$$

$$= (-1)^k (\Delta^k b)_n > 0 \quad (n, k = 0, 1, 2, \dots)$$

$$(\Delta^0 b)_n \equiv b_n \quad (\Delta^1 b)_n \equiv b_{n+1} - b_n \quad (\Delta^k b)_n \equiv \Delta^1(\Delta^{k-1} b)_n$$

$$b_n > 0$$

$$b_n - b_{n+1} > 0$$

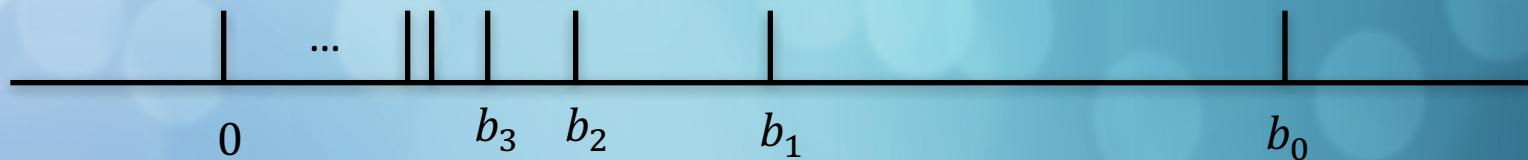
$$b_n - b_{n+1} - (b_{n+1} - b_{n+2}) > 0$$

completely
monotonic
sequence

$$b_n - b_{n+1} - 2(b_{n+1} - b_{n+2}) + b_{n+2} - b_{n+3} > 0$$

$$b_n - b_{n+1} - 3(b_{n+1} - b_{n+2}) + 3(b_{n+2} - b_{n+3}) - (b_{n+3} - b_{n+4}) > 0$$

for example $\{b_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$



the converse is also true (Hausdorff theorem): a sequence $\{b_n\}$ obeying

$$(-1)^k (\Delta^k b)_n \geq 0 \quad \text{for all non-negative } n \text{ and } k$$

is a sequence of moments corresponding to a unique measure $\frac{1}{\pi} \operatorname{Im} \tilde{T}(s, x) dx$

exercise

in our previous example

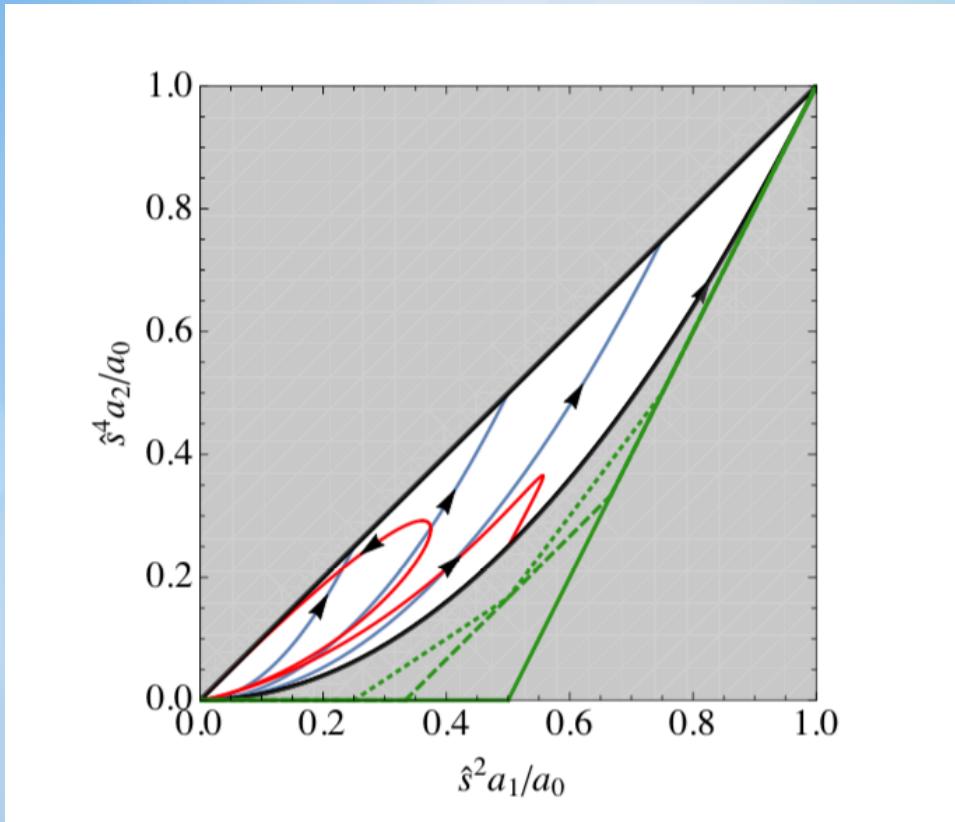
$$b_n = \frac{2m_\sigma^2}{v^2} \left(\frac{s}{m_\sigma^2} \right)^{2n+2}$$

■ find $\operatorname{Im} T(s', 0)$ and identify the measure $\frac{1}{\pi} \operatorname{Im} \tilde{T}(s, x) dx$

■ show that $(-1)^k (\Delta^k b)_n = \frac{2m_\sigma^2}{v^2} \left(\frac{s}{m_\sigma^2} \right)^{2n+2} \left(1 - \frac{s^2}{m_\sigma^4} \right)^k$

■ discuss the domain of validity of the condition $(-1)^k (\Delta^k b)_n \geq 0$

bounds on first three arcs. $b_n \equiv (s - 2m^2)^{2n+2} a_n(s)$



when we are interested in few $a_n(s)$ there are optimization tools and only a finite number of inequalities are sufficient

exercise

which operator have we bounded?

