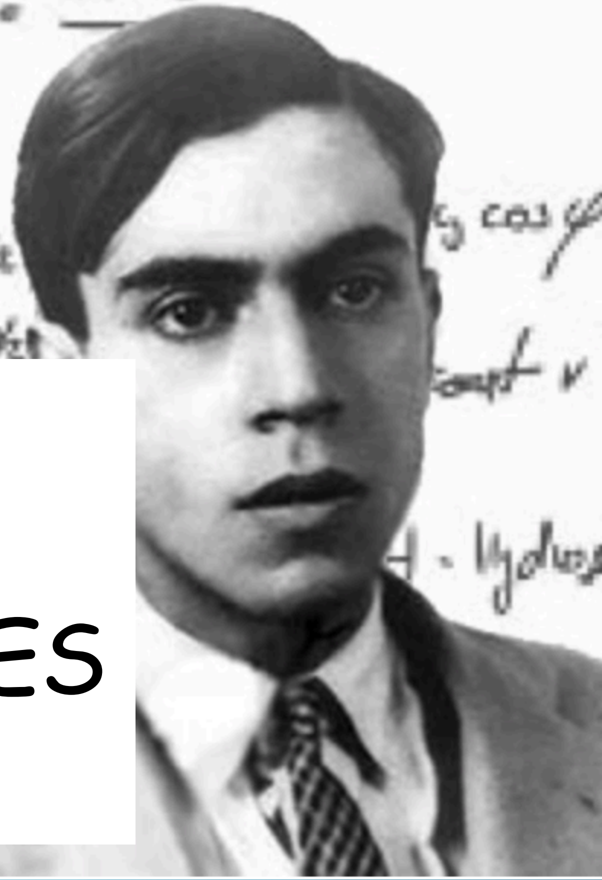
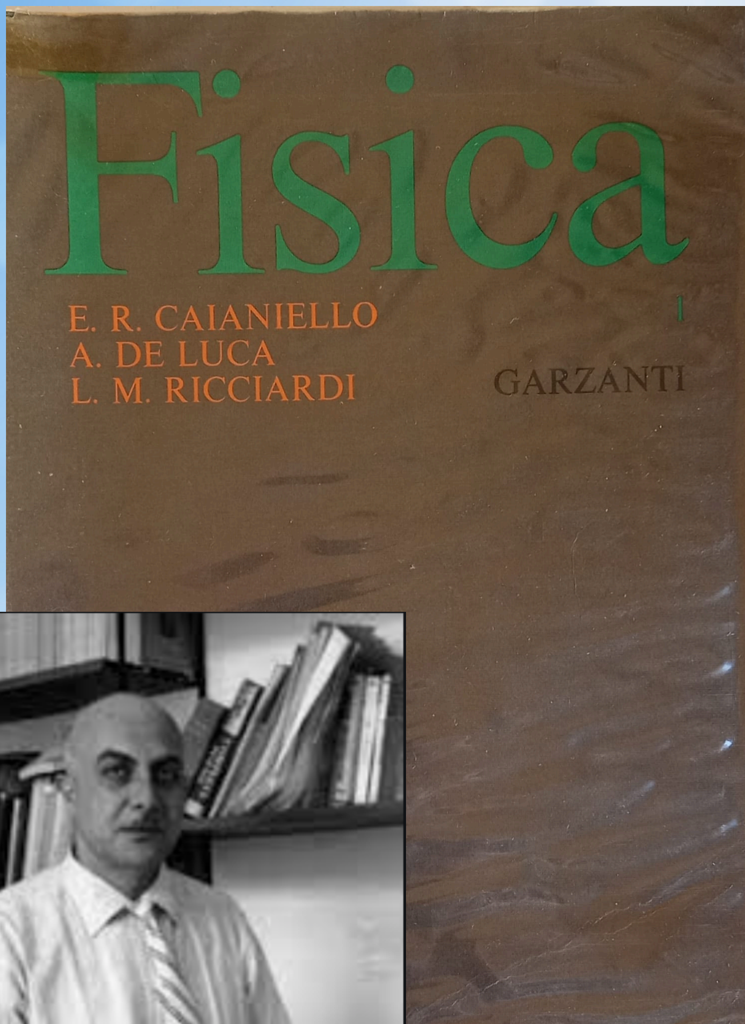
XI series of Majorana Lectures

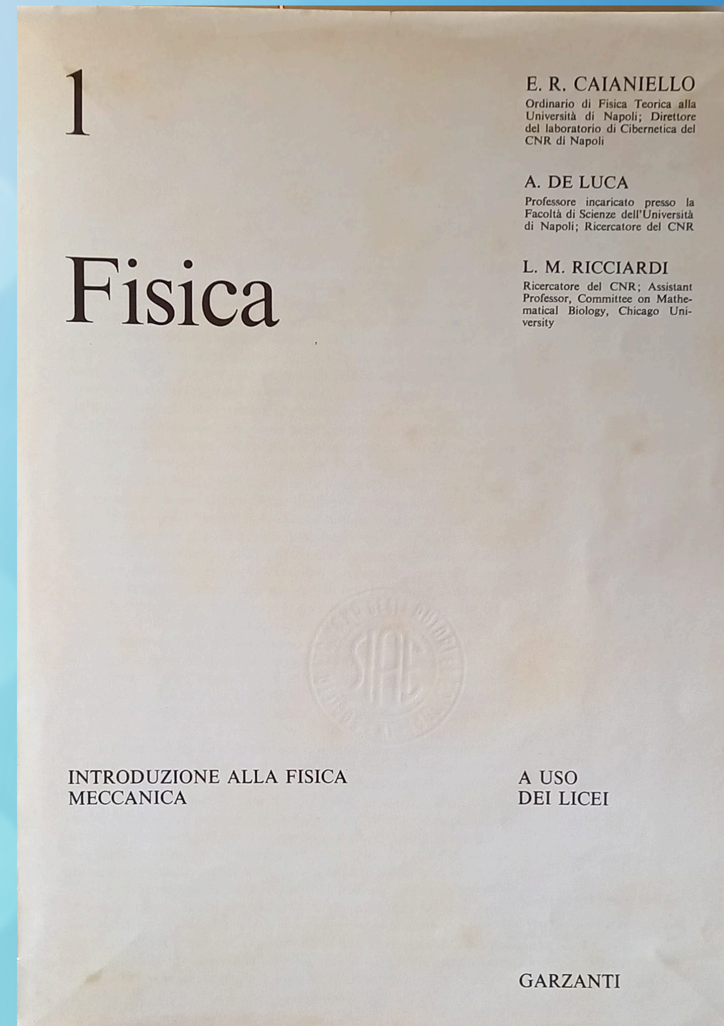


CONSISTENT EFFECTIVE FIELD THEORIES

Ferruccio Feruglio
INFN Padova

Universita' di Napoli Federico II
22, 23, 24 Febbraio 2022





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Fisica

INTRODUZIONE ALLA FISICA
MECCANICA

A USO
DEI LICEI

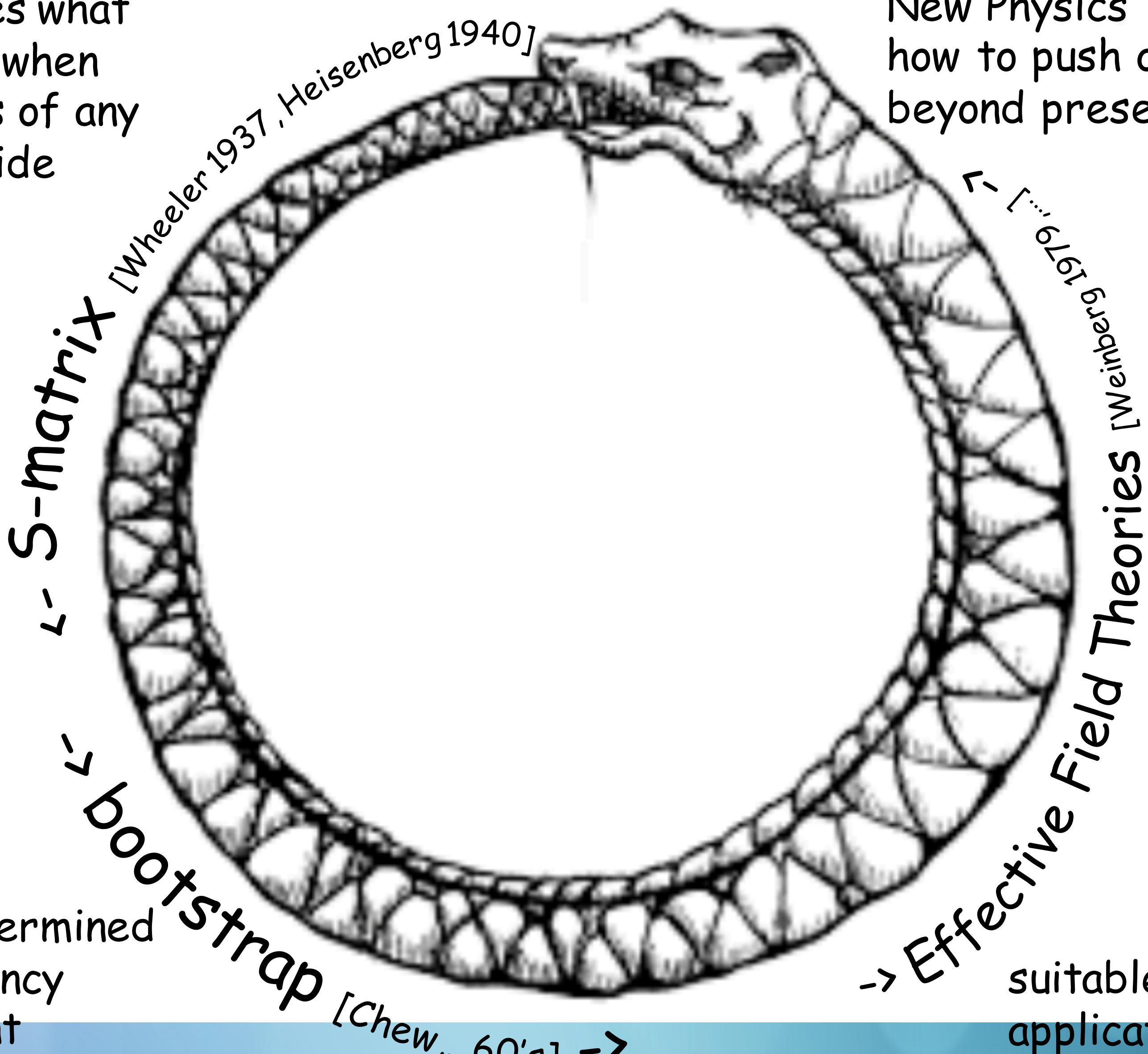
GARZANTI

Eduardo Renato Caianiello
(Napoli, 25 giugno 1921 – Napoli, 22 ottobre 1993)

historical recurrences

describes what happens when particles of any sort collide

>2000: no signal of New Physics
how to push our knowledge beyond present limits?



theory determined by consistency requirement

suitable to practical application

Plan

1. Introduction to ET and EQFT
2. Basic Principles in QFT (causality, unitarity…)
3. Positivity Bounds: impact of Basic Principles on EQFT
4. Applications: Euler-Heisenberg EQFT
5. EQFT and anomalies
6. RG flow in 4d

Questions are very welcome!

