# Climbing the mountain: the electron $g$-2

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Inspired by precision
Bologna
10 Dec 2021

# Anomalous magnetic moment of a lepton

$$\vec{\mu} = g \frac{e\hbar}{2mc} \vec{s} \qquad \text{magnetic moment } \mu \ , \quad \text{spin } s$$

$g$ giromagnetic ratio (adimensional)

$$g = \begin{cases} 1 & \text{classical result} \quad \text{(rotating charged sphere, no intrinsic spin)} \\ 2 & \text{from Dirac equation} \quad \text{(no self-interaction)} \\ 2.002319\ldots & \text{Quantum ElectroDynamics} \end{cases}$$

$$g = 2(1 + a)$$

$$\boxed{a = \frac{g-2}{2} \qquad \text{anomaly}}$$

$$a_e^{SM} = a_e^{QED}(\text{ mass-independent}) + a_e^{QED}(\text{ mass-dependent}) + a_e(\text{hadr}) + a_e(\text{weak})$$

- The mass independent part is universal for all leptons, and notoriously the most difficult to calculate.

$$a^{\text{QED}}(\text{ mass-independent}) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$C_i$ pure numbers $\qquad$ $\alpha$ fine structure constant

$$a_e^{SM} = a_e^{QED}(\text{ mass-independent}) + a_e^{QED}(\text{ mass-dependent}) + a_e(\text{hadr}) + a_e(\text{weak})$$

$$a_e^{QED}(x) = C_1^{(x)}\left(\frac{\alpha}{\pi}\right) + C_2^{(x)}\left(\frac{\alpha}{\pi}\right)^2 + C_3^{(x)}\left(\frac{\alpha}{\pi}\right)^3 + C_4^{(x)}\left(\frac{\alpha}{\pi}\right)^4 + C_5^{(x)}\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$ $\alpha$ fine structure constant

| | |
|---|---|
| QED 1-loop = | $1\ 161\ 409\ 731.851 \pm 0.093^{\alpha} \times 10^{-12}$ (error due to $\alpha$) |
| QED 2-loop = | $-\ 1\ 772\ 305.060 \pm 0.000 \times 10^{-12}$ |
| QED 3-loop = | $14\ 804.203 \times 10^{-12}$ |
| QED 4-loop = | $-\ 55.667 \times 10^{-12}$ |
| QED 5-loop = | $0.456 \pm 0.011 \times 10^{-12}$ |
| QED, 2-loop, $\mu$ = | $2.738 \times 10^{-12}$ |
| QED, 2-loop, $\tau$ = | $0.009 \times 10^{-12}$ |
| hadronic v.p.,2-loop = | $1.8490 \pm 0.0108 \times 10^{-12}$ |
| hadronic v.p.,3-loop = | $-\ 0.2213 \pm 0.0012 \times 10^{-12}$ |
| hadronic v.p.,4-loop = | $0.0280 \pm 0.0002 \times 10^{-12}$ |
| hadronic l-l = | $0.0370 \pm 0.0050 \times 10^{-12}$ |
| weak = | $0.03053 \pm 0.00023 \times 10^{-12}$ |





$\alpha^{-1}(Rubidium : 2020) = 137.035\ 999\ 206(11)$ $(0.08\ ppb)$

- Dominated by QED mass independent contributions

The precision of the most precise measurement of $a_e$ is

$$a_e^{exp} = \quad 1\,159\,652\,180.730 \pm 0.280 \times 10^{-12} \qquad (0.24\ ppb) \qquad \text{(Gabrielse 2008)}$$

Before 2018, the precision of $a_e^{exp}$ was higher than that of the measurements of $\alpha$, so the relation between $a_e$ and $\alpha$ was inverted in order to infer a value of $\alpha$ from $a_e^{exp}$, assuming the validity of the theory. Since 2018 the situation has changed. There are currently two new measurements of $\alpha$ more precise

$$\alpha^{-1}(Cs : 2018) = 137.035\,999\,046(27) \qquad (0.20\ ppb)$$

$$\alpha^{-1}(Rb : 2020) = 137.035\,999\,206(11) \qquad (0.08\ ppb)$$

The two determinations of $\alpha$ are in disagreement each other at $5.4\sigma$.