$$\mathcal{L}_{UV} = \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma + \frac{1}{2} \left(1 + \frac{\sigma}{v}\right)^2 \partial^\mu \varphi \partial_\mu \varphi - \frac{\lambda}{4} (\sigma^2 + 2 v \sigma)^2$$

show that, integrating out the heavy mode σ (not in the static approximation) we get

$$\begin{aligned} \mathcal{L}_{IR} &= \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \\ &\quad \frac{1}{2v^2 m_\sigma^2} \partial^\mu \varphi \partial_\mu \varphi \left(\gamma_2 + \gamma_3 \frac{\partial^\mu \partial_\mu}{m_\sigma^2} + \gamma_4 \left(\frac{\partial^\mu \partial_\mu}{m_\sigma^2} \right)^2 + \dots \right) \partial^\mu \varphi \partial_\mu \varphi \\ &\quad + \text{higher orders in } \partial^\mu \varphi \partial_\mu \varphi \end{aligned}$$

determine the coefficients γ_k

write the relation between γ_k and c_k

do the moments $\{b_n\}$ bound the coefficient γ_3 ?

contribution from γ_3 grows like s^3

crossing symmetry requires $s^3 + t^3 + u^3 = -3(s^2t + st^2)$

we need a dispersion relation at $t \neq 0$

forward scattering cannot bound all possible operators

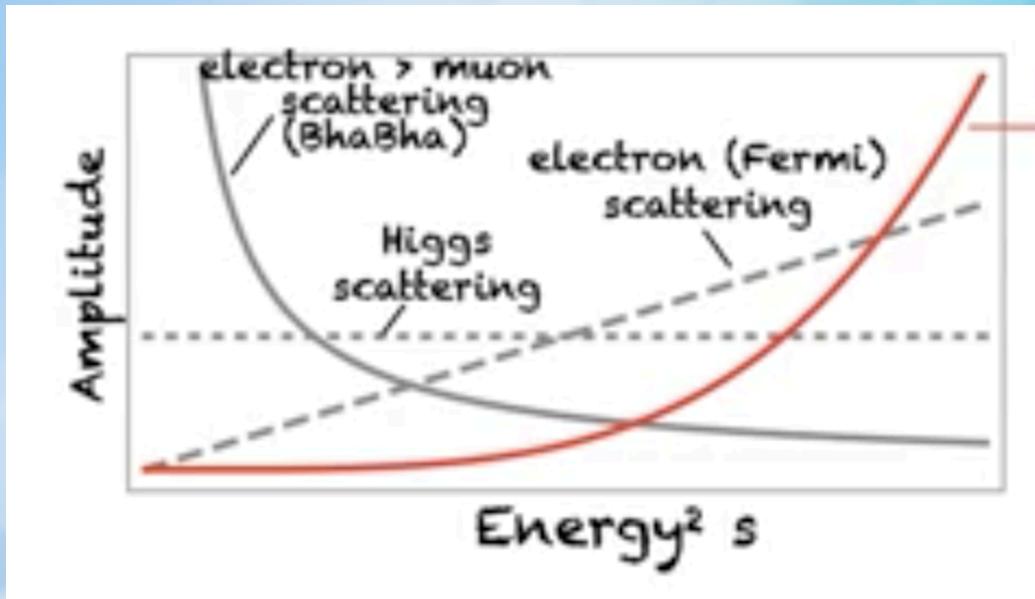
super soft behavior

$$T(s, 0) = c_0 + c_2 s^2 + c_4 s^4 + c_6 s^6 + \dots$$

here $m = 0$

super softness: $T(s, 0)$ dominated by s^n , $n > 2$

can this be true at least in some energy range?



$$c_{2n+2} = a_n(s)$$

$$c_2 s^2 > c_4 s^4$$

super soft behavior always subdominant

recent directions

beyond forward scattering: $t \neq 0$

$$\begin{aligned}\mathcal{M}_{\text{low}}(s, t) = & -g^2 \left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right] - \lambda \\ & + g_2(s^2 + t^2 + u^2) + g_3(stu) + g_4(s^2 + t^2 + u^2)^2 + g_5(s^2 + t^2 + u^2)(stu) \\ & + g_6(s^2 + t^2 + u^2)^3 + g'_6(stu)^2 + g_7(s^2 + t^2 + u^2)^2(stu) + \dots\end{aligned}$$

$$\tilde{g}_3 = g_3 \frac{M^2}{g_2}, \quad \tilde{g}_4 = g_4 \frac{M^4}{g_2}, \quad \tilde{g}_5 = g_5 \frac{M^6}{g_2},$$

$$0 \leq \frac{g_2}{(4\pi)^2} \leq \frac{0.794}{M^4}$$

not only lower bounds

$$-10.346 \leq \tilde{g}_3 \leq 3, \quad 0 \leq \tilde{g}_4 \leq \frac{1}{2}, \quad -4.0960 \leq \tilde{g}_5 \leq \frac{5}{2}.$$

not only even
coefficients

Extremal Effective Field Theories

Simon Caron-Huot (McGill U.), Vincent Van Duong (McGill U.) (Nov 5, 2020)

Published in: *JHEP* 05 (2021) 280 • e-Print: [2011.02957 \[hep-th\]](#)