Part I

an introduction to EFT and its basic principles

what is an EFT ?

DEFINITION

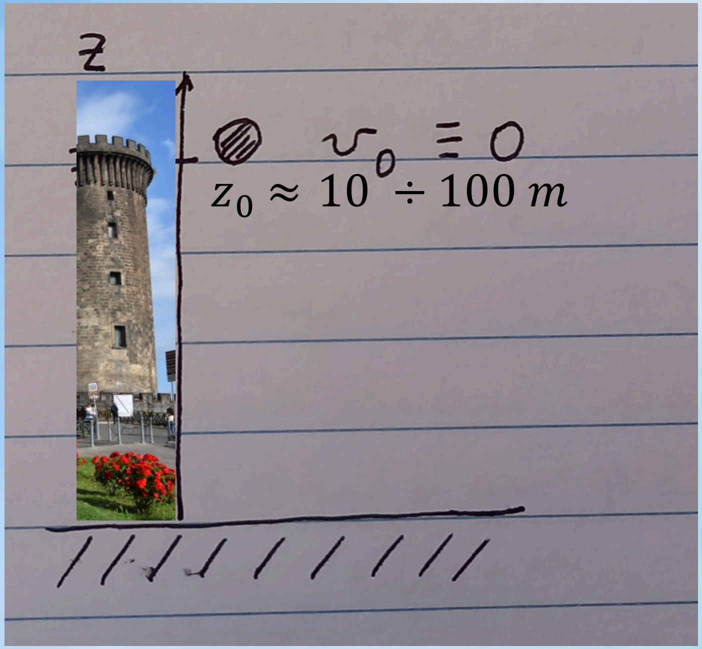
EFT as a sort of "incomplete" theory, as opposed to a "fundamental" one

Wikipedia definition:
In physics, an effective field theory is a type of approximation, or effective theory, for an underlying physical theory,…

the domain of validity of an ET is limited
there is a boundary in some measurable variable (length, energy, …)
beyond which we cannot apply the theory
in the "forbidden" region we need another theory*)

*) isn't this true for all known theories in physics?

Example



a falling body near the Earth surface

task: compute the arrival time

$$t = \sqrt{\frac{2z_0}{g}}$$

physics law? $F = \frac{GMm}{r^2}$

$$F = \frac{GMm}{(R+z)^2} = mg\left[1 - 2\frac{z}{R} + O\left(\frac{z}{R}\right)^2\right]$$

$$g = \frac{GM}{R^2}$$

our effective theory $F = mg$

if our instruments are very precise and z is sufficiently large, we can include the 1st correction

Properties

0. Use of the "full" theory leads to unnecessary complications [we do not need QM to build a house]

1. DOF: 1 massive body m

2. SYMMETRY:
   - translations in x and y directions
   - rotation around the z axis

   differ from that of the "full" theory: SO(3)

3. EXPANSION PARAMETER:
   z/R tell us where the ET breaks down

4. All the details of the "full" theory go in $g=GM/R^2$ a parameter that can be determined with arbitrary precision within the ET

1. + 2. + 3. DEFINE the EF

most general theory with 1 DOF, invariant under T(2) x SO(2), with expansion parameter z/R

$$F = F_0[1 + c_1 \frac{z}{R} + c_2(\frac{z}{R})^2 + \ldots]$$

all constants F_0 , c_1 , c_2 ,… can be determined from measurements without reference to the full theory

DEFINITION of EF

A theory characterized by

- a set of DOF
- a symmetry
- a set of expansion parameters

allowing predictions to a given precision in terms of a finite number of parameters, directly accessible by the experiments

- we went TOP – DOWN [TD] $F = \frac{GMm}{r^2}$ ⇒ $F = mg$

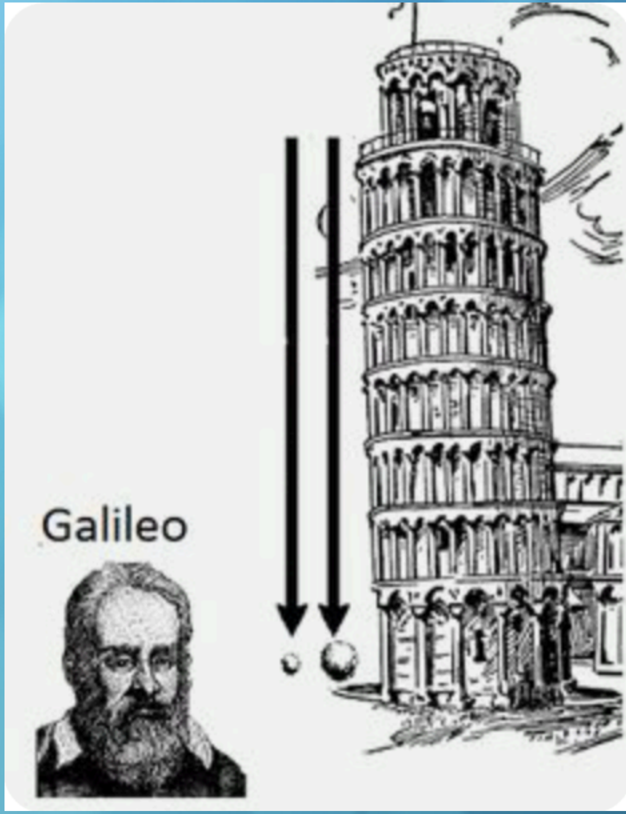
- we can also go BOTTOM – UP [BU]

assume we do not know the full theory,
but our experiments discovered 2.
[assuming F(z) regular in z=0] we can write

$$F = F_0[1 + c_1\frac{z}{L} + c_2(\frac{z}{L})^2 + \ldots]$$

we can do precision tests and measure
F_0, c_1/L, c_2/L^2, … looking for New Physics

major difference:



we went TOP – DOWN [TD] $F = \frac{GMm}{r^2}$ $\Rightarrow$ $F = mg$

we can also go BOTTOM – UP [BU]

assume we do not know the full theory,
but our experiments discovered 2.
[assuming F(z) regular in z=0] we can write

$$F = F_0[1 + c_1 \frac{z}{L} + c_2 (\frac{z}{L})^2 + \ldots]$$

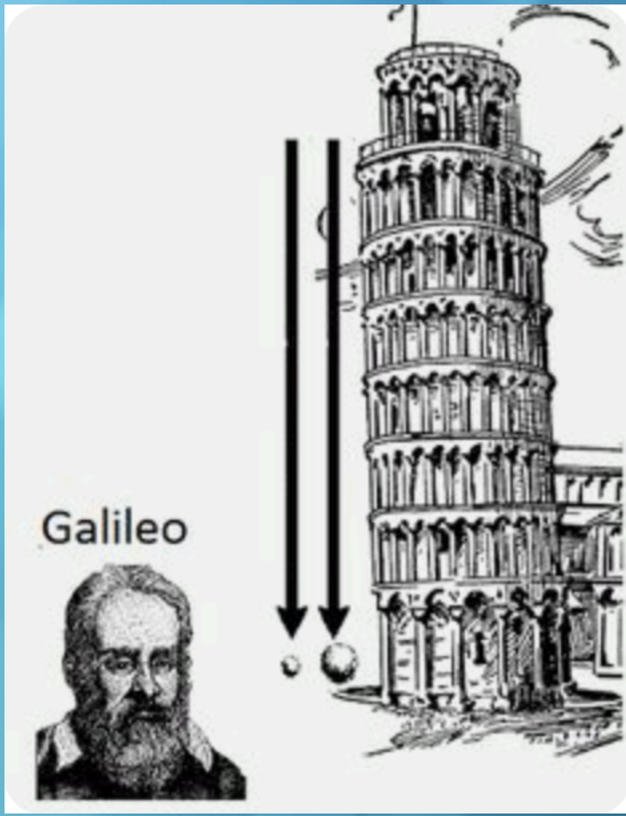
we can do precision tests and measure
F_0 , c_1/L , c_2/L^2 ,… looking for New Physics

