Axi-symmetric neutral dust coupled to gravity

In the Newtonian and general relativistic regimes

Davide Astesiano - 04/11/2021- Università dell'Insubria

This talk is mainly based on

D. Astesiano, S. L. Cacciatori and F. Re,
``Towards a full general relativistic approach to galaxies,''
[arXiv:2106.12818 [gr-qc]].

Abstract:
In order to make realistic models for spiral galaxies we are studying axi-symmetric neutral dust solutions in Einstenian gravity. Our work for now is mainly from the theoretical perspectives.

Reason:
Dark matter has been postulated to solve the discrepancies between the observations and the accepted theories of gravity.
One of these observations is the measured rotation curve of galaxies, that does not seems to be reproduced in the Newtonian theory.
Here we try to study the more general solution of the Einstein theory and then take the low energy limit without restricting it to the Newtonian framework.

The low energy limit

It is obtained imposing the followings

- The possibility of referring the space-time to global coordinates x^0, x^1, x^2, x^3 in which the metric reads $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$;

- We call $\mathscr{I}$ the reference frame (RF) defined by $x^0 =$ var. From the RF point of view, the velocities are little compared to the speed of light: $\dfrac{v^i}{c} \approx h_{\mu\nu}$; and the fields are almost-stationary: $\partial_{x^0} h_{\mu\nu} \approx 0$;

- The dominant contribution to gravitation comes from the inertial mass of the body: $T_{00} = \mu v_0 v_0 = \mu c^2$.

The newtonian limit

We analyse "passive" and "active" aspects of gravitation

The Newtonian limit is selected choosing a diagonal form of the metric.

- In the RF $\mathscr{I}$ a free particle accelerates according to $\dfrac{d^2x^i}{d\tau} = -\Gamma^i_{\nu\rho}u^\nu u^\rho = -\Gamma^i_{00}$. From this result, in analogy with Newton, we get $\Gamma^i_{00} = \partial_{x^i}\Phi$. Then, Γ^i_{00} possesses the information about the Newtonian limit!

- On the other hand, we have $R_{00} = \dfrac{1}{c^2}\Delta\Phi = 4\pi G\mu$.