New positivity bounds from full crossing symmetry

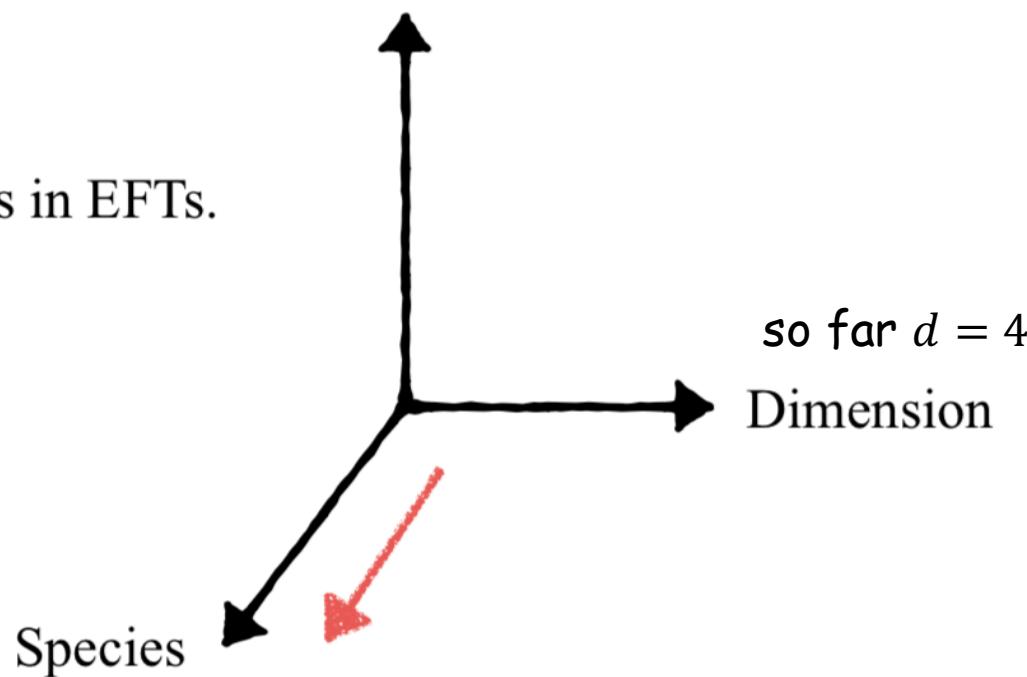
Andrew J. Tolley (Imperial Coll., London and Case Western Reserve U., CERCA), Zi-Yue Wang (USTC, Hefei), Shuang-Yong Zhou (USTC, Hefei and Hefei, CUST and PCFT, Hefei) (Nov 4, 2020)

Published in: *JHEP* 05 (2021) 255 • e-Print: [2011.02400 \[hep-th\]](#)

recent directions

fields carrying spin
Angular Momentum

Space of interactions in EFTs.



more scalar fields,....

Euler-Heisenberg EFT

in classical theory, light propagation obey the superposition principle

W. Heisenberg, "Comments on the Dirac theory of the positron", Zeit. f. Phys. **90**, 209 (1934)

"Halpern and Debye have already independently drawn attention to the fact that the Dirac theory of the positron leads to the scattering of light by light, even when the energy of the photons is not sufficient to create pairs."

On the scattering of light by light in Dirac's theory ¹⁾

By Hans Euler

Translated by D. H. Delphenich

effective Lagrangian:

$$L = \frac{\mathfrak{E}^2 - \mathfrak{B}^2}{2} + \frac{1}{90\pi} \frac{\hbar c}{e^2} \frac{1}{E_0^2} [(\mathfrak{E}^2 - \mathfrak{B}^2)^2 + 7(\mathfrak{E}\mathfrak{B})^2]$$



Folgerungen aus der Diracschen Theorie des Positrons.

Von W. Heisenberg und H. Euler in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

effects

tiny cross-section, yet unobserved

$$\sigma = \frac{973}{10125\pi} \frac{\alpha^4}{m_e^2} \left\{ \frac{\omega}{m_e} \right\}^6, \quad \omega \ll m_e$$

$$\sigma = 7 \times 10^{-70} m^2 = 7 \times 10^{-42} b. \quad \text{for visible light } \omega \approx 1 \text{ eV}$$

birefringence of the vacuum: light propagation velocity depend on both direction and polarization. Not yet detected
[common in some materials, e.g. calcite]

.... even when the energy of the photons is not sufficient to create pairs.



low-energy effective Lagrangian for electromagnetic field $A_\mu(x)$

$\text{U}(1)$ gauge invariance $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x)$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad F_{\mu\nu}(x) \rightarrow F_{\mu\nu}(x)$$

CP invariance $A_\mu(x) \rightarrow -A^\mu(x_P)$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

dim = 6 $F_{\mu\rho} F^{\rho\sigma} F_\sigma^\mu \quad \partial_\rho F^{\rho\sigma} \partial_\mu F_\sigma^\mu \quad F_{\mu\nu} \partial^\sigma \partial_\sigma F^{\mu\nu}$ candidates

exercise why we can neglect these?

dim = 8

$$(F_{\mu\nu} F^{\mu\nu})^2$$

$$(F_{\mu\nu} \tilde{F}^{\mu\nu})^2$$

$$F_{\mu\rho} F^{\rho\nu} F^{\mu\sigma} F_{\sigma\nu}$$

exercise

show that $4F_{\mu\rho} F^{\rho\nu} F^{\mu\sigma} F_{\sigma\nu} = (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + 2(F_{\mu\nu} F^{\mu\nu})^2$

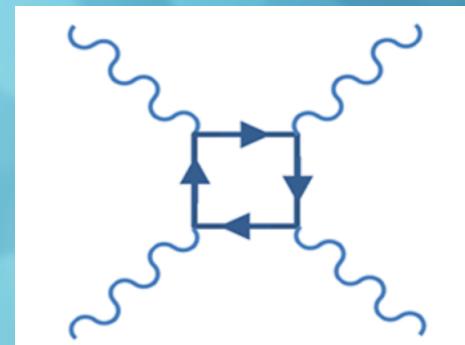
$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\varrho\sigma} F^{\varrho\sigma}$$

$$\epsilon^{0123} = -\epsilon_{0123} = +1$$

$\mathcal{L}_{EH} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c_1}{16} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{c_2}{16} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \dots$