major difference:
we do not know what L is <-> invariance

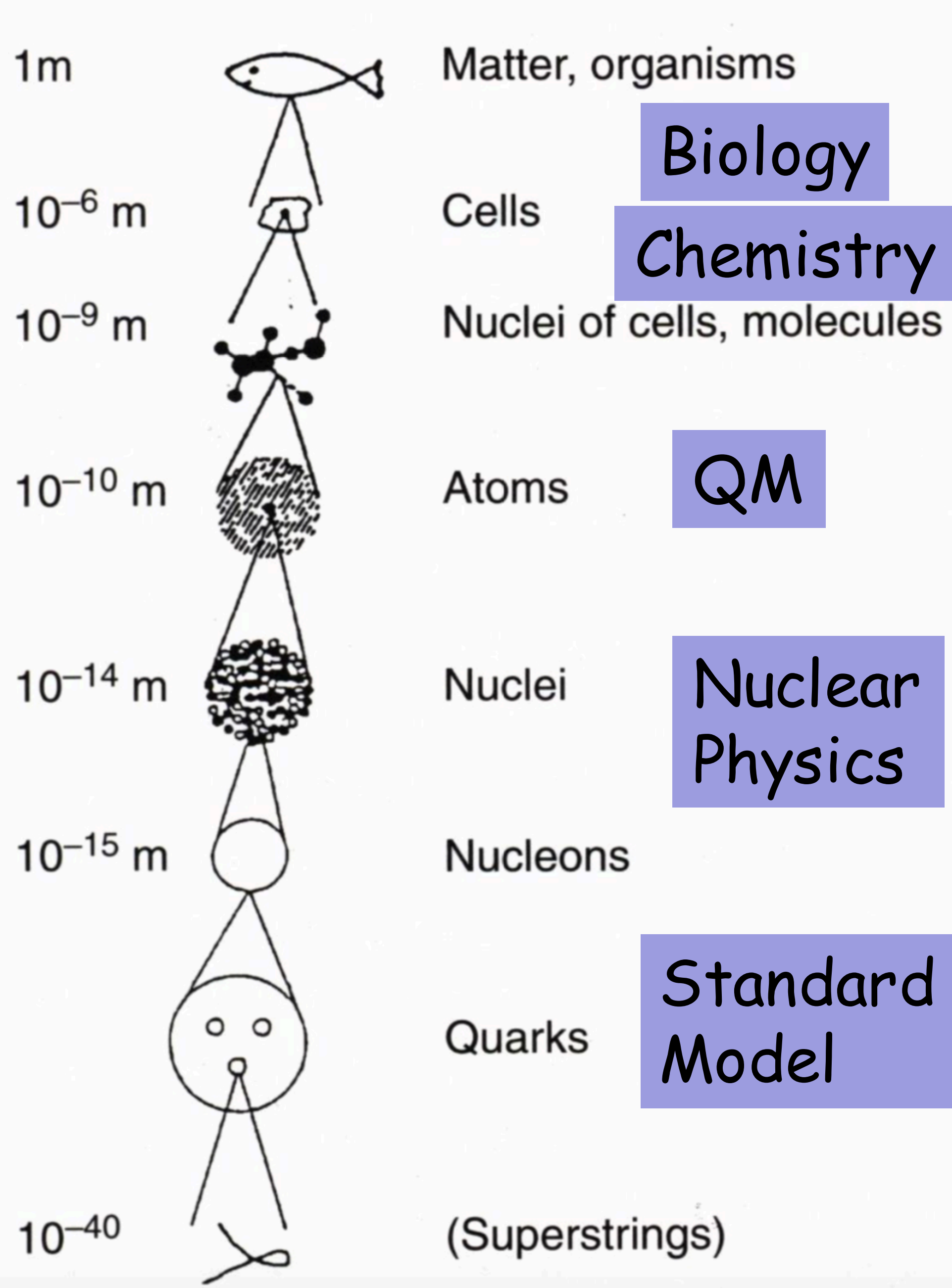
$$L \to \lambda L$$
$$c_1 \to \lambda c_1$$
$$c_2 \to \lambda^2 c_2$$
$$\ldots$$

from exp: $\frac{c_1}{L} = -0.31 \times 10^{-6}\ m^{-1}$

expansion parameter z/L needs a guess:
e.g. assuming $c_1 = O(1)$ leads to $L \approx 3000\ Km$



is there any theory covering the full ℓ range ?



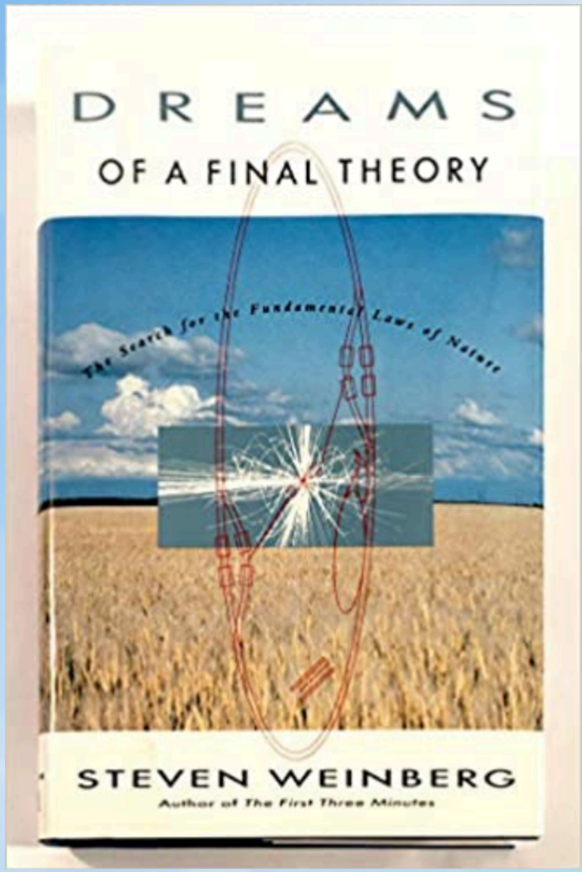
the full range is divided into layers, each with its own "incomplete" theory

each layer largely autonomous

incompleteness as an advantage, rather than a limitation

old view point: reductionism
- there is an hierarchy between layers
- QM is more fundamental than Chemistry or Biology
- upper layers can be derived from the Theory of Everything

reductionism have great supporters



"The dream of a final theory inspires much of today's work in high-energy physics, and though we do not know what the final laws might be or how many years will pass before they are discovered, already in today's theories we think we are beginning to catch glimpses of the outlines of a final theory. The"

― **Steven Weinberg, Dreams of a Final Theory: The Scientist's Search for the Ultimate Laws of Nature** but also great detractors

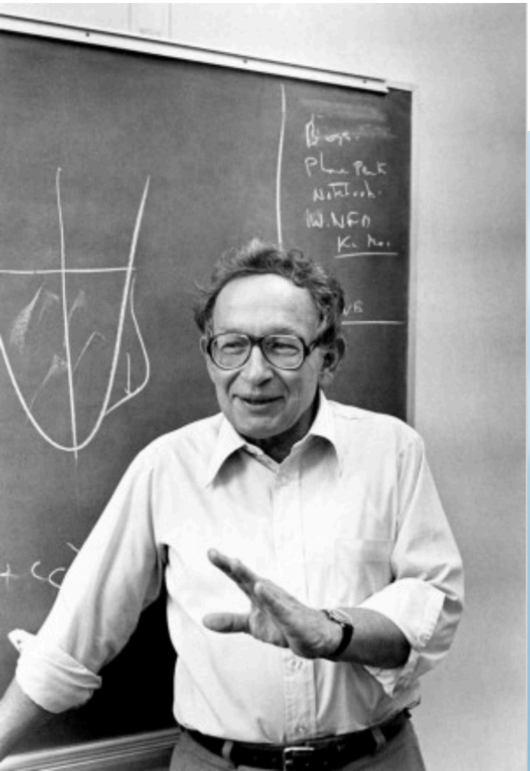
4 August 1972, Volume 177, Number 4047

SCIENCE

More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson



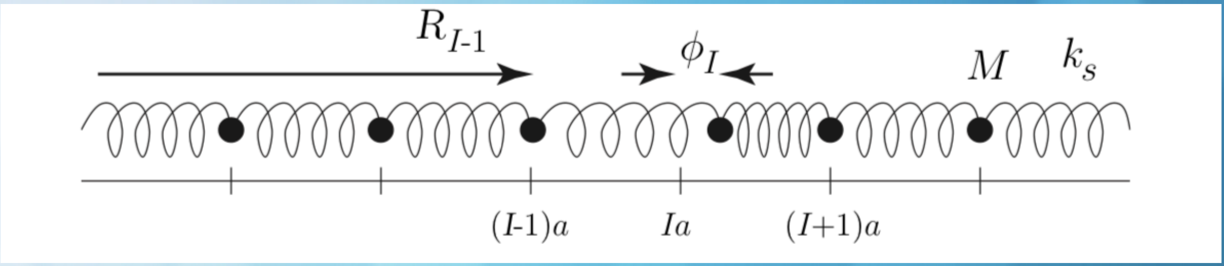
X	Y
solid state or many-body physics	elementary particle physics
chemistry	many-body physics
molecular biology	chemistry
cell biology	molecular biology
•	•
•	•
•	•
psychology	physiology
social sciences	psychology

But this hierarchy does not imply that science X is "just applied Y." At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry.

Emergent properties

relation between contiguous layers not obvious

toy model for lower layer:
1D N-atom chain



$$L = \sum_{n=1}^{N} \left[\frac{1}{2} m\, \dot{x}_n^2 - \frac{1}{2} k (x_{n+1} - x_n - a)^2\right] + \cdots$$

[periodic boundary conditions]

specific heat, transport properties: go to the continuum limit

$$x_n - na = \frac{1}{\sqrt{ka}} \varphi\left(\frac{x}{a}\right)\bigg|_{x=na}$$

$$x_{n+1} - x_n - a = \sqrt{\frac{a}{k}} \frac{\partial}{\partial x} \varphi\left(\frac{x}{a}\right)\bigg|_{x=na} + \ldots$$

$$\sum_{n=1}^{N} \to \frac{1}{a} \int_0^{\ell} dx$$

get a field theory giving rise to a 1D Klein-Gordon equation

$$L = \int_0^{\ell} dx \frac{1}{2} \partial^{\mu} \varphi \partial_{\mu} \varphi$$

metric $\eta_{\mu\nu} = \mathrm{diag}(\frac{1}{v}, -1)$ $\quad v = \sqrt{\frac{ka^2}{m}}$

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \varphi(x,t) = 0$$

$$E = p\, v$$

describe sound waves or phonons, when quantized

	DOF	symmetries	
"upper" layer	∞	1D translations	$x \to x + c$
		Lorentz-like	$x^{\mu} \to \Lambda^{\mu}_{\nu} x^{\nu}$
		shift symmetry	$\varphi(x,t) \to \varphi(x,t) + C$
"lower" layer	N	1D translations	$x_n \to x_n + c$

emergent properties

Emergent properties

Emergent = appropriate for the level under investigation

Particle Physics
$10^{-18}\, m \leq \ell \leq 10^{-15}\, m$
"upper" layer

DOF

∞

symmetries

4D translations
SO(3) rotations

$x^\mu \to x^\mu + c^\mu$

Lorentz-symmetry

$x^\mu \to \Lambda^\mu_\nu\, x^\nu$

emergent properties ?

gauge symmetry

$\ell \leq 10^{-18}\, m$
"lower" layer

???

???

