Nine ways of evaluating a 3-loop diagram

David Broadhurst, Open University, UK, 10 December 2021, virtually at
Inspired by Precision, held in Bologna in honour of **Ettore Remiddi**

1. I begin with a **3-loop** result from Ettore and **Juan Alberto Mignaco** in **October 1968**, when I began my graduate work.

2. I describe a meeting in France in **January 1992**, where I was kidnapped by Ettore. This resulted in a stimulating week in **Bologna**, where we discussed integration by parts for **massive** 3-loop diagrams in **dimensional regularization**.

3. I recall the monumental 3-loop result of Ettore and **Stefano Laporta** in **February 1996** for the electron's magnetic moment.

4. The **Greek** for Ettore is $\mathcal{E}\kappa\tau\omega\rho$: "he who holds all things together". Accordingly, I exhibit **9 ways** of evaluating a **3-loop massive** diagram. Some of the **tools** were ably forged by Ettore.

1: Inspired by precision in 1968

Juan Alberto Mignaco and **Ettore Remiddi**, *Fourth-order vacuum polarization contribution to the sixth-order electron magnetic moment,* CERN TH.953, **30 October 1968**, Il Nuovo Cimento A 60 (1969) 519-529.

Abstract: Il contributo della polarizzazione del vuoto al quarto ordine alla correzione radiativa al **sesto ordine** al momento magnetico dell'elettrone è calcolata **analiticamente** e risulta essere $0.055(\alpha/\pi)^3$.

Their **exact** coefficient of $(\alpha/\pi)^3$ for this **3-loop** contribution was

$$\frac{269}{81} - \frac{434\zeta_2}{135} + \frac{61\zeta_3 - 8\pi^2 \log 2}{18} + \frac{\zeta_4 - 16U_{3,1}}{3}$$
$$= 0.055429177412284346130272752652834435623140476941711076254576 97\ldots$$

with an **alternating double sum** of **weight 4**:

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{1}{2}\int_0^1 \frac{\log^2(x)\log(1-x)\mathrm{d}x}{1+x}$$
$$= \frac{\zeta_4}{2} + \frac{(\pi^2 - \log^2 2)\log^2 2}{12} - 2\sum_{n>0}\frac{1}{2^n n^4}$$
$$= -0.1178759996505093268410139508341376187152175131759750633222452\ldots$$

2: Visiting Bologna in 1992

In January 1992, **Denis Perret-Gallix** (1949-2018) organized a workshop *Artificial Intelligence in High Energy and Nuclear Physics* near Toulon, at La-Londe-Les-Maures. I recall rather vividly **Ettore's greeting**, said with a friendly smile: "you are the person who has been giving seminars saying that I have wasted **20 years of my life** by not using **dimensional** regularization for **massive** diagrams." Of course, I had said no such thing.

I explained that **integration by parts** in $D = 4 - 2\varepsilon$ dimensions had been useful for me, working with massive two-loop two-point integrals and massive three-loop vacuum diagrams. It gave **recurrence relations** that related many terms to a few **master integrals**, which I had obtained as hypergeometric series with parameters of the form **half integers** plus multiples of ε. Then the **Laurent expansion** in ε gave **alternating sums** as well as zeta values. Ettore seemed interested and invited me to join him for a week in **Bologna**, after the workshop. I explained that my family was expecting me to return home. "You must tell them that you have been kidnapped by someone called Hector!"

It was a very enjoyable week in Bologna. I grouped diagrams into families that might have the same number content. I guessed that the only numbers up to weight 4 would be those found by Ettore in 1968. At **weight 5**, I was agnostic.

3: Inspired by precision in 1996

The **magnetic moment** of the electron, in Bohr magnetons, has electrodynamic contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L=3$ loops by

$$
\begin{aligned}
a_0 &= 1 \quad [\textbf{Dirac}, 1928]\\
a_1 &= \frac{1}{2} \quad [\textbf{Schwinger}, 1947]\\
a_2 &= \frac{197}{144} + \frac{\zeta_2}{2} + \frac{3\zeta_3 - 2\pi^2\log 2}{4} \quad [\textbf{Petermann}, \textbf{Sommerfield}, 1957]\\
a_3 &= \frac{28259}{5184} + \frac{17101\zeta_2}{135} + \frac{139\zeta_3 - 596\pi^2\log 2}{18} - \frac{39\zeta_4 + 400U_{3,1}}{24}\\
&- \frac{215\zeta_5 - 166\zeta_3\zeta_2}{24} \quad [\textbf{Laporta}\,\&\,\textbf{Remiddi}, 1996]
\end{aligned}
$$

with 3-loops terms up to weight 4 containing the constants encountered by Ettore in **1968**, as expected, and terms of **weight 5** that contain only **zeta values**, which is a pleasing simplicity for the sum of so many difficult diagrams.

Stefano's amazing semi-analytical **4-loop** result, in **February 2017**, gives **4800 digits** of a_4 and contains many new constants, two of which are **Bessel moments**.

4: Three-loop sunrise, 2007-2018

As a showcase, I choose the 3-loop equal-mass sunrise diagram in $D = 2$ space-time dimensions. In momentum space it gives

$$J(p^2) \propto \left(\prod_{n=1}^{4} \frac{\mathrm{d}^2 k_n}{k_n^2 - m^2 + \mathrm{i}\epsilon}\right) \delta^{(2)}\left(p - \sum_{n=1}^{4} k_n\right)$$

with **4 massive particles** in the intermediate state. For the magnetic moment of the electron, no diagram has an intermediate state with an even number of massive particles, since fermion number is conserved. Nevertheless, Stefano Laporta and I identified this integral, at $p^2 = m^2$, as an instructive **stepping stone**, en route to the 5-fermion intermediate states at 4 loops. The **number theory** at $D = 2$ is the same as for minimal subtraction at $D = 4$, where there are UV divergences.