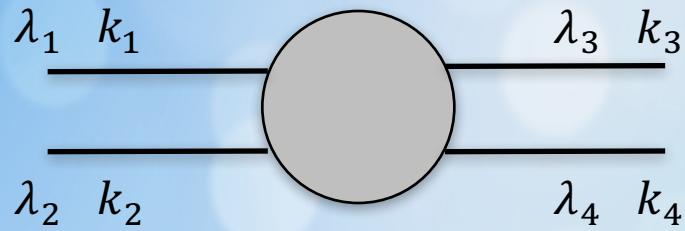
we expect

$$c_{1,2} \approx \frac{\alpha^2}{m_e^4}$$



$$\sigma = \frac{973}{10125\pi} \frac{\alpha^4}{m_e^2} \left\{ \frac{\omega}{m_e} \right\}^6, \quad \omega \ll m_e$$

light by light scattering



$$= T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (s, t, u)$$

16 amplitudes

λ_i photon helicities

symmetries

P $T_{-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}(s, t, u) = T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$

T $T_{\lambda_3 \lambda_4 \lambda_1 \lambda_2}(s, t, u) = T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$

B
o
s
e

B $T_{\lambda_2 \lambda_1 \lambda_4 \lambda_3}(s, t, u) = T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$

crossing

$$\begin{array}{l} s \leftrightarrow t \\ \lambda_2 \leftrightarrow -\lambda_3 \end{array}$$

$$T_{\lambda_1 - \lambda_3 - \lambda_2 \lambda_4}(t, s, u) = T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$$

$$\begin{array}{l} s \leftrightarrow u \\ \lambda_2 \leftrightarrow -\lambda_4 \end{array}$$

$$T_{\lambda_1 - \lambda_4 \lambda_3 - \lambda_2}(u, t, s) = T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$$

exercise



five independent S-matrix elements, out of 16

$$T_{++++}(s, t, u) = T_{+-+-+}(t, s, u) = T_{+-+-}(u, t, s)$$

$$T_{+-+-}(s, t, u) = \dots$$

$$T_{+-+}(s, t, u) = \dots$$

$$T_{++--}(s, t, u) \text{ fully } (s, t, u) \text{ symmetric}$$

$$T_{+++-}(s, t, u) \text{ fully } (s, t, u) \text{ symmetric}$$

forward scattering $t = 0$

exercise

$$T_{+--+}(s, 0, -s) = T_{+++-}(s, 0, -s) = 0$$

angular momentum
conservation

we are left with

$$T_{++++}(s) = T_{+-+-}(-s)$$

$$T_{XYZW}(s, 0, -s) \equiv T_{XYZW}(s)$$

$$T_{+---}(s) = T_{+---}(-s)$$

two independent functions, one even in s

not directly usable in the dispersion relation

($s \leftrightarrow u$) crossing relates amplitudes

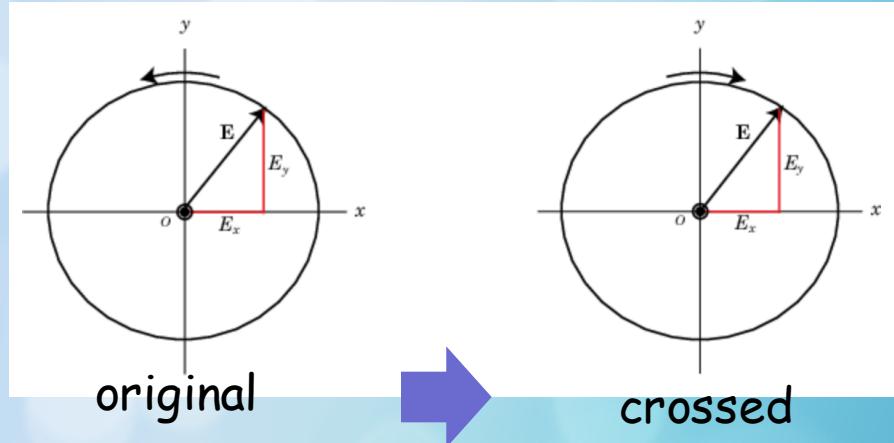
- with different polarization or
- where initial and final states have different polarization

forward **elastic** scattering needed:
same polarization of initial and final states

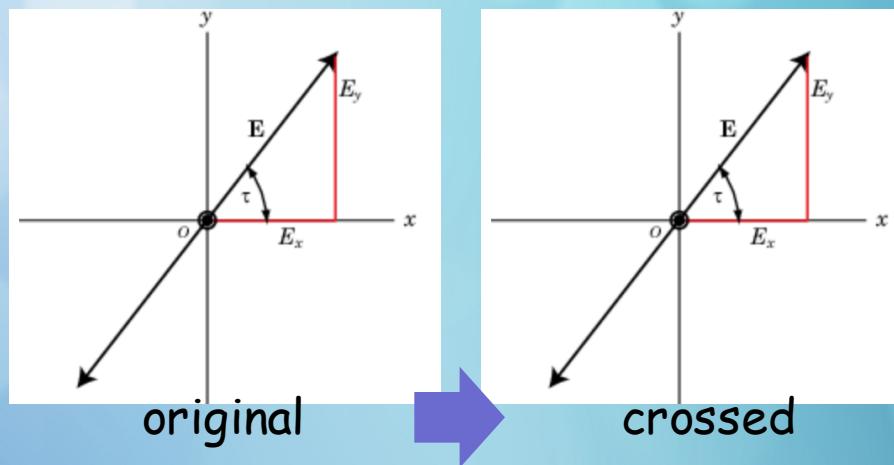


$$T_{ABAB}(s) = T_{ABAB}(-s)$$

crossed particle = original particle propagating backward in time:
momenta, helicities and conserved charges flip their signs



linear polarizations left unchanged



$$T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s) = \epsilon_{\lambda_1}^{\alpha_1}(k_1) \epsilon_{\lambda_2}^{\alpha_2}(k_2) \epsilon_{\lambda_3}^{\alpha_3*}(k_1) \epsilon_{\lambda_4}^{\alpha_4*}(k_2) \times \\ \times \mathcal{M}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(k_1, k_2)$$