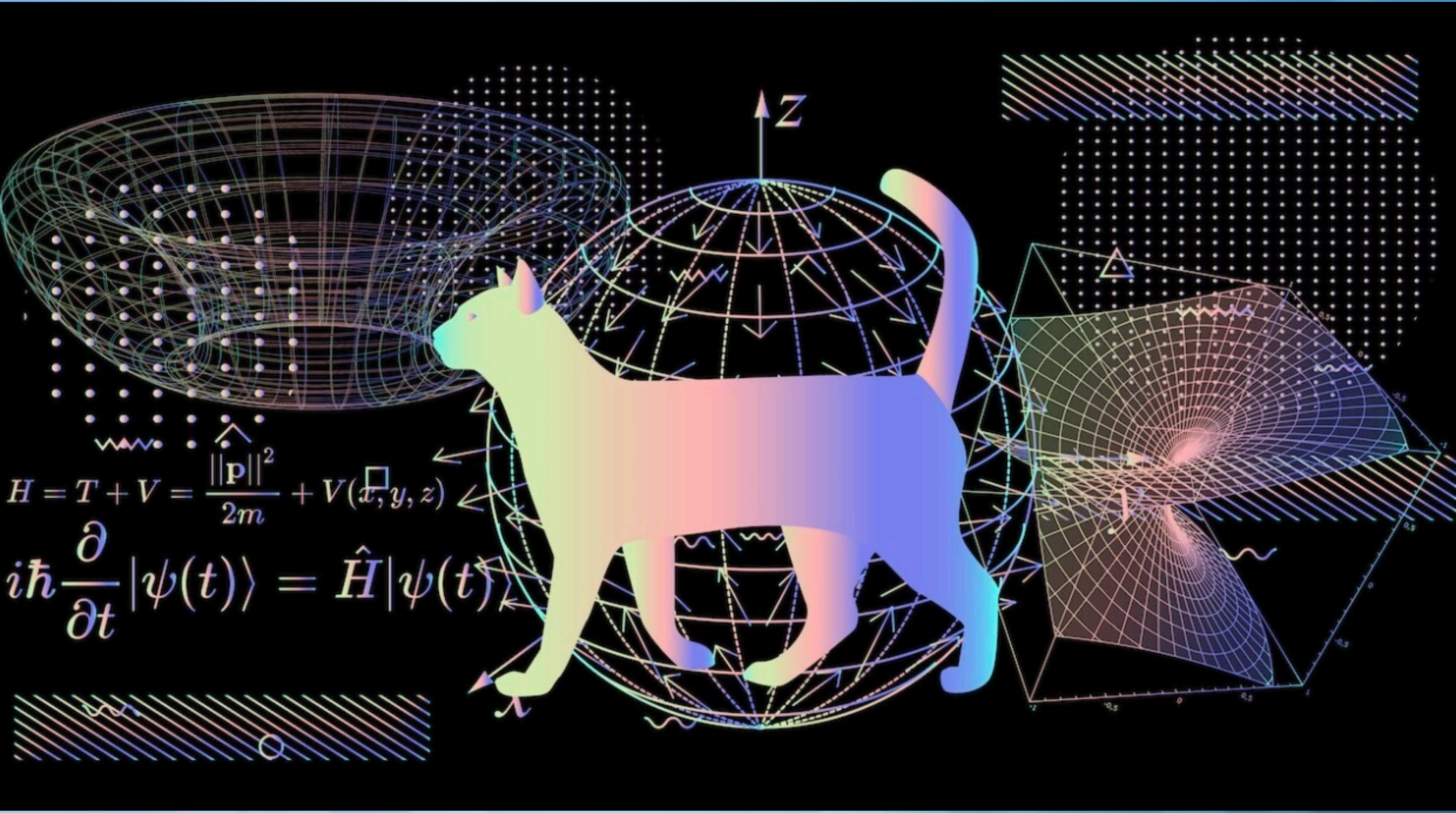
???

food for thought

Back to our toy model in the continuum limit

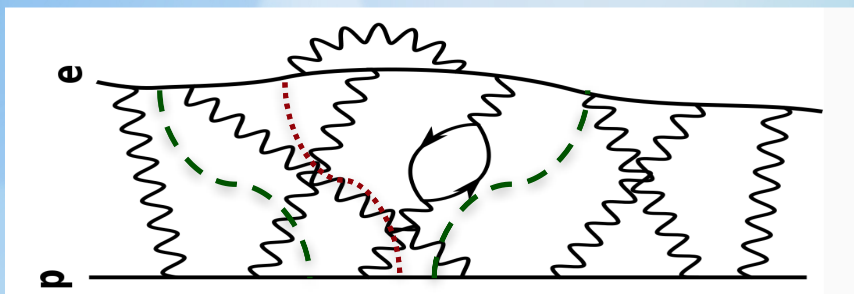
- what is the expansion parameter in the EFT?
- is Lorentz symmetry good to any order of the expansion parameter ?
- is the shift symmetry good to any order of the expansion parameter ?

Effective Quantum Field Theories



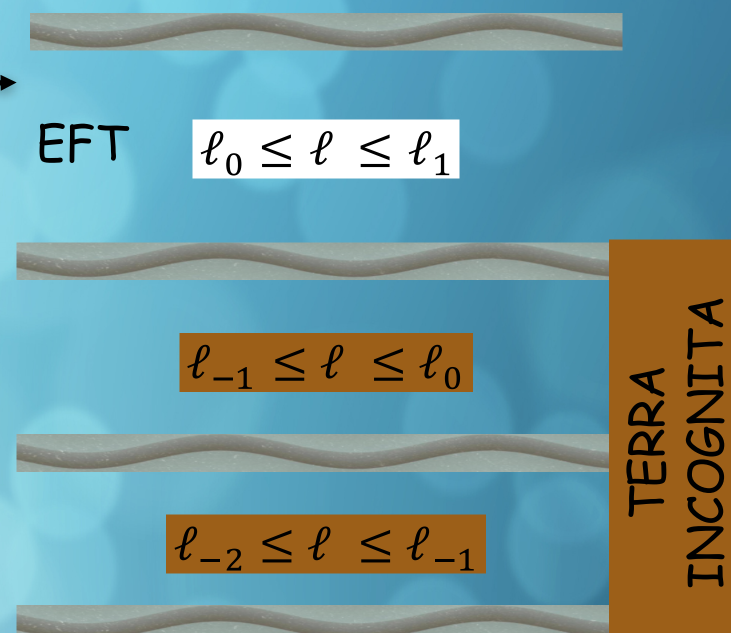
problem

predictions in QM from summing over all possible intermediate states

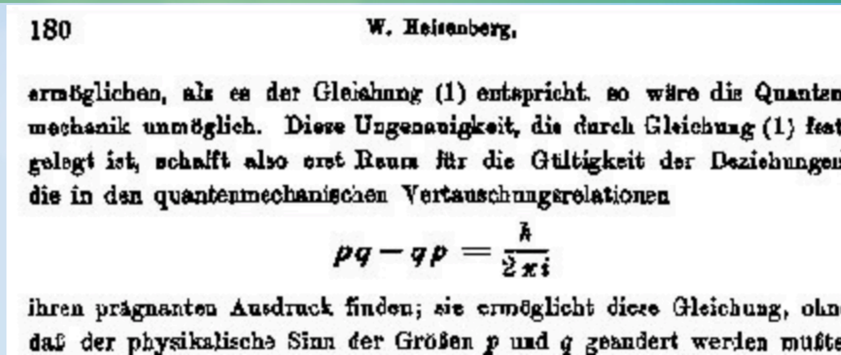


how to isolate a layer,
if we should know about

ALL DOF and
THEIR INTERACTIONS?



solution: the uncertainty principle

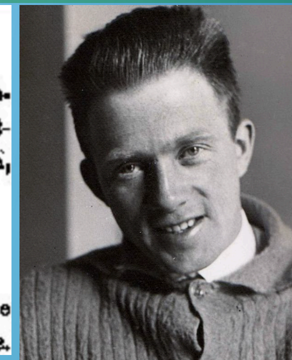


180 W. Heisenberg,

ermöglichen, als es der Gleichung (1) entspricht, so wäre die Quantenmechanik unmöglich. Diese Ungenauigkeit, die durch Gleichung (1) festgelegt ist, schafft also erst Raum für die Gültigkeit der Beziehungen, die in den quantenmechanischen Vertauschungsrelationen

$$pq - qp = \frac{h}{2\pi i}$$

ihren prägnanten Ausdruck finden; sie ermöglicht diese Gleichung, ohne daß der physikalische Sinn der Größen p und q geändert werden müßte.

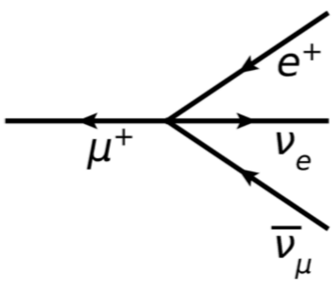


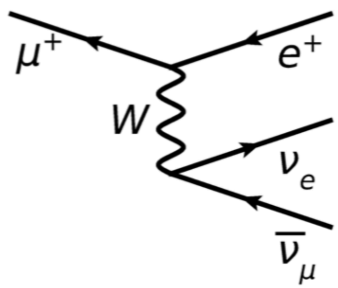
- high-energy effects are short range $\Delta x \ll \ell_0$

  if our experiments cannot resolve Δx,
  high-energy effects becomes local ≡
  look like some term in a local Lagrangian

$$g = \frac{GM}{R^2}$$

- Example: Fermi theory of weak interactions





$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

$$g^2 \int d^4x \int d^4y \, (\bar{\psi}\gamma^\mu\psi)(x)\Delta(x-y)(\bar{\psi}\gamma_\mu\psi)(y) \to \frac{g^2}{M^2}\int d^4x \; (\bar{\psi}\gamma^\mu\psi\,\bar{\psi}\gamma_\mu\psi)(x)$$

high-energy effects are absorbed in parameters c_k , M,… determined by experiment

most general theory with given DOF and SYMMETRY:

$$\mathcal{L}_{EFT} = \mathcal{L}_{\leq 4}(c_{\leq 4}, M, \varphi) + \frac{1}{\Lambda}\mathcal{L}_5(c_5, \varphi) + \frac{1}{\Lambda^2}\mathcal{L}_6\,(c_6, \varphi) + +\ldots$$

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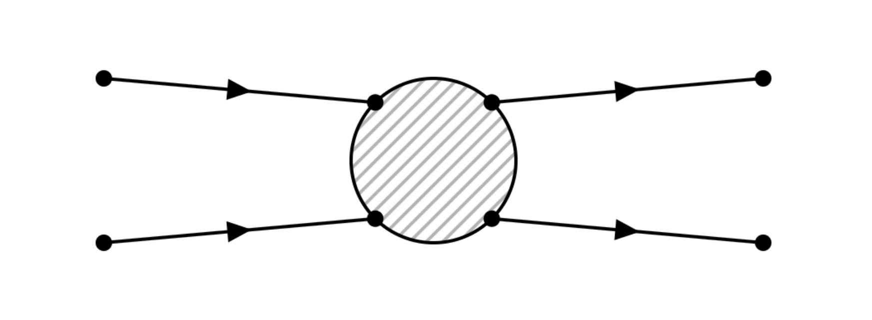
PHENOMENOLOGICAL LAGRANGIANS*