$$a_e^{SM}(\alpha) = \quad 1\,159\,652\,181.606 \pm \overset{\text{5-loop}}{0.011} \pm \overset{hadr}{0.012} \pm \overset{\alpha}{0.228} \times 10^{-12} \qquad (Cs : 2018)$$

$$a_e^{SM}(\alpha) = \quad 1\,159\,652\,180.254 \pm 0.011 \pm 0.012 \pm 0.093 \times 10^{-12} \qquad (Rb : 2020)$$

$$a_e^{exp} - a_e^{SM}(\alpha) = -0.88 \pm 0.36 \times 10^{-12} \quad (2.4\ \sigma) \qquad (Cs : 2018)$$

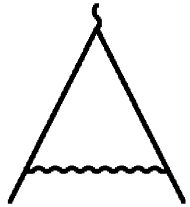
$$a_e^{exp} - a_e^{SM}(\alpha) = +0.47 \pm 0.30 \times 10^{-12} \quad (1.6\ \sigma) \qquad (Rb : 2020)$$

Note the different signs

| loop | vertex diagrams | number of internal lines |
|---|---|---|
| 1 | 1 | 3 |
| 2 | 7 | 6 |
| 3 | 72 | 9 |
| 4 | 891 | 12 |
| 5 | 12672 | 15 |
| 6 | 202770 | 18 |

Number of diagrams grows factorial-like

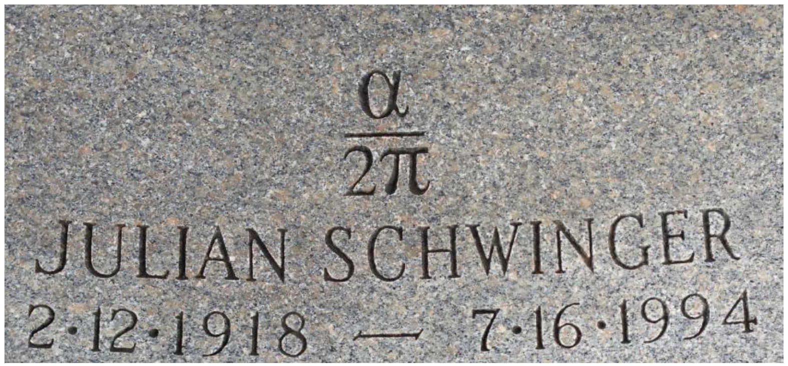
$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$



1 diagram

$$C_1 = \frac{1}{2}$$

Obtained by Julian Schwinger in 1948



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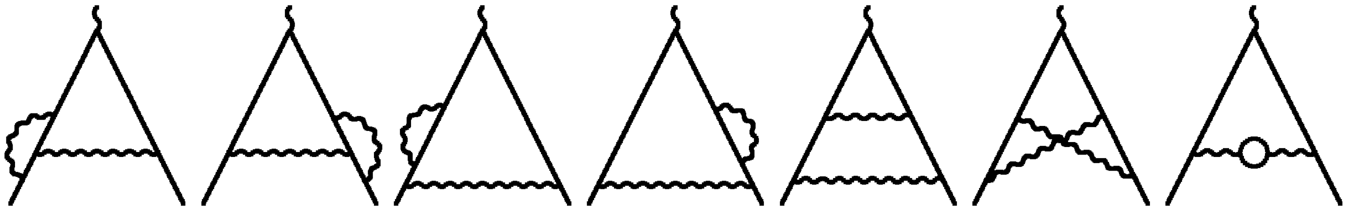