from helicities to linear polarization

$$\epsilon_X^\alpha(k) = \frac{1}{\sqrt{2}} [\epsilon_-^\alpha(k) - \epsilon_+^\alpha(k)]$$

$$\epsilon_Y^\alpha(k) = \frac{i}{\sqrt{2}} [\epsilon_-^\alpha(k) + \epsilon_+^\alpha(k)]$$

two independent elastic channels

$$T_{XXXX}(s) = \frac{1}{4} [T_{++--}(s) + T_{++++}(s) + T_{+-+-}(s)]$$

$$T_{XYXY}(s) = \frac{1}{4} [T_{++--}(s) - T_{++++}(s) - T_{+-+-}(s)]$$

both satisfying $T_{AB}(s) = T_{AB}(-s)$

notation $T_{AB}(s) \equiv T_{ABAB}(s)$

$$T_{AB}(s) = \epsilon_A^{\alpha_1}(k_1)\epsilon_B^{\alpha_2}(k_2)\epsilon_A^{\alpha_3*}(k_1)\epsilon_B^{\alpha_4*}(k_2) \times \mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1, k_2)$$

$$\mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1, k_2) = A(s)g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4} +$$

$$B(s)g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3} +$$

$$B(-s)g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4} + [k_{1\alpha_1}k_{1\alpha_2}g_{\alpha_2\alpha_4} + \dots]$$

exercise: show that

$$\begin{aligned} s &\leftrightarrow -s \\ 2 &\leftrightarrow 4 \end{aligned}$$

$$T_{AB}(s) = A(s) + B(s)\delta_{AB} + B(-s)\delta_{AB}$$

use $\epsilon_X^\alpha(k_1) = (0, +1, 0, 0) \quad \epsilon_X^\alpha(k_2) = (0, -1, 0, 0)$
 $\epsilon_Y^\alpha(k_1) = (0, 0, +1, 0) \quad \epsilon_Y^\alpha(k_2) = (0, 0, +1, 0)$

dispersion relation

$$\left. \frac{1}{2!} \frac{d^2 T_{AB}(s, 0)}{ds^2} \right|_{s=0} = 4 \int_0^{+\infty} \frac{ds'}{\pi} \sigma_{AB}(s') \frac{1}{(s')^2} > 0$$

exercise: evaluate $A(s)$ and $B(s)$ from \mathcal{L}_{EH}

$$\mathcal{L}_{EH} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c_1}{16} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{c_2}{16} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \dots$$

hint: derive the contribution to the Feynman rule proportional to

$$a g_{\alpha_1 \alpha_3} g_{\alpha_2 \alpha_4} + b g_{\alpha_1 \alpha_4} g_{\alpha_2 \alpha_3} + c g_{\alpha_1 \alpha_2} g_{\alpha_3 \alpha_4}$$

a, b, c being linear combination of c_1 and c_2

$$A(s) = c_2 s^2$$



$$T_{XX}(s) = c_1 s^2$$

$$B(s) = \frac{1}{2} (c_1 - c_2) s^2$$

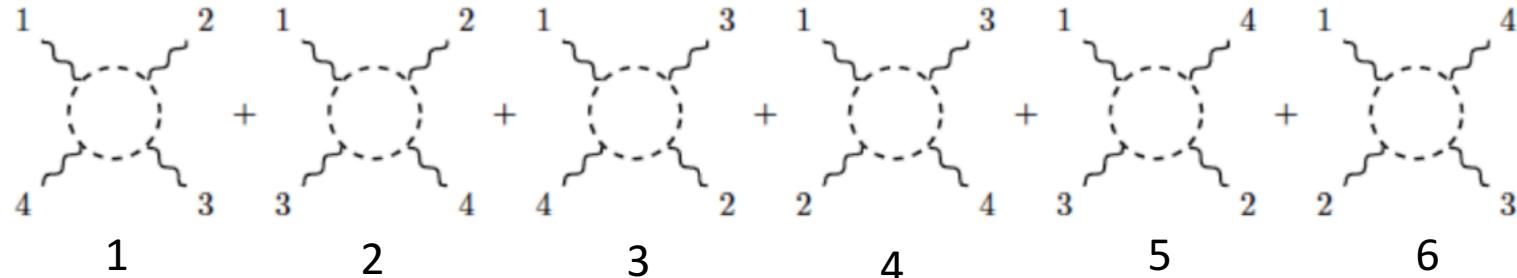
$$T_{XY}(s) = c_2 s^2$$

$$\left. \frac{1}{2!} \frac{d^2 T_{XX(XY)}(s, 0)}{ds^2} \right|_{s=0} = c_{1(2)} > 0$$

given an UV theory we can compute c_1 and c_2

QED [internal electron loop]

$$\mathcal{M}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(k_1, k_2) =$$



	$g_{\alpha_1 \alpha_3} g_{\alpha_2 \alpha_4}$	$(g_{\alpha_1 \alpha_4} g_{\alpha_2 \alpha_3} + g_{\alpha_1 \alpha_2} g_{\alpha_3 \alpha_4})$
(1,6)	+78	-62
(2,4)	+59	+45
(3,5)	+59	-25
Tot x 2	+392	-84

units
 $\frac{\alpha^2}{315 m_e^4} (k_1 \cdot k_2)^2$

low-energy expansion $s \ll m_e^2$

$$\frac{\alpha^2}{315 m_e^4} [+392 g_{\alpha_1 \alpha_3} g_{\alpha_2 \alpha_4} - 84(g_{\alpha_1 \alpha_4} g_{\alpha_2 \alpha_3} + g_{\alpha_1 \alpha_2} g_{\alpha_3 \alpha_4})] (k_1 \cdot k_2)^2 + \dots$$