STEVEN WEINBERG

in the context of perturbation theory: if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible *S*-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not been proved, but any counterexamples would be of great interest, and I do not know of any.

a new view on renormalizability

- $\mathcal{L}_{EFT} = \mathcal{L}_{\leq 4}(c_{\leq 4}, M, \varphi) + \frac{1}{\Lambda}\mathcal{L}_5(c_5, \varphi) + \frac{1}{\Lambda^2}\mathcal{L}_6\,(c_6, \varphi) + +\ldots$



$$\text{Probability} \propto |\mathcal{A}|^2 = \left|\mathcal{A}_{\leq 4}(c_{\leq 4}) + \frac{E}{\Lambda}\mathcal{A}_5(c) + \frac{E^2}{\Lambda^2}\mathcal{A}_6(c) + \cdots\right|^2$$

$[M \ll E \ll \Lambda]$

- typical expansion parameter: $\frac{E}{\Lambda}$ domain of validity: $E \leq \Lambda$

- $E \ll \Lambda$ Probability $\propto |\mathcal{A}|^2 = |\mathcal{A}_{\leq 4}(c_{\leq 4}, M)|^2$

all theories look like renormalizable at low energy

a journey in coupling space

$$F = F_0[1 + c_1 \frac{z}{L} + c_2 (\frac{z}{L})^2 + \ldots]$$

$$L \rightarrow \lambda L$$
$$c_1 \rightarrow \lambda c_1$$
$$c_2 \rightarrow \lambda^2 c_2$$
$$\ldots$$

what happens if we slightly lower the cut-off?

$$\Lambda \to e^{-t}\Lambda < \Lambda$$

[assumption: DOF and SYMMETRY do not change]

EFT $\ell_0 \leq \ell \leq \ell_1$

EFT' $\ell'_0 \leq \ell \leq \ell_1$

$\ell_{-1} \leq \ell \leq \ell_0$

$\ell_{-1} \leq \ell \leq \ell'_0$

$\ell_{-2} \leq \ell \leq \ell_{-1}$

$\ell_{-2} \leq \ell \leq \ell_{-1}$

$$S(c_k;\Lambda) \to S(c_k(t), e^{-t}\Lambda)$$

the two descriptions agree when $t > 0$

effect absorbed in shift of mass/coupling

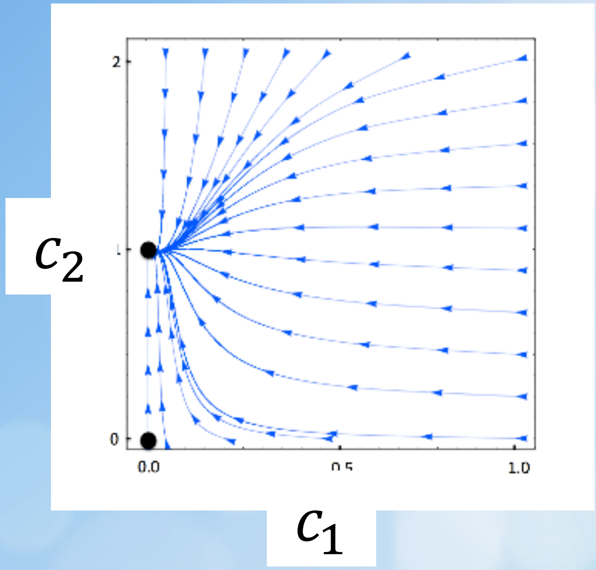
PHYSICAL REVIEW B VOLUME 4, NUMBER 9 1 NOVEMBER 1971

Renormalization Group and Critical Phenomena.
I. Renormalization Group and the Kadanoff Scaling Picture*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

(Received 2 June 1971)

lowering the cut-off generates a motion in the space of couplings c_k



[arrows toward low-energy]

$$\frac{dc_i}{dt} = \beta_i(c)$$

at special points c_i^* the motion stops

$$\beta_i(c^*) = 0$$

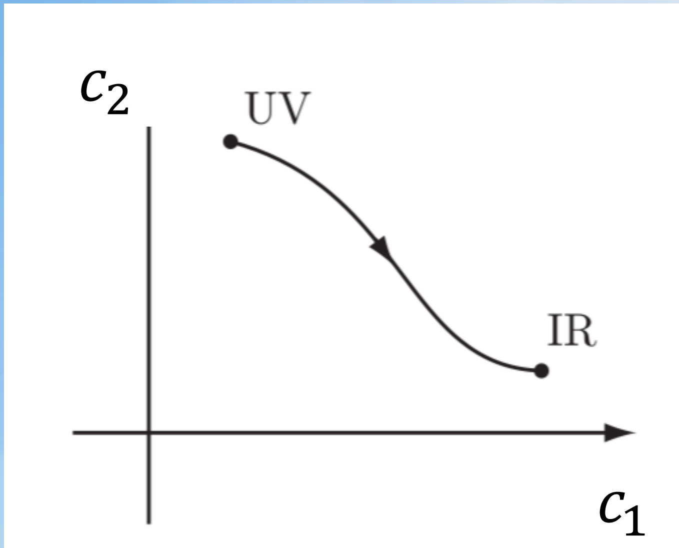
fixed point = FP

at FP the theory is scale invariant

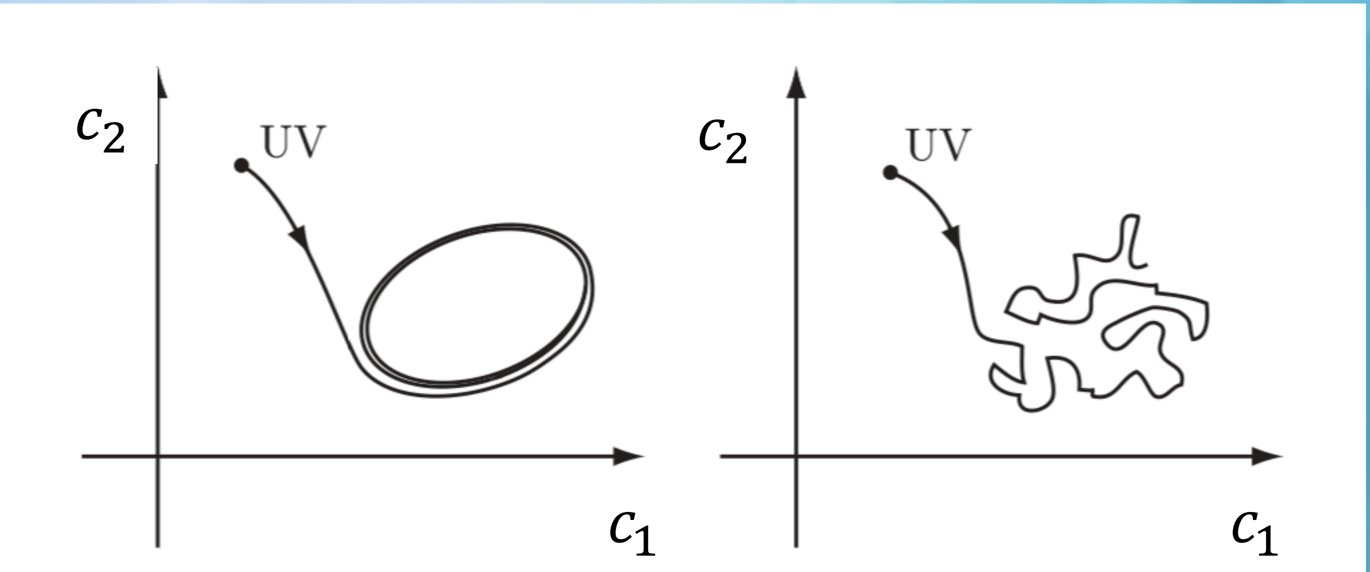
around a FP we linearize the motion

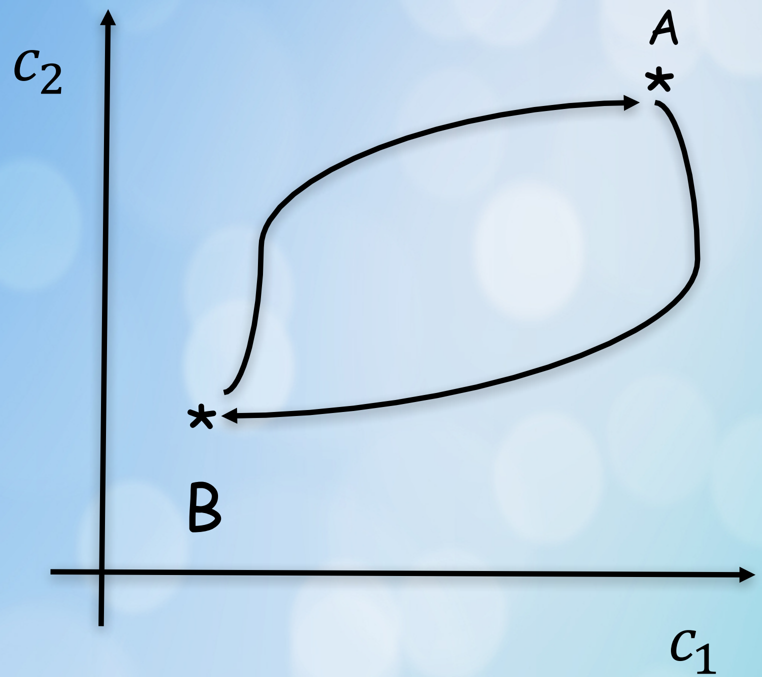
$$\beta_i(c) = \beta_i(c^*) + \left.\frac{\partial \beta_i}{\partial c_j}\right|_{c=c^*} \left(c_j - c_j^*\right) + \cdots$$

$\left.\frac{\partial \beta_i}{\partial c_j}\right|_{c=c^*}$ has all negative(positive) eigenvalues = stable(unstable) FP = IR(UV) most plausible motion is from one stop to another stop



but other possibilities are not excluded





can we have something like this?

no, if the the motion is described by a gradient flow

$$\beta_i(c) = -\frac{\partial V}{\partial c_i}$$

$$0 = V_A - V_A = \int dV = \int \frac{\partial V}{\partial c_i}\frac{dc_i}{dt}dt = -\sum \int |\beta_i|^2 dt < 0$$

in this case the flow is irreversible

we have a quantity that decreases monotonically along the flow

$$dV = -|\beta_i|^2 dt$$

measures the loss of information when the layer between Λ and $(1 - dt)\Lambda$ is removed from EFT

RGE flow in 2d

"Irreversibility" of the flux of the renormalization group in a 2D field theory

A. B. Zomolodchikov
L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 20 May 1986)
Pis'ma Zh. Eksp. Teor. Fiz. **43**, No. 12, 565–567 (25 June 1986)

There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

Some of the information on the ultraviolet behavior of the field theory is lost under renormalization transformations with $t > 0$, since in the field theory it is not legitimate to examine correlations at scales smaller than the cutoff. We would therefore expect that a motion of the space Q under the influence of the renormalization group would become an "irreversible" process, similar to the time evolution of dissipative systems.

what happens in 4d ?

summary

our focus

Effective Quantum Field Theory

- a set of DOF: fields $\varphi(x)$
- a SYMMETRY: including the Lorentz group
- an EXPANSION PARAMETER: E / Λ

dynamics specified by

$$\mathcal{L}_{EFT} = \mathcal{L}_{\leq 4}(c_{\leq 4}, M, \varphi) + \frac{1}{\Lambda}\mathcal{L}_5(c_5, \varphi) + \frac{1}{\Lambda^2}\mathcal{L}_6\,(c_6, \varphi) + +...$$

[most general local Lagrangian with given DOF and SYMMETRY]

general question:

- under which conditions on the fields $\varphi(x)$ and the parameters c_k , M, the above theory is consistent ?
- is the RG flow of c_k irreversible in 4d ?