The gravitomagnetic limit

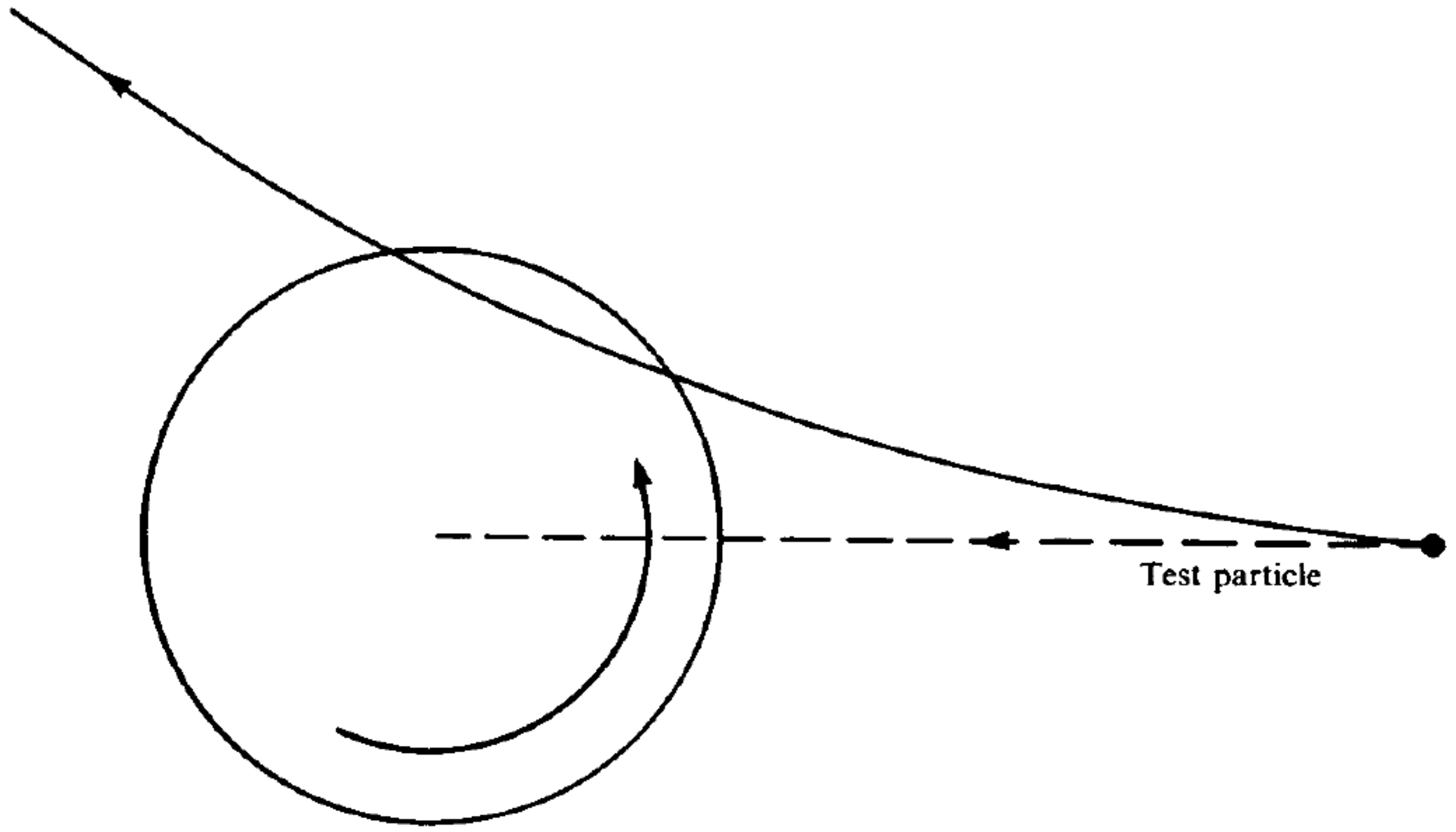
There is no need to choose the specific diagonal (Newtonian) form of the metric. Usually when we allow for non diagonal elements we call it the gravitomagnetism.

First section:
we investigate the differences between the Newtonian theory and General Relativity, when they are coupled to dust in presence of axi-symmetry.

1- Non newtonian effect for stationary axially symmetric systems

Dragging

Let us study a particle approaching the system in the plane $z = 0$



$$M = g_{\phi t}\, p^t + g_{\phi\phi}\, p^\phi$$

2- Non newtonian effect for stationary axially symmetric systems (1/4)

Non newtonian “forces” (W B Bonnor 1977 J. Phys. A: Math. Gen. 10 1673)

- Steady, axially simmetric motion of dust is necessarily cilindrically simmetric according to the newtonian theory.
  Hence the motion is the same in every plane $z =$ constant.

$$(\overrightarrow{v} \cdot \nabla)\overrightarrow{v} = -\nabla\Phi, \qquad \nabla^2\Phi = 4\pi\rho(r, z)$$

$$\partial_z \rho = 0.$$

2- Non newtonian effect for stationary axially symmetric systems (2/4)

Non newtonian “forces”- gravitoelectromagnetism

- This result is not true in GR. Consider a massive body with centre O, rotating about Oz: then, it is possible for a test particle P to describe a circle in the plane perpendicular to Oz, with centre C on Oz but different from O. The spin of the central body causes a force to act on P parallel to the z-axis which can balance the component of the gravitational attraction in that direction.

2- Non newtonian effect for stationary axially symmetric systems (3/4)

Non newtonian “forces”- gravitoelectromagnetism

The standard solution of the linear approximation to the vacuum equations in GR, will apply to any stationary rotating spherical mass

$$ds^2 = -(1-2\frac{m}{r})dt^2 + (1+2\frac{m}{r})dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - 4\frac{ma}{r}\sin^2\theta dt d\phi\,.$$

There are geodesics which are non-equatorial circular orbit.