7 diagrams

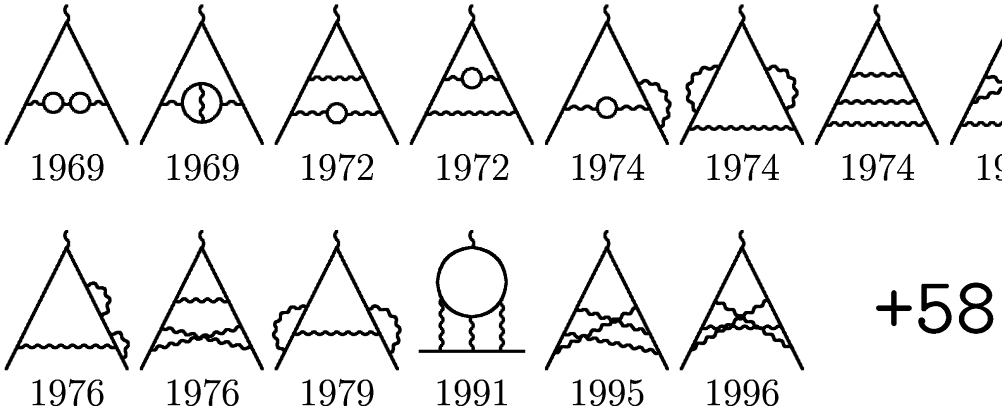
$$\begin{aligned} C_2 = & \ \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \\ = & \ -0.328\,478\,965\,579\ldots \end{aligned}$$

obtained independently by Petermann and Sommerfield in 1957.

(The two-loop coefficient was also computed analytically by Karplus and Kroll in 1950, but unfortunately their result was wrong)

# QED Mass-independent term: 3-loop contribution

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$




72 diagrams

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$
$$= 1.181\ 241\ 456\ 587\ 200\ 006\ldots$$

- The final analytical expression was obtained by S.L. and Ettore Remiddi in 1996.
- Ettore Remiddi begun the analytical calculation of $C_3$ in 1969. I joined him and his group in Bologna in 1989 as a graduate student.
- In 1989 there were 21 diagrams (3groups) still not known analytically. It took us 7 years to complete the analytical calculations.

At the beginning of 1984, I was undergraduate student in his last year. I was in search for a thesis. Sandro Turrini taught the exercise part of the Theoretical Physics course, whose main part was held by Roberto Odorico. Sandro was a collaborator of Ettore and he advertised a lot for their calculations of Feynman diagrams. So, one afternoon I meet Ettore for the first time in order to ask for a thesis. I remember he drew one family of diagrams (the 3-loop corner-ladder) that he was computing at that time. Unfortunately he was about to leaving for USA, for a sabbatical year. He said that following a new student via mail was possible, even if complicated, and his collaborators (e.g. Turrini) were available for explanations. After some discussion with Turrini I decided that the situation had too complications, and instead I asked for (phenomenological) thesis to Prof.Roberto Odorico. I regret that decision. By the way, in the published paper with R.Odorico that followed the thesis work to my surprise this appeared:

The inadequacy of the $O(\alpha)$ approximation in the treatment of soft and collinear radiation in $Z^\circ$ and $W^\pm$ leptonic decays has not stopped some authors [3] from making a quantitative use of it in kinematic regions which lie beyond its actual limit of applicability. The ensuing claims can only be taken at a qualitative level*.

* We must point out that the statement made in the abstract of ref. [3] (and repeated in the text), according to which with the definition of elastic events there given the ratio of bremstrahlung to elastic events "increases remarkably", is false. Rather, the opposite is true. Also, the passing affirmation made at the end of ref. [3], that collinear singularities are hard to be properly dealt with by Monte Carlo techniques, is inconsistent with all the literature on the subject dealing with QED and QCD applications.

[3] M. Caffo, R. Gatto and E. Remiddi, Phys. Lett. 139B (1984) 439

In the first days of 1989, after the military service, I had won the selection for the Ph.D., and I met Ettore for the second time asking to work with him. He was *very understanding* about the paper with the criticism. It was a strange time as a couple of weeks before, a collaborator of him, Alfred Hill, was killed in the Lockerbie bombing. He accepted me, with the argument of my work being the calculation of the $g$-2 contribution from the 3-loop"light-light"diagrams; As my first assignment he presented me an integral symbolically indicated

$$db * irootb * irab1 = \int_4^{(\sqrt{a}-1)^2} \frac{db}{\sqrt{b(b-4)}R(a,b,1)}$$

(and elliptic integral) and asked me to write a routine to compute numerically it, giving also to me his code for gaussian integration. The morning after I presented him instead program which used the arithmetic-geometric mean, and apparently he was satisfied.