Part I

basic principles

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in the context of perturbation theory: if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not been proved, but any counterexamples would be of great interest, and I do not know of any.

counterexample

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + g\left(\partial^\mu\varphi\partial_\mu\varphi\right)^2 + \cdots \qquad g < 0$$

symmetries: Poincare' and $\varphi(x) \to \varphi(x) + c$

what is wrong with $g < 0$?

equations of motion have a family of translationally invariant solutions

$$\partial_\mu \varphi_0(x) = C_\mu \text{ (constant)}$$

Exercise

we can quantize the fluctuation of the theory around one of these solutions

$$\sigma(x) = \varphi(x) - \varphi_0(x)$$

linearized EOM for the fluctuations

$$\partial_\mu \partial^\mu \sigma(x) + 8\,g\, C_\mu C_\nu \partial^\mu \partial^\nu \sigma(x) = 0$$

Exercise

dispersion relation for quanta

$$\omega^2 = \vec{k}\cdot\vec{k} - 8g\left(C_\mu k^\mu\right)^2$$

speed of quanta is superluminal
unless $g \geq 0$

… one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles…

$$\mathcal{L}_{EFT} = \mathcal{L}_{\leq 4}(c_{\leq 4}, M, \varphi) + \frac{1}{\Lambda}\mathcal{L}_5(c_5, \varphi) + \frac{1}{\Lambda^2}\mathcal{L}_6\,(c_6, \varphi) + +\ldots$$

… calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with:

CAUSALITY: the effect cannot precede the cause

UNITARITY: conservation of probability in QM setting

LORENTZ INVARIANCE: we deal with relativistic theories

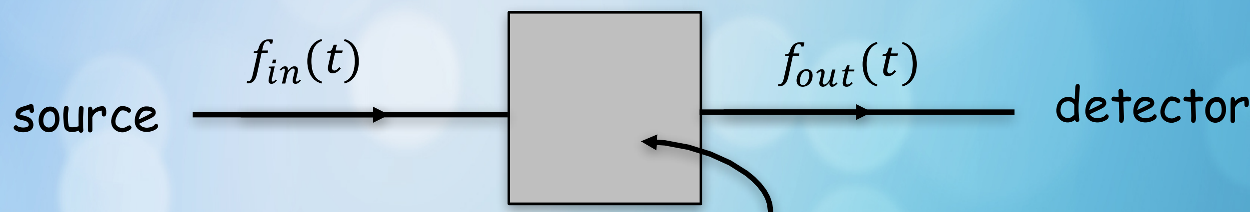
GAUGE INVARIANCE: describe em, weak and strong interactions

how do these principles affect the DOF $\{\varphi\}$ and the parameter space $\{c_k\}$?

[here: neglect gravitational interactions]

causality and analyticity

signal model [no spatial coordinates]



assume:

- output depends linearly on input
- time-translational invariance

$$f_{out}(t) = \int_{-\infty}^{+\infty} dt' \; S(t-t') f_{in}(t')$$

switch to Fourier space $\quad y(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} d\omega \; y(\omega) \; e^{-i\omega t}$

$$\boxed{f_{out}(\omega) = S(\omega) f_{in}(\omega)}$$

causality: output cannot precede the input

$$\boxed{f_{in}(t') = 0 \qquad t' < T} \;\Rightarrow\; \boxed{f_{out}(t) = 0 \qquad t < T}$$

$$0 = \int_T^{+\infty} dt' \; S(t-t') f_{in}(t') \;\Rightarrow\; S(t-t') = 0 \qquad t-t' < 0$$

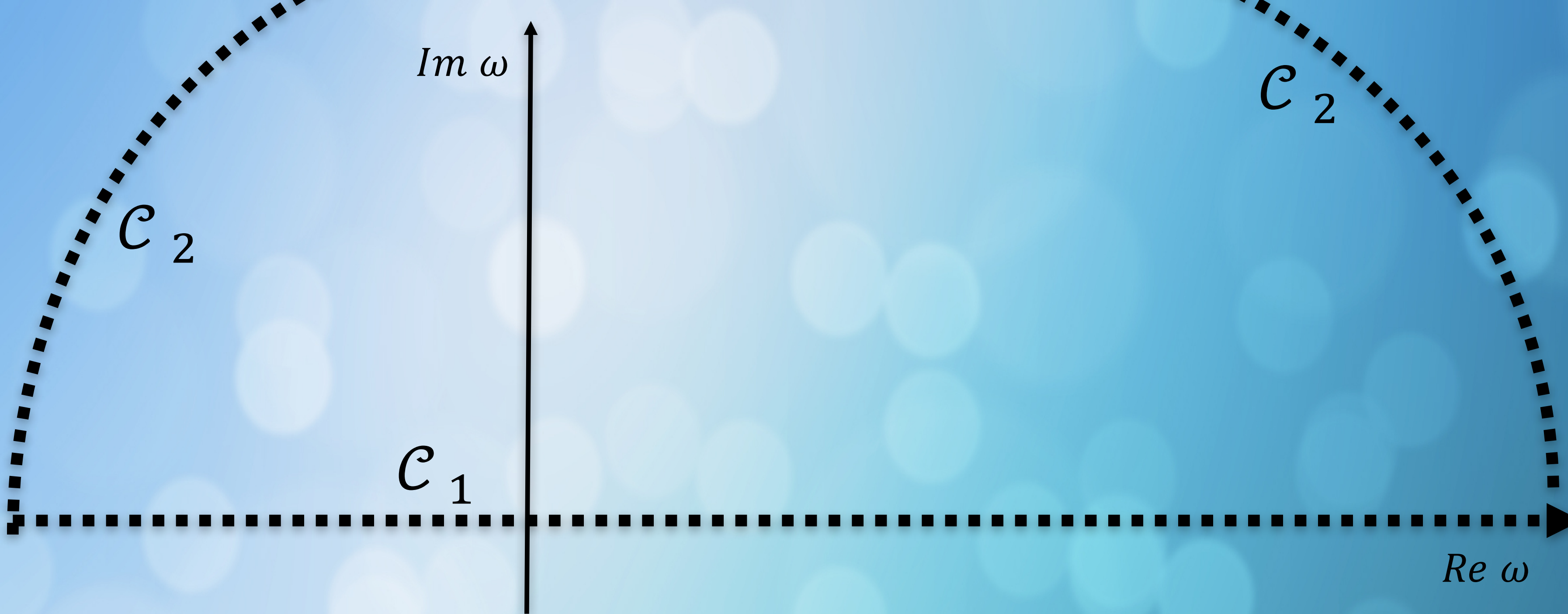
$$S(\omega) = \int_0^{+\infty} dt \, S(t) e^{i\omega t}$$ can be analytically continued in the UHP

$\omega \to \omega + i\epsilon$, $\epsilon > 0$ $\qquad e^{i\omega t} \to e^{i\omega t} e^{-\epsilon t}$ improves convergence of the integral

$S(\omega)$ is analytic in the UHP $\qquad$ define $\qquad S_{phys}(\omega) = \lim_{\epsilon \to 0^+} S(\omega + i\epsilon)$

analyticity in UHP of $S(\omega)$ is a necessary condition for causality

is it sufficient?



- analyticity ⇒ $$0 = \frac{1}{2\pi}\oint_{\mathcal{C}_1+\mathcal{C}_2} d\omega\, S(\omega)\, e^{+i\omega|t|}$$ no poles inside the contour

- $$S(-|t|) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, S(\omega)\, e^{+i\omega|t|} = -\frac{1}{2\pi}\int_{\mathcal{C}_2} d\omega\, S(\omega)\, e^{+iu|t|}e^{-v|t|}$$ $\omega = u + iv$

to prove causality, $S(-|t|) = 0$,
we need sub-exponential behavior of $S(\omega)$ at large $|\omega|$

unitarity and polynomial boundness

unitarity (weak form)

we ask $$\int_{-\infty}^{+\infty} dt\,|f_{out}(t)\,|^2 \leq \int_{-\infty}^{+\infty} dt\,|f_{in}(t)\,|^2$$

we allow for absorption

this implies [exercise]

$$|S(\omega)\,| \leq 1 \qquad Im\ \omega \geq 0$$

when $|S(\omega)|$ decreases sufficiently rapidly at infinity we can write a

dispersion relation

$$S(\omega + i\epsilon) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega' \frac{S(\omega')}{(\omega' - \omega - i\epsilon)}$$

$$\frac{1}{(\omega' - \omega - i\epsilon)} = PP\frac{1}{(\omega' - \omega)} + i\pi\delta(\omega - \omega')$$