2- Non newtonian effect for stationary axially symmetric systems (4/4)

Non newtonian “forces”- gravitoelectromagnetism

The geodesic can be written as

$$r = r_0, \qquad \theta = \theta_0, \qquad \phi = 4\frac{ma}{r_0^3}[1-(2\frac{m}{r_0^3})]^{-1/2}s, \qquad t = [1-(2\frac{m}{r_0^3})]^{-1/2}s\,.$$

A non-newtonian force, arising from the spin h of the central body, permits non-equatorial circular orbits. At first order in h, the force on a test particle of mass M is

$$F_{NN} = 6GMhc^{-2}r^{-2}\Omega\sin^2\theta\cos\theta,$$

Ω being the test particle’s orbital angular velocity. This is true under the condition

$$\sin^2\theta_0 = r_0^3(24ma^2)^{-1}\,.$$

3- Non newtonian effect for stationary axially symmetric systems

Gravitoelectromagnetism - Ludwig, G.O. Galactic rotation curve and dark matter according to gravitomagnetism. ***Eur. Phys. J. C*** **81, 186 (2021)**

Using cylindrical coordinates r, ϕ, z and in presence of stationarity and symmetry around ϕ, in the gravitomagnetic limit the z-balance equation becomes

$$g_{0i} = \overrightarrow{S}, \qquad v = v_\phi \hat{\phi}, \qquad \partial_z \Phi + \frac{v_\phi}{2} \partial_z S_\phi = 0.$$

v_{ϕ} is the velocity of the dust and S_{ϕ} are off diagonal elements of the metric.

Second section: General relativistic regime

General relativity coupled with stationary axi-symmetric neutral dust

Exact Solutions of Einstein's Field Equations. ***By H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt*****. Second Edition. 732 p., Cambridge University Press, Cambridge, 2003.**

$$u = (-H)^{-1/2}(\partial_t + \Omega\partial_\phi), \qquad H = H(\eta),$$

$$\Omega = \frac{1}{2}\int H' \frac{d\eta}{\eta},$$

$$\mathscr{F} = 2\eta + r^2 \int \frac{H'}{H}\frac{d\eta}{\eta} - \int \frac{H'}{H}\eta\, d\eta,$$

$$\begin{cases} g_{tt} &= \dfrac{(H-\eta\Omega)^2 - r^2\Omega^2}{H} \\ g_{t\phi} &= \dfrac{\eta^2 - r^2}{-H}\Omega + \eta \\ g_{\phi\phi} &= \dfrac{r^2-\eta^2}{-H} \end{cases},$$

$$\mathscr{F}_{rr} - \frac{1}{r}\mathscr{F}_r + \mathscr{F}_{zz} = 0$$

General relativity coupled with stationary axi-symmetric neutral dust-Corotating case (H=-1)

H. Balasin and D. Grumiller, ``Non-Newtonian behavior in weak field general relativity for extended rotating sources,'' Int. J. Mod. Phys. D 17 (2008)

In the co-rotating case, the system boils down to the following equations

$$u = \partial_t, \qquad ds^2 = -\left(dt - Nd\phi\right)^2 + r^2 d\phi^2 + e^{\nu}\left(dr^2 + dz^2\right)$$

with the condition

$$N_{rr} - \frac{1}{r}N_r + N_{zz} = 0$$

General relativity coupled with stationary axi-symmetric neutral dust-Corotating case (H=-1) (1/5)

H. Balasin and D. Grumiller, ``Non-Newtonian behavior in weak field general relativity for extended rotating sources,'' Int. J. Mod. Phys. D 17 (2008)

Using the separation ansatz N= R(r) Z(z), one arrives at

$$N(r,z) = \int_0^\infty d\lambda cos(\lambda z)(r\lambda)\left[A(\lambda)K_1(\lambda r) + B(\lambda)I_1(\lambda r)\right]$$

The functions K_1 and I_1 are modified Bessel functions of the first and second kind. We note that I_1 blows up exponentially for large values of r.