2.1.1 by Hiren H. Patel
hpatel6@ucsc.edu

In[1]:= << X`

Package-X v2.1.1, by Hiren H. Patel
For more information, see the guide

In[2]:= rule1 = {k1.k1 → 0, k2.k2 → 0};

In[3]:= LScalarQ[x] = True; LScalarQ[y] = True;

In[4]:= (* Compute the 1st diagram - 1234 *)

In[5]:= traccia1 = Spur[γ_{a1}, γ.k + m₁, γ_{a2}, γ.k - y γ.k2 + m₁, γ_{a3}, γ.k + x γ.k1 - y γ.k2 + m₁, γ_{a4}, γ.k + x γ.k1 + m₁];

In[6]:= I1 = LoopIntegrate[traccia1, k, {k, m}, {k - y k2, m}, {k + x k1 - y k2, m}, {k + x k1, m}];

In[7]:= I1 = I1 /. rule1;

In[8]:= I1 = Coefficient[Expand[I1], g_{a1,a3}];

In[9]:= I1 = Coefficient[Expand[I1], g_{a2,a4}];

In[10]:= I1 = LoopRefineSeries[I1, {x, 0, 2}, {y, 0, 2}];

In[11]:= C1 = Coefficient[Expand[I1], x^2 y^2]

$$\frac{26 (k1.k2)^2}{105 m^4}$$

QED
Euler
Heisenberg

c_1

c_2

$$\frac{\alpha^2}{m_e^4} \frac{8}{45}$$

$$\frac{\alpha^2}{m_e^4} \frac{14}{45}$$

1-loop

scalar
QED

$$\frac{\alpha^2}{m_e^4} \frac{7}{90}$$

$$\frac{\alpha^2}{m_e^4} \frac{1}{90}$$

1-loop

axion

0

$$\frac{\alpha^2}{m_a^2 f_a^2} \frac{g_a^2}{2\pi^2}$$

tree-level

$$\mathcal{L}_a = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial^\mu a \partial_\mu a - \frac{1}{2} m_a^2 a^2 + g_a \frac{\alpha}{4\pi f} a F_{\mu\nu} \tilde{F}^{\mu\nu}$$

birefringence of vacuum

refractive index depends on both direction and polarization.



light propagation $\leftrightarrow A_\mu$ equations of motion

$$\mathcal{L}_{EH} = -\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 + \dots$$

$$\mathcal{F} \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

exercise: derive the EOM

$$\mathcal{G} \equiv \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

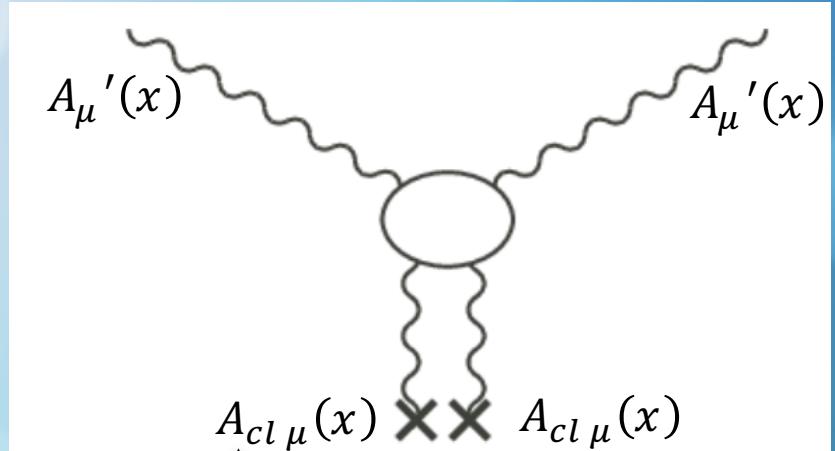
$$(1 - 2c_1 \mathcal{F}) \partial_\alpha F^{\alpha\beta} = 2c_1 (\partial_\alpha \mathcal{F}) F^{\alpha\beta} + 2c_2 (\partial_\alpha \mathcal{G}) \tilde{F}^{\alpha\beta} +$$

nonlinear EOM, reducing to Maxwell when $c_{1,2} = 0$

propagation of light through
a given E and/or B field

linearize the EOM by
decomposing

$$A_\mu(x) = A_{cl\ \mu}(x) + A_\mu'(x)$$



assume constant E and B fields

$$\partial_\alpha(F_{cl}^{\mu\nu}) = 0$$

exercise: derive the linearized EOM

$$\begin{aligned} & (1 - 2c_1 \mathcal{F}_{cl})(\partial^\lambda \partial_\lambda g^{\mu\nu} - \partial^\mu \partial^\nu) A'_\nu = \\ & = 2 \left(c_1 F_{cl}^{\alpha\mu} F_{cl}^{\beta\nu} + c_2 \tilde{F}_{cl}^{\alpha\mu} \tilde{F}_{cl}^{\beta\nu} \right) \partial_\alpha \partial_\beta A'_\nu + \dots \end{aligned}$$

go to Lorentz gauge $\partial^\nu A'_\nu = 0$ and to momentum space

$$b^\mu \equiv F_{cl}^{\alpha\mu} k_\alpha \quad \tilde{b}^\mu \equiv \tilde{F}_{cl}^{\alpha\mu} k_\alpha$$

$$(1 - 2c_1 \mathcal{F}_{cl}) k^2 A'^\mu = 2(c_1 b^\mu b^\nu + c_2 \tilde{b}^\mu \tilde{b}^\nu) A'_\nu$$