I had to adapt to a new strange world. A morning I was in the terminal room, and he went in, and he saw me looking alternatively to two adjacent monitors.

Ettore:"What are you doing?"
Me:"I'm copying a 1-line expression from a terminal to another"
Ettore:"Don't do it in this way, it is dangerous, you could make a mistake. Send it via mail and include the file"
It seemed to me a over-complication, but in time I learned that such a level of safety (and precision) was necessary for such a calculations.

At the end of 1989 I was accumulating the first analytical results from light-light diagrams Ettore suggested me this kid of identities: (excerpt from my Ph.D. thesis)

$$\frac{\partial}{\partial q_\mu}\frac{1}{(q^2+1)((q-k_1)^2+1)((q-k_2)^2+1)} =$$

$$= \frac{V_\mu}{(q^2+1)((q-k_1)^2+1)((q-k_2)^2+1)}$$

$$V_\mu = -2\left(\frac{q_\mu}{(q^2+1)} + \frac{q_\mu - k_{1\mu}}{((q-k_1)^2+1)} + \frac{q_\mu - k_{2\mu}}{((q-k_2)^2+1)}\right) \qquad (35.1.5)$$

Integrando in $q$ si ha:

$$\int \frac{d^4q}{(q^2+1)((q-k_1)^2+1)((q-k_2)^2+1)} V_\mu = 0 \qquad (35.1.6)$$

Si può saturare questa identità con $p_\mu, k_{1\mu}, k_{2\mu}$, moltiplicandola per un fattore non dipendente da $q$, poi integrare su $d^4k_1$ , $d^4k_2$ . Come esempio:

$$\int \frac{d^4q}{(q^2+1)((q-k_1)^2+1)((q-k_2)^2+1)} V_\mu p_\mu = 0 \Rightarrow$$

$$I\left(\frac{pq}{(q^2+1)}\right) + I\left(\frac{pq}{(q-k_1)^2+1}\right) + I\left(\frac{pq}{(q-k_2)^2+1}\right) - I\left(\frac{pk_1}{(q-k_1)^2+1}\right) +$$

$$-I\left(\frac{pk_2}{(q-k_2)^2+1}\right) = 0 \qquad (35.1.7)$$

Saturando con $q_\mu$ l'identità (35.1.6) diviene leggermente differente:

$$\frac{\partial}{\partial q_\mu}\frac{q_\mu}{(q^2+1)((q-k_1)^2+1)((q-k_2)^2+1)} = (4 + q_\mu V_\mu) \qquad (35.1.8)$$

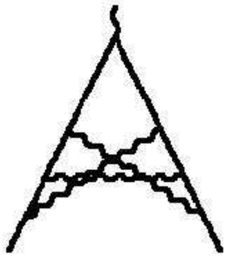
These actually were simple integration by parts identities in 4 dimensions.

In 1992, after the visit of David Broadhurst, Ettore (very prudently) suggested me to try to generate *a single* identity between integrals in $d$ dimensions, insert in it the analytical values already known, and look for something useful (new) in 4 dimensions.

Instead, I did not follow his suggestion (sorry Ettore!), and I did the exact opposite.

I considered the generation of *all possible* $d$-dimensional i.b.p. identities, without regarding to 4 dimension results. And this was (in some sense) the very beginning of my algorithm for the resolution for system of i.b.p. identities.

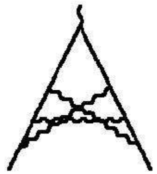
Even in the my first approach (using FORM, editing by hand), the generation required still some human manipulation, so it was difficult to generate a large number; for this reason I was not able to solve some system. The cause was lack of crucial identities, but I realized this only later, in 1994.



After my analytical calculation of corner-ladder diagrams was completed (yes, they were those depicted by Ettore 10 years before) only a single family of diagrams remained unknown, containg a triple cross of photons and being not planar.

We didn't know what approach was working for them. In 1994 Ettore left for a sabbatical year to CERN, so we decided to attack in parallel. He at CERN following the longer (3 cuts) but safer recipe of 2 loops dispersive and 1 loop hyperspherical, me in Bologna attacking it with the more risky recipe 1-loop dispersive and 2-loop hyperspherical. The goal: calculate analytically the simplest scalar integral while avoiding elliptic unfeasible integrations in the intermediate steps.