General relativity coupled with stationary axi-symmetric neutral dust-Corotating case (H=-1) (2/5)

A Toy model for galaxies

A toy model can be obtained imposing

$$N(r,z) = V_0(R - r_0) + \frac{V_0}{2}\sum_{\pm}\left(\sqrt{(z \pm r_0^2) + r^2} - \sqrt{(z \pm R^2) + r^2}\right)$$

using the bulge radious r_0 and R bigger than the size of the stellar disk.

General relativity coupled with stationary axi-symmetric neutral dust-Corotating case (H=-1) (3/5)

A Toy model for galaxies

It is possible to show that the 3-velocity as seen by an observer who is at rest with respect to the center of the galaxy is

$$V(r,z) = \frac{N(r,z)}{r},$$

then in this specific case, in the galaxy z=0 equatorial plane we have

$$V(r,0) = \frac{V_0}{r}\left((R - r_0) + \sqrt{r_0^2 + r^2} - \sqrt{R^2 + r^2}\right)$$

A Toy model for galaxies

then in this specific case, in the galaxy z=0 equatorial plane we have

$$V(r,0) = \frac{V_0}{r}\left((R - r_0) + \sqrt{r_0^2 + r^2} - \sqrt{R^2 + r^2}\right)$$

$$r_0 = 1kpc, \quad R = 100kpc$$

This reproduces the behaviour of the velocity profile!

Energy density

The ratio between the newtonian and the relativistic densities are

$$\frac{\rho_E}{\rho_N}(r, z = 0) = \beta \left(1 + \frac{r^2 v'^2}{v^2 + 2rvv'} \right).$$

Clearly, at each point in the galaxy we may achieve perfect agreement between Newtonian and GR predictions by choosing β appropriately. This reflects the local validity of the Newtonian approximation. However β can only be chosen once.

Anyhow, the co-rotating class is very small compared to all the possible solutions. Let us focus on the differentially rotating solutions.

General relativity coupled with stationary axi-symmetric neutral dust- Differentially rotating case (1/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

Let us compare the possible velocities field between the two classes.
We find for the differentially rotating and the corotating cases respectively

$$l'\left(\frac{1}{v}-v\right)[r^2v_z^2+(rv_r+v)^2]+l\left[\left(\frac{1}{v}-v\right)(rv_{zz}+rv_{rr}+3v_r)-\left(\frac{1}{v}+v\right)\frac{r}{v}(v_z^2+v_r^2)+\frac{2}{r}\right]+2\left(v_{zz}+v_{rr}+\frac{v_r}{r}-\frac{v}{r^2}\right)=0,$$

and

$$\left(v_{zz}+v_{rr}+\frac{v_r}{r}-\frac{v}{r^2}\right)=0,$$

where

$$l(\eta(r,z)) := \partial_\eta H/H\,.$$

General relativity coupled with stationary axi-symmetric neutral dust-Differentially rotating case (2/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

Let us compare the possible velocities field between the two classes.
We define the deformation and the vorticity tensor

$$\mathbf{P} = (u_{\mu;\nu} + u_{\nu;\mu})dx^\mu \otimes dx^\nu, \quad \mathbf{W} = (u_{\mu;\nu} - u_{\nu;\mu})dx^\mu \wedge dx^\nu,$$

Choosing for simplicity the "slightly" non co-rotating case

$$H(\eta) := -e^{p^2\eta^2} = -1 - p^2\eta^2 + o(p^2),$$

the squares of these vectors become

$$P^2 := P_{\mu\nu}P^{\mu\nu} = 2p^4\frac{r^2}{g_{rr}}\left(\eta_r^2 + \eta_z^2\right) + o(p^4), \quad W^2 := W^{\mu\nu}W_{\mu\nu} = \frac{2}{r^2 g_{rr}}\left(\eta_r^2 + \eta_z^2\right)(1 - p^2\eta^2) + o(p^2)\,.$$

General relativity coupled with stationary axi-symmetric neutral dust- Differentially rotating case (3/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