	$c_{1,2} = 0$	standard dispersion relation	$k^2 = 0$
--	---------------	------------------------------	-----------

$$k^\mu = (\omega, \vec{k}) \quad \omega = |k|$$



	$c_{1,2} \neq 0$ & $F_{cl}^{\alpha\mu}, \tilde{F}_{cl}^{\alpha\mu} \approx \varepsilon \ll 1$	$k^2 = O(\varepsilon^2 \omega^2)$
--	---	-----------------------------------

$$b^\mu, \tilde{b}^\mu \approx \varepsilon \omega \quad \mathcal{F}_{cl} k^2 \approx \varepsilon^4 \omega^2$$

$$(k^2 g^{\mu\nu} - 2c_1 b^\mu b^\nu - 2c_2 \tilde{b}^\mu \tilde{b}^\nu) A'_\nu = O(\varepsilon^4 \omega^2)$$

up to terms of $O(\varepsilon^4 \omega^2)$

1

$$b^\mu \tilde{b}_\mu = 0$$

2

$$b^\mu b_\mu = \tilde{b}^\mu \tilde{b}_\mu = -\omega^2 Q^2 \leq 0 \quad Q^2 \approx \varepsilon^2$$

$$b^\mu b_\mu = -\omega^2 |\vec{E}|^2 + (\vec{k} \cdot \vec{E})^2 - |\vec{k}|^2 |\vec{B}|^2 + (\vec{k} \cdot \vec{B})^2 + 2\omega \vec{k} \cdot (\vec{E} \cdot \vec{B})$$

$2\omega \vec{k} \cdot (\vec{E} \times \vec{B}) \leq 0$ maximized when $\vec{k}, \vec{E}, \vec{B}$ form an orthogonal set

$$(b^\mu b_\mu)_{MAX} \leq -(\omega |\vec{E}| - |\vec{k}| |\vec{B}|)^2 \leq 0$$



two linearly independent solutions

$$A'_\mu = \beta b_\mu + \tilde{\beta} \tilde{b}_\mu$$

dispersion relations

$$A'_\mu = b_\mu$$

$$k^2 + 2c_1 \omega^2 Q^2 = 0$$

$$A'_\mu = \tilde{b}_\mu$$

$$k^2 + 2c_2 \omega^2 Q^2 = 0$$

dashed lines depend on k^μ

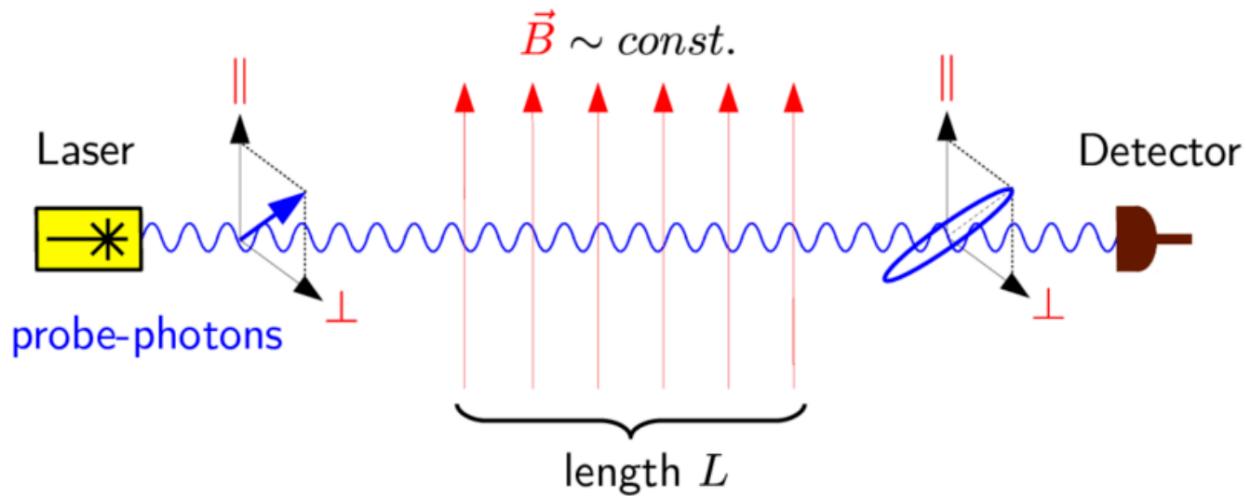
refractive index $k^\mu = (\omega, n \omega \hat{k})$