He won, using a elegant change of variable.
All my tryings in my approach encountered elliptic integrals at some stage of the calculations, so I interrupted my work when he finished. But I published some of the partial results obtained, and some results have been recently found useful in hadronic v.p. calculations (Frael,Passera).



The first triple-cross integral was so computed, and the result was impossibly simple, containg $\pi^4$ and $\pi^2 \ln^2 2$.

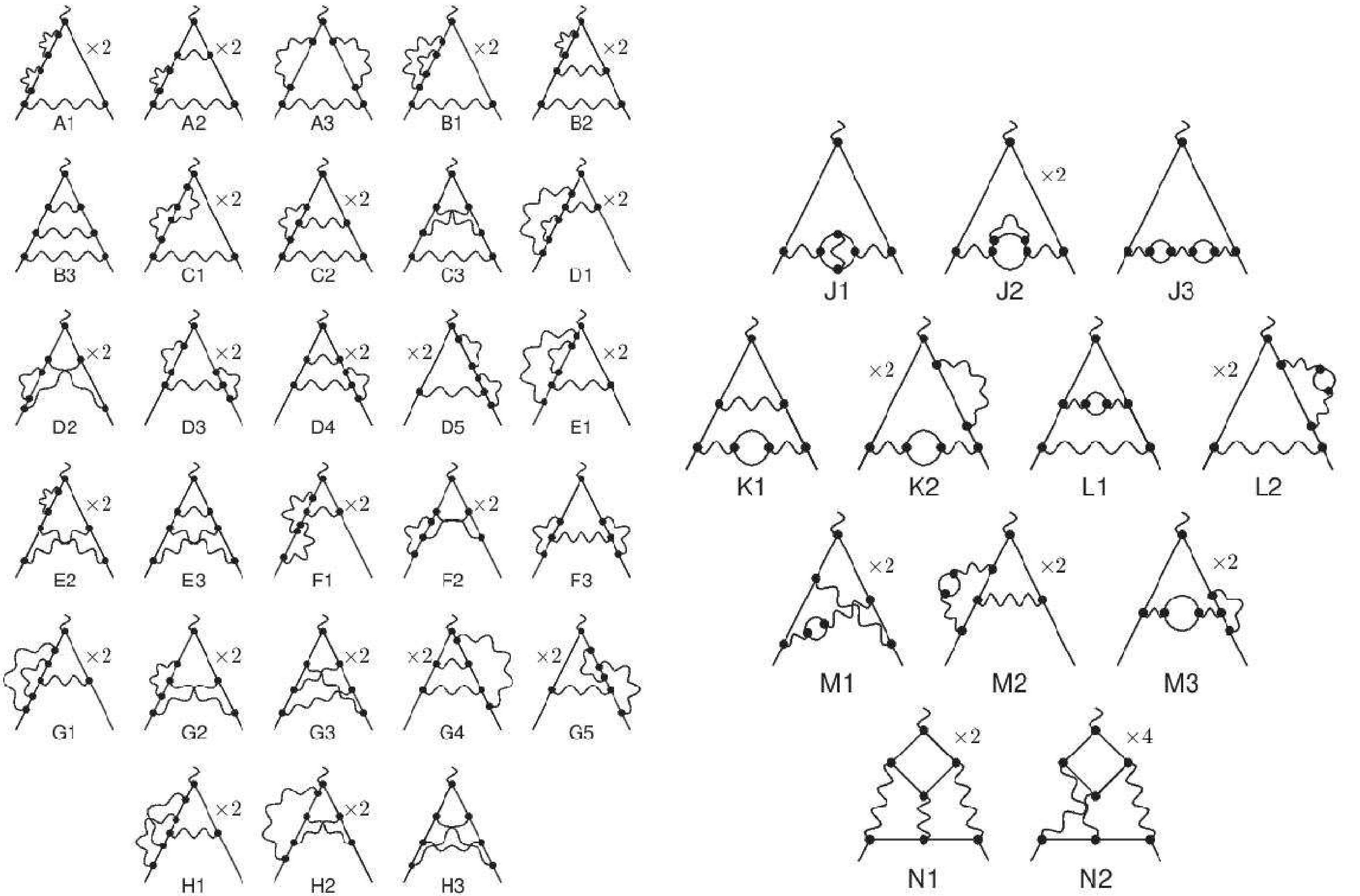
So there was the problem of computing all the other integrals. I decided to give a real try to i.b.p. identities. At that time I had also an all-in-one FORM program which was generating and solving the system, by expanding the rational coefficients, around $d = 4$. It was not clear to me at that time if all identities were useful, so I divided arbitrarily them in classes, throwing away what I didn't like. Only later I understood that no identity is useless. Ettore was informed only that I was processing i.b.p. identities without any details. And at the end I finished, all integrals were reduce to 18 master integrals, and the values othe integrals (Laurent series in $(d-4)$) were obtained from several already known 4-dimensional results. Actually an identity between two master integrals was hidden in the bin of "useless i.b.p. identities", so the master integrals were actually 17. I discovered this to my chagrin one year later.