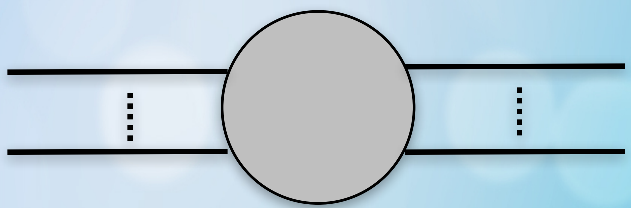
$$Re\ S(\omega) = +\frac{1}{\pi} PP \int_{-\infty}^{+\infty} d\omega' \frac{ImS(\omega')}{(\omega' - \omega)}$$

$$Im\ S(\omega) = -\frac{1}{\pi} PP \int_{-\infty}^{+\infty} d\omega' \frac{Re\ S(\omega')}{(\omega' - \omega)}$$

e.g. Kramer-Kronig relations for complex refractive index

$$S(\omega) = n(\omega) - 1$$

application to QFT

$$|in'\rangle$$  $$|out\rangle$$

S-matrix elements

Probability $\propto |\langle out|in'\rangle|^2$

$$\langle out|in'\rangle = \langle in|\hat{S}|in'\rangle$$

$$\hat{S}\hat{S}^{+} = \hat{S}^{+}\hat{S} = 1$$

$$\hat{S} = 1 + i\,\hat{T}$$

$$-i(\hat{T} - \hat{T}^{+}) = \hat{T}^{+}\hat{T}$$

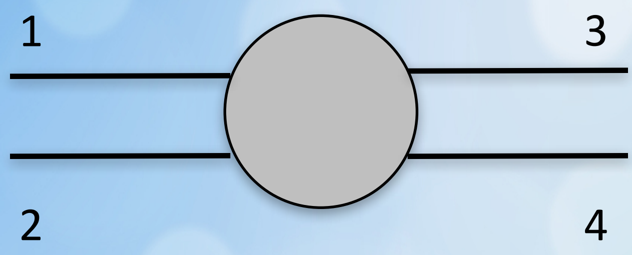
$$\langle f|\hat{T}|i\rangle \equiv (2\pi)^4 \delta^4(P_f - P_i)\,\tilde{T}_{fi}$$

unitarity as a non-linear (and nonperturbative) relation among matrix elements

$$(\tilde{T}_{fi} - \tilde{T}^{*}_{if}) = i(2\pi)^4 \sum_n \delta^4(P_n - P_i)\tilde{T}^{*}_{nf}\,\tilde{T}_{ni}$$

kinematics recap

consider a theory describing a single scalar particle of mass m



elastic scattering

$$\tilde{T}_{fi} = T(s,t)$$

Mandelstam variables

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

$$s + t + u = 4m^2$$

physical region

$$s \geq 4m^2$$

$$4m^2 - s \leq t, u \leq 0$$

in C.O.M. frame

$$s = 4\,E^2$$
$$t = -2\,k^2(1 - \cos\vartheta)$$
$$u = -2\,k^2(1 + \cos\vartheta)$$

$$p_1 = (E, 0, 0, k)$$

$$\cos\vartheta = 1 + \frac{2t}{s - 4m^2}$$

unitarity relation at $t = 0$

forward elastic scattering

$$t = 0 \qquad \vartheta = 0$$

$$p_1 = p_3 \qquad p_2 = p_4$$

$$|f\rangle = |i\rangle$$

$$(\tilde{T}_{fi} - \tilde{T}^*_{if}) = i(2\pi)^4 \sum_n \delta^4(P_n - P_i)\tilde{T}^*_{nf}\,\tilde{T}_{ni}$$

$$2i\, Im\tilde{T}_{ii}$$

$$2\,\sigma_i(s) = \frac{1}{2\sqrt{s(s-4m^2)}} \sum_n (2\pi)^4 \delta^4(P_n - P_i)\left|\tilde{T}_{ni}\right|^2$$

total cross-section

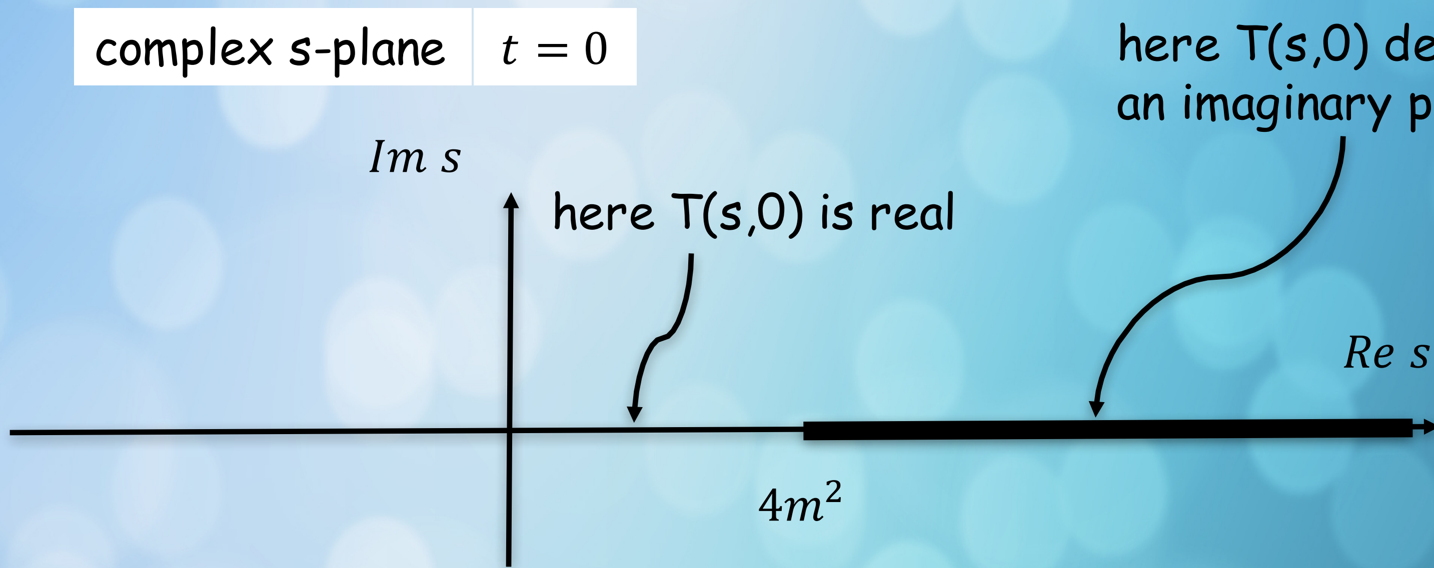
optical theorem

$$Im\,T(s,0) = 2\sqrt{s(s-4m^2)}\,\sigma_i(s) > 0$$

$$s \geq 4m^2$$

analytic extension in complex s-plane at $t = 0$

- assume $T(s, 0)$ has no singularities other than those implied by the unitarity relation [more on this, later on]



below real axis, analytic continuation by the Schwartz Reflection Principle

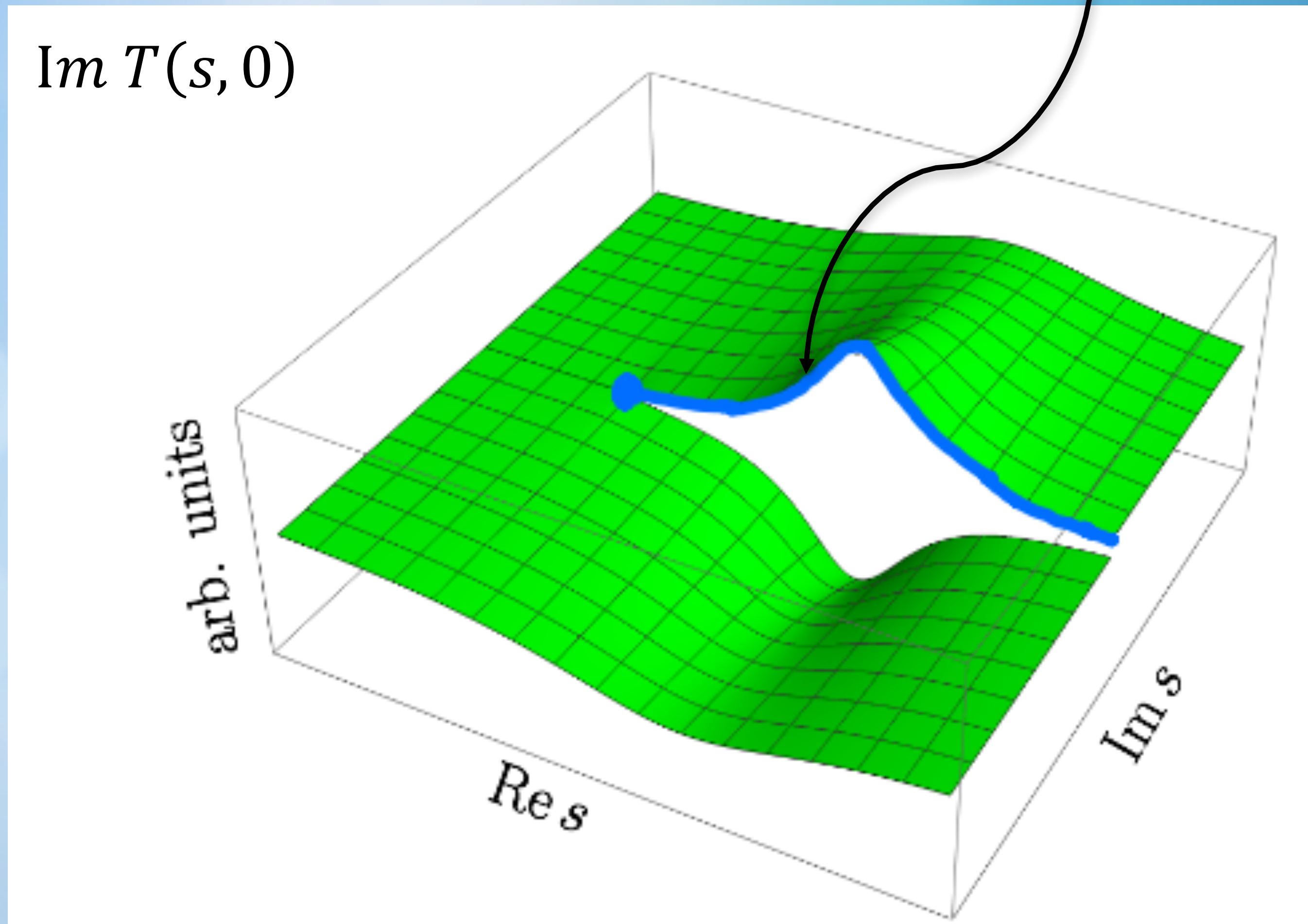
$$T(s^*, 0) = [T(s, 0)]^*$$

$$Im\, T(s, 0) = -Im\, T(s^*, 0)$$

➡ discontinuity (cut) along the real line, starting at $s \geq 4m^2$

define
$$T_{phys}(s, 0) = \lim_{\epsilon \to 0^+} T(s + i\epsilon, 0)$$

$$T_{phys}(s,0) = \lim_{\epsilon \to 0^+} T(s + i\epsilon, 0)$$



(a) first Riemann sheet

Exercise

- show that for $s < 4m^2$, $T(s, 0)$ is real, at any order in perturbation theory