The deformation and the vorticity squares tensor are in the “slightly” non co-rotating case

$$H(\eta) := -e^{p^2\eta^2} = -1 - p^2\eta^2 + o(p^2),$$

$$P^2 := P_{\mu\nu}P^{\mu\nu} = 2p^4\frac{r^2}{g_{rr}}\left(\eta_r^2 + \eta_z^2\right) + o(p^4), \quad W^2 := W^{\mu\nu}W_{\mu\nu} = \frac{2}{r^2 g_{rr}}\left(\eta_r^2 + \eta_z^2\right)(1 - p^2\eta^2) + o(p^2)\,.$$

One may ask if the differential rotation is just a different gauge from the co-rotating case, but from these invariants we see that this is not the case.
While in the co-rotating case the fluid behaves like a “rigid body”,
in the differentially rotating case we introduce a physical local additional degree of freedom.

General relativity coupled with stationary axi-symmetric neutral dust-Differentially rotating case (4/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

In spiral galaxies the velocity profile shows an approximately flat profile in a certain region, it is interesting to see that this solution is allowed

$$H(r,z) = -B\, r^{\frac{2v^2}{1+v^2}} \exp(-A\, r^{1-2\gamma^2}),$$

$$E = E_{\frac{4v^2}{(v^2+1)^2}}\left(Ar^{-\frac{v^2+1}{1-v^2}}\right) \quad s.t. \quad E_n(z) = \int_1^\infty e^{-zt}/t^n dt\,.$$

$$g_{tt} = -\frac{B}{4}\frac{1-v^2}{v^2} r^{\frac{2v^2}{v^2+1}} e^{-Ar^{\frac{v^2+1}{1-v^2}}} \times \left(2e^{Ar^{\frac{v^2+1}{1-v^2}}} E + \frac{\left(1-v^2\right)^2 E^2}{\left(1+v^2\right)^2} + e^{2Ar^{\frac{v^2+1}{1-v^2}}}\right)$$

$$g_{t\phi} = \frac{r}{2\left(v^3+v\right)}\left[\left(1-v^2\right)^2 e^{Ar^{\frac{v^2+1}{1-v^2}}} E + \left(v^2+1\right)^2\right]$$

$$g_{\phi\phi} = -\frac{\left(1-v^2\right)}{B} r^{\frac{2}{v^2+1}} e^{-Ar^{\frac{v^2+1}{1-v^2}}}$$

General relativity coupled with stationary axi-symmetric neutral dust-Differentially rotating case (5/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

An interesting class of approximated solutions is given by

$$H(r,z) = -e^{lvr} \approx -(1+lvr), \quad H' \approx -l, \quad H'' \approx 0, \qquad l = av_c/R_g$$

In the inner part, where r << R_g we are in the co-rotating regime.
Where, on the other hand, r is comparable with R_g, the approximately constant velocity solution is allowed

$$ds^2 = -\left(dt - Nd\phi\right)^2 + r^2 d\phi^2 + \zeta\left(dr^2 + dz^2\right), \quad N = \frac{a}{2} v_c \log(r/r_0)\frac{r^2}{R_G} + vr, \quad u = \left(\partial_t - \frac{a}{2}\frac{v_c}{R_G}\log\left(\frac{r}{r_0}\right)\partial_\phi\right).$$

General relativity coupled with stationary axi-symmetric neutral dust-Differentially rotating case (6/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

In this class of approximated solutions

$$H(r,z) = -e^{lvr} \approx -(1+lvr), \quad H' \approx -l, \quad H'' \approx 0, \qquad l = av_c/R_g$$

the density in this regime compared to the co-rotating density with the same velocity profile is

$$\frac{\rho_{nCor}}{\rho_{Cor}} = \frac{\left(4 - a^2(r/R_G)^2(v_c/v)^2\right)}{4} \approx 1 - \frac{a^2}{4}\left(\frac{r}{R_G}\right)^2\left(\frac{v_c}{v}\right)^2$$

the non co-rotating case further decrease the amount of energy density required for explaining the velocity profile compared to the co-rotating case.