I was very concerned about a possible theft of the final result, so I decided to do the final steps at home. I informed Ettore that I had obtained the final result, but I showed him only the numerical value and not the analytical expression. He complained *a bit* ~~*a lot*~~, but he was trusting me, and I gave him the final expression only 2 months after, while we were writing the paper.

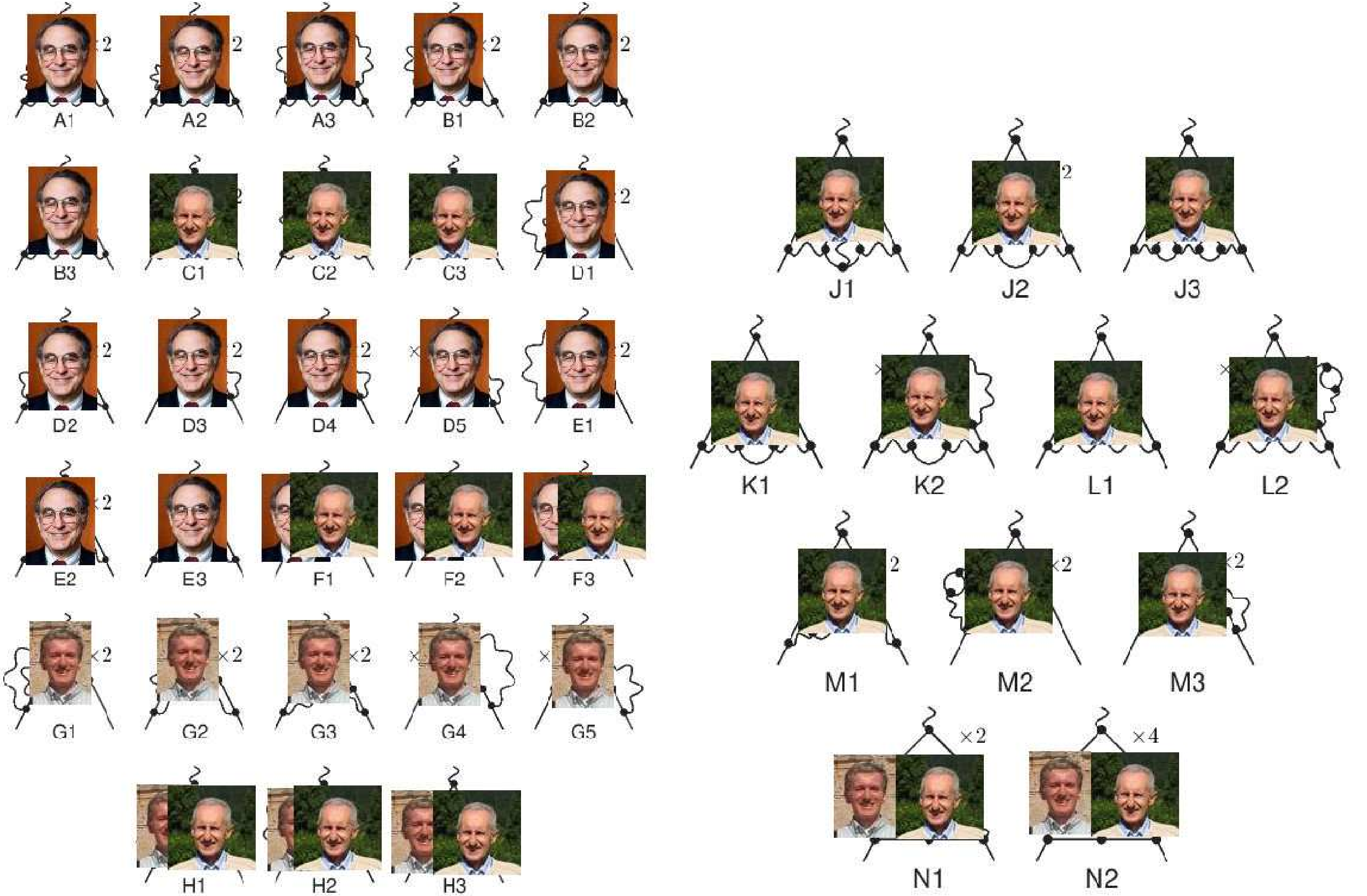
## Brief chronology of the determinations of the 3-loop coefficient