$$n_{1,2} \approx 1 + c_{1,2} Q^2$$

$$\frac{v_{1,2}}{c} \approx 1 - c_{1,2} Q^2$$

superluminality
unless $c_{1,2} \geq 0$

birefringence



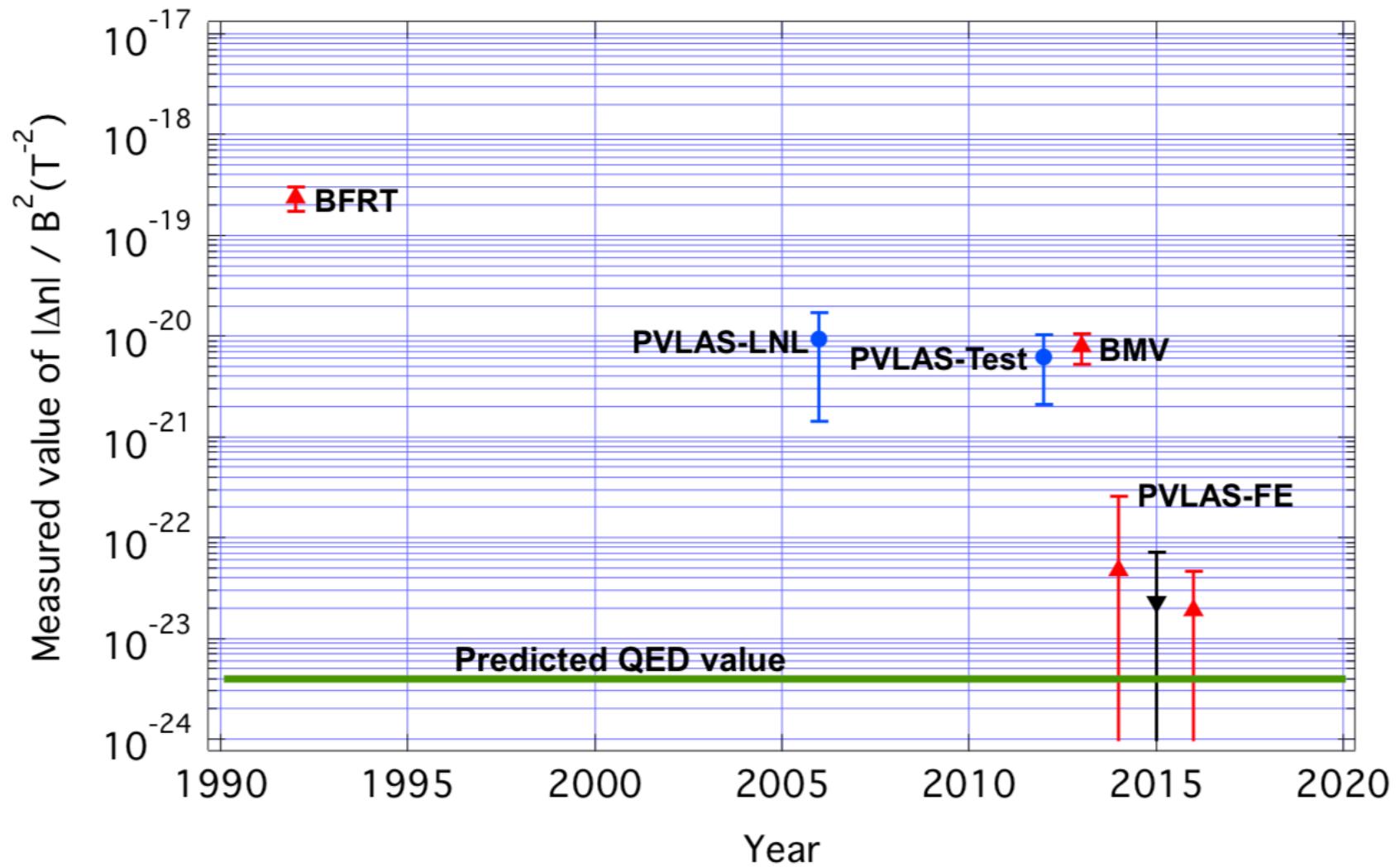
x and z components of em field propagate with different velocities and accumulate a phase difference

$$(c_2 - c_1)_{QED} \approx 4 \times 10^{-24} T^{-2}$$

$$\Delta\varphi = (k_2 - k_1)L = \frac{2\pi}{\lambda}(n_2 - n_1)L \approx \frac{2\pi}{\lambda}(c_2 - c_1)B^2 L$$

$$\Delta\varphi = 2.3 \times 10^{-16} \left(\frac{B}{3 T}\right)^2 \left(\frac{L}{1 m}\right) \left(\frac{1 \mu m}{\lambda}\right) \text{rad}$$

enhanced to $10^{-11} \text{ rad} \ll 10^{-7} \text{ rad}$ [present experimental sensitivity]



Backup Slides


$$T(s, 0) = -\frac{1}{v^2} \left(\frac{s^2}{s - m_\sigma^2 + i\epsilon} + \frac{s^2}{-s - m_\sigma^2 + i\epsilon} \right)$$

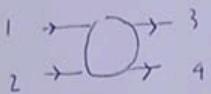

$$\text{Im } T(s', 0) = \frac{\pi {s'}^2}{v^2} [\delta(s' - m_\sigma^2) + \delta(-s' - m_\sigma^2)]$$


$$\frac{1}{\pi} \text{Im } \tilde{T}(s, x) dx = \frac{2m_\sigma^2}{v^2} x \delta\left(x - \frac{s^2}{m_\sigma^4}\right)$$


$$\frac{1}{2v^2 m_\sigma^2} \partial^\mu \varphi \partial_\mu \varphi \left(1 - \frac{\partial^\mu \partial_\mu}{m_\sigma^2} + \left(\frac{\partial^\mu \partial_\mu}{m_\sigma^2} \right)^2 - \left(\frac{\partial^\mu \partial_\mu}{m_\sigma^2} \right)^3 \dots \right) \partial^\mu \varphi \partial_\mu \varphi$$

if $S < 4m^2$ (T scalar particle mass m)

no internal line in a diagram can be on
the mass-shell (in a $2 \rightarrow 2$ process)



- ① cut the diagram intercepting the internal
line assumed to be on-shell

$$p_1 \quad 1 \quad \text{---} \quad q_1 \quad 3 \\ p_2 \quad 2 \quad \text{---} \quad p \quad \text{---} \quad q_n \quad 4$$
$$q = \sum_i q_i$$

momentum conservation: $p_1 + p_2 = p + q$

$$p = p_1 + p_2 - q$$

$$p^2 = (p_1 + p_2)^2 - 2q(p_1 + p_2) + q^2$$
$$\downarrow$$
$$m^2 \leq 4m^2$$

Work in COM

$$p_1 + p_2 = (2E, \vec{0})$$

$$q = (E_q, \vec{q})$$
$$p = (\sqrt{m^2 + \vec{q}^2}, -\vec{q})$$

$$m^2 \leq 4m^2 - 4EE_q + E_q^2 - \vec{q}^2$$

$$(m^2 + \vec{q}^2) \leq 4m^2 + \underbrace{(2E - E_q)^2}_{m^2 + \vec{p}^2} - 4E^2$$

$$0 \leq 4m^2 - 4E^2 \leq 4m^2 - 4(m^2 + \vec{p}^2)$$
$$\equiv -4\vec{p}^2$$

satisfied only if particles are at rest

$$T(s,t) = c_0 + c_{2,0}s^2 + c_{4,0}s^4 + c_{6,0}s^6 + \cdots + c_{2,1}s^2t + c_{4,1}s^4t + \cdots$$