General relativity coupled with stationary axi-symmetric neutral dust-Differentially rotating case (7/7)

D. Astesiano, S. Cacciatori and F. Re, ``Towards a full general relativistic approach to galaxies,'' [arXiv:2106.12818 [gr-qc]]

In general the density is given by

$$l(\eta(r,z)) := \partial_\eta H/H, \qquad 8\pi G\rho = \frac{v^2(2-\eta l)^2 - r^2 l^2}{4g_{rr}} \frac{\eta_r^2+\eta_z^2}{\eta^2}.$$

here we can see the effect of the choice of H, and how it can decrease the total amount of energy density. Following this interpretation we claim that the gravitational field is pulling the dust, decreasing the needed amount of density compared to the newtonian and co-rotating model: the "energy-momentum" of the gravitational field is providing in this sense dark matter effects.

Third section: Conclusions

There are others studies reaching analog conclusions, as

1-F.I. Cooperstock, S. Tieu, Galactic dynamics via general relativity: a compilation and new developments. Int. J. Mod. Phys. A 22, 2293–2325 (2007)

2- H. Balasin and D. Grumiller, ``Non-Newtonian behavior in weak field general relativity for extended rotating sources,'' Int. J. Mod. Phys. D 17 (2008)

3-Mariateresa Crosta, Marco Giammaria, Mario G Lattanzi, Eloisa Poggio, On testing CDM and geometry-driven Milky Way rotation curve models with *Gaia* DR2, *Monthly Notices of the Royal Astronomical Society*, Volume 496, Issue 2, August 2020, Pages 2107–2122

4-Ludwig, G.O. Galactic rotation curve and dark matter according to gravitomagnetism. *Eur. Phys. J. C* 81, 186 (2021)

Although these works reached analog conclusions, our research make use of an additional degree of freedom which can be choose independently from the matter. This allows us to draw some extra conclusions.

1- In the light of our analysis, we claim that the gravitational field can account for dark matter effects.

2- This freedom in the choice of H or, as we suggested, the “energy-momentum” of gravity is totally in accordance with the fact that different galaxies seem to show different amount of dark matter.

3- The equations are also suggesting something regarding the evolution of these systems, in fact…

4- Since Crosta et al. (2018) shows that the Milky Way can be well described by the Balasin-Grumiller model, we can conclude that our galaxy is almost co-rotating with the gravity. This is someway intuitive, because it is a big and old galaxy, so that the matter and the field had enough time to exchange angular momentum, possibly reaching the same rotation. Such a mechanism would include a breaking of stationarity and would deserve further study, which we mean to tackle in a following work. If true, it would also suggest an higher probability to find more dark component in younger galaxies, a fact that could be investigated.
More work is in order, which could possibly allows us to better understand the dynamics introduced by the momentum and inertia of the gravitational field, providing new explanations for some unsatisfactory models.

Since the co-rotating model and the differentially rotating model can have the same velocity profile, how can we decide which one is better to model a galaxy?
1) Measure the components of P and W.
2) Measure the energy density.
3) Using the precession of gyroscopes.

$$e^0 = \frac{r}{\sqrt{g_{\phi\phi}}}dt, \qquad e^1 = \sqrt{g_{\phi\phi}}(d\phi - \chi dt), \qquad e^2 = e^{\mu/2}\,dr, \qquad e^3 = e^{\mu/2}\,dz,$$

$$\chi \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{H\eta}{(r^2 - \eta^2)} + \Omega\,.$$

The corresponding gyroscopes precess relatively to the orthonormal frame with angular velocity

$$\omega \propto \partial_\alpha \chi, \qquad \alpha = r, z\,.$$

$$\Delta\omega = \omega_{nCor} - \omega_{Cor} \propto \frac{\eta}{r^2 - \eta^2}\partial_\alpha H + \partial_\alpha \Omega + \Delta H\,\partial_\alpha\left(\frac{\eta}{r^2 - \eta^2}\right),$$

$$\Delta H = H_{nCor} - H_{Cor}$$

Thanks for your attention!