$C_3 = 1.49(20)$ (Levine Wright 1971)
$C_3 = 1.17611(42)$ (Kinoshita 1990)
$C_3 = 1.181259(40)$ (Kinoshita et al. 1995) $+12\sigma$ shift due to an error in a counterterm
$C_3 = 1.181241456587200006\ldots$ (S.L., Remiddi 1996) analytical

# The 3-loop diagrams



# The 3-loop diagrams (with photos)



In 1996, the 4-loop calculation was to be considered. On his suggestion, Ettore and I wrote a paper on using finite master integrals instead of the divergent integrals with Laurent series that we used in the 3-loop paper (and that's when I discovered that the master integrals were actually 17). My fixed-time position in Bologna was unexpectedly terminated in advance, but I was determined to continue my calculation at home. I bought a new PC for the occasion, where to develop my monolythic all-comprehensive program `SYS`. In that period my contacts with Ettore became much more rare. In 1997 he published a paper on differential equations for Feynman diagrams, topic that he had frequently introduced in our discussions since 1993. At a certain point, considering more complicate diagrams, he tried unsuccessfully to solve some systems of i.b.p. identities, and he contacted me asking clarifications, as he didn't know the details of the program used for the triple-cross reduction.

Me: "The trick is ordering the identities: if you get together a system sufficiently large, you reach a critical mass of identities and the system can be solved"

Ettore: "That is, The number of identities grows faster that the number of integrals, then at a certain point the system becomes over-determined and can be solved"

As always, his understanding contained a more insightful and clearer view of the topic.

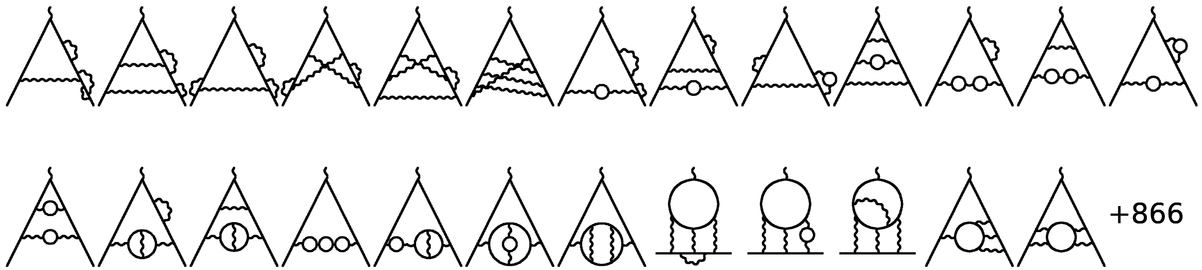
# After the completion of 3-loop calculation

In 1999 I finished working at home the program and the corresponding papers, whose publication speed was slowed for some reasons. More or less in the same time he published a paper with Thomas Gehrmann which contained the information on how solving system of identities. Nevertheless, he always presented the algorithm as originally mine, and advertized it everywhere in the years.
He was developing a series of programs for the solution of system to be applied to the 4-loop reduction, but he stopped the development when he saw at the point my program was advanced.