each vertex carries $\pm i$
each propagator carries $\pm i$
each loop, after Wick rotation, carries $+i$

consider a general diagram with N_V vertices, N_I propagators, N_L loops
up to an overall sign we have

$$\frac{(i)^{N_V+N_I+N_L}}{\prod_I (k_I^2 - m^2 + i\epsilon)} = \frac{(i)^{2N_I+1}}{\prod_I (k_I^2 - m^2 + i\epsilon)} \qquad [N_L = N_I - N_V + 1]$$

prove than none of the internal particles can be on-shell

$$\frac{(i)^{2N_I+1}}{\prod_I (k_I^2 - m^2 + i\epsilon)} \xrightarrow{\epsilon \to 0} \frac{i}{\prod_I (k_I^2 - m^2)} \Rightarrow T(s, 0) \text{ real}$$

unitarity relation at $t \neq 0$

Partial Wave Expansion

$$T(s,t) = 16\pi \sum_{J=0}^{\infty} (2J+1) f_J(s) P_J(\cos\vartheta)$$

Legendre Polynomials

$$f_J(s) = \frac{1}{32\pi} \int_{-1}^{+1} dz\, P_J(z)\, T(s, t(z)) \qquad t \equiv (z-1)\frac{(s-4m^2)}{2}$$

unitarity relation in elastic region: $4m^2 \leq s \leq 9m^2$
[total cross-section -> elastic cross-section]

single relation replaced by an infinite series

$$Im f_J(s) = \sqrt{\frac{s-4m^2}{s}} \left|f_J(s)\right|^2$$

$4m^2 \leq s \leq 9m^2$

$J = 0,1,2,\ldots$

$$Im f_J(s) \geq \sqrt{\frac{s-4m^2}{s}} \left|f_J(s)\right|^2$$

$J = 0,1,2,\ldots$

Exercise

- show that the unitarity relation in the elastic region can be written as

$$|S_J(s)| = 1$$ with $$S_J(s) \equiv 1 + 2i\sqrt{\frac{s-4m^2}{s}}\, f_J(s)$$

- use the previous result to show that in the elastic region the most general solution of the unitarity equation reads

$$f_J(s) = \sqrt{\frac{s}{s-4m^2}}\, sin\delta_J(s) e^{i\delta_J(s)}$$

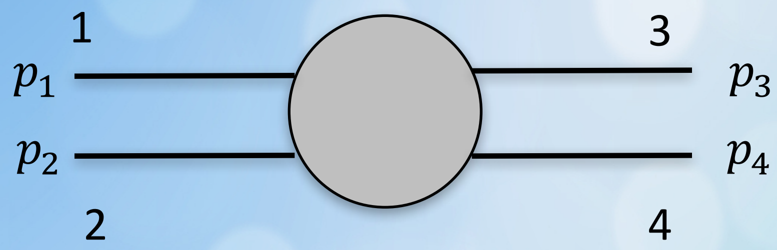
$\delta_J(s)$ (real) scattering phase

- show that the total cross-section can be written as:

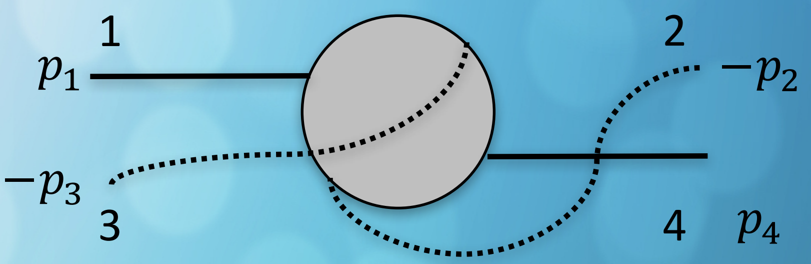
$$\sigma_i(s) = \frac{16\pi}{s}\sum_{J=0}^{\infty}(2J+1)\left|f_J(s)\right|^2$$

crossing symmetry

particles are indistinguishable from anti-particles with the opposite energy and momentum



s-channel



t-channel

$$s = (p_1 + p_2)^2 \to (p_1 - p_3)^2 = t$$
$$t = (p_1 - p_3)^2 \to (p_1 + p_2)^2 = s$$

$$T^{(s)}(s,t) = T(s,t)$$

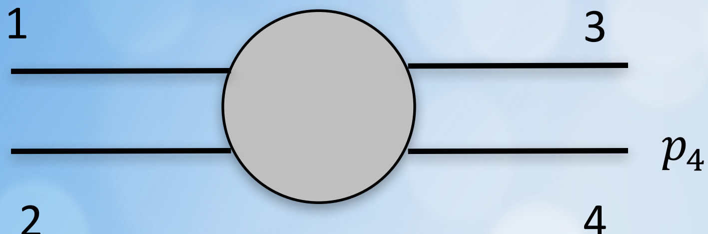
$$T^{(t)}(s,t) = T(t,s)$$

$$T^{(s)}(s,t) = T^{(t)}(s,t)$$

$$T(s,t) = T(t,s)$$

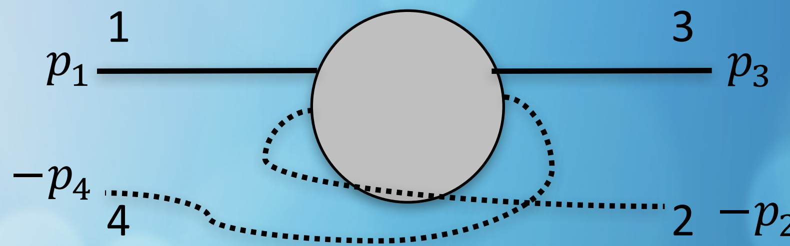
if s,t physical in LHS, they are unphysical in RHS and viceversa

to be precise, the meaning of the equality sign above is that there exists a complex analytic function $T(s,t)$ whose boundary values in their respective physical regions are the two scattering amplitudes.



s-channel

$T^{(s)}(s,t) = T(s,t)$



u-channel

$T^{(u)}(s,t) = T(u,t)$

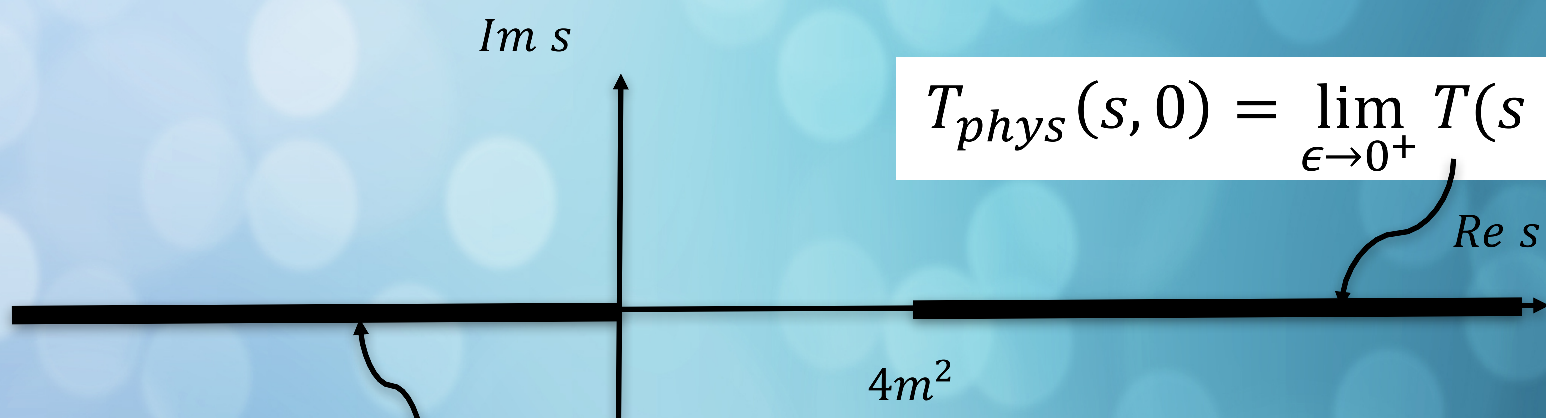
$$T^{(s)}(s,t) = T^{(u)}(s,t)$$

$$T(s,t) = T(u,t)$$

$$\mathrm{u} = 4m^2 - s - t$$

by unitarity this implies $T(s,t)$ has a cut $s \leq -t$

complex s-plane $t = 0$



$$T_{phys}(s,0) = \lim_{\epsilon \to 0^+} T(s + i\epsilon, 0)$$

$$T^{(u)}{}_{phys}(4m^2 - s, 0) = \lim_{\epsilon \to 0^+} T(4m^2 - s + i\epsilon, 0)$$

analyticity

results of a huge collective effort in late 50's and early 60's

1. $T(s,t)$ at fixed $t<0$ is analytic in the complex s–plane, except for the cuts related to unitarity

2. $T(s,t)$ is polynomially bounded in s at large $|s|$

$$|T(s,t)| < |s|^N \qquad |s| \gg m^2$$

1+2 ⟺ dispersion relations

3. analyticity and polynomial boundness of $T(s,t)$ in s can be extended in the region $|t| < 4m^2$

[Martin]

4. $$\big|\big|T(s,0)\big|\big| < 2\pi \frac{s}{m^2} \log^2 \frac{s}{m^2}$$

Froissart bound

$$\lim_{|s|\to\infty} \frac{|T(s,t)|}{s^2} = 0 \qquad |t| < 4m^2$$

working assumption (still to be proven)

Lighest Particle Maximal Analyticity: The $2 \to 2$ scattering amplitude of the lightest particles in the theory, $T(s,t)$, is analytic on the physical sheet for arbitrary complex s and t, except for potential bound-state poles, a cut along the real axis starting at $s = 4m^2$, and the images of these singularities under the crossing symmetry transformations.

Backup Slides

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Exercise

include NLO term in the continuum limit

$$x_{n+1} - x_n - a = \sqrt{\frac{a}{k}} \frac{\partial}{\partial x} \varphi\left(\frac{x}{a}\right)\bigg|_{x=na} + \frac{a}{2}\sqrt{\frac{a}{k}} \frac{\partial^2}{\partial x^2} \varphi\left(\frac{x}{a}\right)\bigg|_{x=na} + \cdots$$

and verify that

- Lorentz-like symmetry is broken

- expansion parameter of the EFT is $\frac{x}{a}$

- what happens with the shift symmetry?