In the years 2000 I was working in Parma, but I remained also in contact with him, collaborating to some paper (on elliptic 2-loop Feynman integrals, and a particular 4-loop master integral), and to some other problems, like when Giorgio Parisi asked Ettore to calculate some complicate multidimensional integrals, and Ettore asked for my help.
In 2007 when I told him that the bottleneck to the completion of 4-loop contribution was the computing power, he asked Thomas Gehrmann, which at the time I had not yet met, to provide me access to the Zurich computing machines. And it was working on that machines (cluster and supercomputers) that in subsequent $\sim 10$ years I was able to finish the 4-loop calculation.

# QED Mass-independent term: 4-loop contribution

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$




891 diagrams

$C_4 =$

-1.91224576492644557415264716743983005406087339065872534517132984800603844398065170614276089270000363158375584153314732700563785149128545391902804327050273822304345578957045562729309941296699760277782211578472033906415190816652709797086743811501215514797227432216427343192797595860740500578373849607018743283140248380251922494607422985589304635061404922526634310944240002356356881280620645494013224977594300429288836761748899236915180878086989705263578533753776964117024536196013497574494361268486175162606832387186747303831505962741878015305514879400536977798369464278684326918431175889581159743566950433048349073613426586499531163878117434753854234883640855844418822372174567068710418233074305174430557394596117155085896114899526126606124699407311840392747234002346496953173548258481799822409737371077365740464513521123091242528111137215302154453721014811121159848970884223279879720484201445122828451516585236561786594592600991733031721302865467212345340500349104700728924487200616044261325449069000431915198230047488181494311038495378299406296758678753852497819469897931321621979757506767011429048979620850 5...

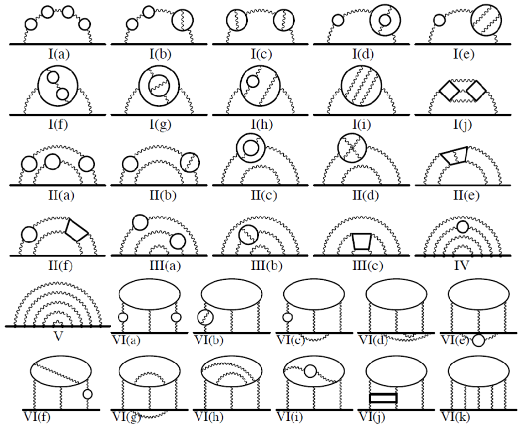
(S.L. 2017)

- This extremely high precision of the result was needed to fit analytically a (very complex) analytical ansatz to the numerical values by using the PSLQ algorithm.
- The successful fit is a strong reliability test of the result.
- 1100 digits is the final total precision; some intermediate fits needed up to 9600 digits of precision

## 4-loop analytical fit: much more complicated than 3-loop result!

$$
\begin{aligned}
C_4 =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
&- \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
&+ \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
&+ \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
&- \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
&+ \sqrt{3}\Big[ -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ 19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&- \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
&+ \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\Big] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
&+ \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&- \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+ \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+ \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\Big(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Big) \\
&+ \sqrt{3}\Big[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\, f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
&+ \pi\Big(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\, f_2(0,0,2) + \frac{5525}{162}\ln 2\, f_2(0,1,1) - \frac{5525}{243}\ln 2\, f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
&- \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\Big)\Big] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\Big(\frac{2541575}{82944}f_1(0,0,2) \\
&- \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\Big) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}.
\end{aligned}
$$

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$




12672 diagrams

$$C_5 = 6.737(159) \quad \text{(Kinoshita et al. 2019)}$$

- Obtained by MonteCarlo numerical integration.
- There is a independent value for the contribution from the subset of all the diagrams without electron loops (Volkov 2019) which disagrees with the corresponding partial result from Kinoshita's group.
- An independent calculation is therefore *very* desiderable. But it would need an huge computing power.

THANKS ETTORE