4.1 Schwinger parameters: Let us set $m = 1$ and normalize the integral as

$$J(t) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{xyz((1+x+y+z)(1+1/x+1/y+1/z) - t - \mathrm{i}\epsilon)}$$

in the cut t-plane with a **branchpoint** at $t = 16$. Since the denominator of the integrand is **quadratic** in all three parameters, we encounter an obstruction to polylogarithmic reduction at the first integration.

4.2 Co-ordinate space: Here we obtain the **Bessel moment**

$$J(t) = 8\int_0^\infty I_0(\sqrt{t}x)K_0^4(x)x\mathrm{d}x, \tag{1}$$

for real $t < 16$. The Bessel function $I_0(z) = \sum_{n\geq 0} z^{2n}/(2^n n!)^2$ grows exponentially at large z, while $K_0(z)$ has a logarithmic singularity at $z = 0$ and falls off exponentially at large z. This **one-dimensional integral** is rather convenient for $16 > t > 0$. In the **space-like** region, with $t < 0$, it becomes less convenient, since $J_0(z) = I_0(\mathrm{i}z)$ is **oscillatory**. On the cut, with $t > 16$, it makes no sense at all.

4.3 Dispersion relation: The discontinuity across the cut is given by **squares** of complete **elliptic integrals** of the first kind and hence by reciprocals of **arithmetic-geometric means** that may be calculated at lightning speed:

$$J(t) = \int_{16}^\infty \frac{f_-^2(s) - 3f_+^2(s)}{s - t - \mathrm{i}\epsilon}\mathrm{d}s, \quad f_\pm(s) = \frac{2\pi}{\mathbf{agm}\left(\sqrt{2\mu}, \sqrt{\mu \pm \sqrt{\nu}}\right)}, \tag{2}$$

$$\mu = \frac{2\alpha + \beta}{2}, \quad \nu = \mu^2 - 48\frac{\alpha - \beta}{\alpha + \beta}, \quad \alpha = \sqrt{s^2 - 4s}, \quad \beta = \sqrt{s^2 - 16s}.$$

I obtained this using a remarkable discovery, made by **Geoff Joyce** in 1973.

4.4 Differential equation: $J(t)$ satisfies a **third-order inhomogeneous** differential equation, with singularities at $t = 0, 4, 16, \infty$. For the **two-loop** sunrise integral $I(w^2) = 4\int_0^\infty I_0(wx)K_0^3(x)x\mathrm{d}x$, the **second-order** differential equation has singularities at $w = 0, 1, 3, \infty$. Joyce's discovery (in condensed matter physics) was that the third-order differential operator is a **symmetric square** of the second-order one, after a transformation of variables. I shall use the transformation

$$t = 10 - w^2 - \frac{9}{w^2}$$

to relate the three-loop and two-loop cases, after **modular parametrization**.

4.5 Modular parametrization at 2 loops: Here the break-through came from **Spencer Bloch** and **Pierre Vanhove** in 2013. I summarize it as follows.

With $q = \exp(2\pi\mathrm{i}z)$ and $\Im(z) > 0$, the **Dedekind eta** function is given by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24} = \frac{\eta(-1/z)}{\sqrt{-\mathrm{i}z}}.$$

With $\eta_n = \eta(nz)$ and $\chi_3(n) = 0, 1, -1$, for $n = 0, 1, 2 \bmod 3$, we may expand

$$\begin{aligned}
\frac{w}{3} &= \frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2} = 1 + 4q + 12q^2 + 28q^3 + 60q^4 + 120q^5 + O(q^6),\\
\frac{w^2-1}{8} &= \frac{\eta_2^9\eta_3^3}{\eta_1^9\eta_6^3} = 1 + 9q + 45q^2 + 171q^3 + 549q^4 + 1566q^5 + O(q^6),\\
\frac{w^2-9}{72} &= \frac{\eta_2\eta_6^5}{\eta_1^5\eta_3} = q + 5q^2 + 19q^3 + 61q^4 + 174q^5 + O(q^6),\\
f &= \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2} = 1 - 6\sum_{n=1}^{\infty}\frac{\chi_3(n)q^n}{1+q^n},\ \text{at weight 1, for an elliptic integral},\\
g &= \frac{\eta_2^5\eta_3^4\eta_6}{\eta_1^4} = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3} = \sum_{n=1}^{\infty}\frac{n^2(q^n - q^{5n})}{1-q^{6n}},\ \text{at weight 3, for 2 loops},\\
h &= \frac{\eta_2^{16}}{\eta_1^8} - 9\frac{\eta_6^{16}}{\eta_3^8} = \sum_{n=1}^{\infty}\frac{n^3(q^n - 8q^{3n} + q^{5n})}{1-q^{6n}},\ \text{at weight 4, for 3 loops.}
\end{aligned}$$

In 1993, Broadhurst, **Fleischer** and **Tarasov** gave the differential equation for the D-dimensional equal-mass **2-loop sunrise** integral. At $D = 2$, this yields

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{I(w^2)}{6f} = g, \quad I(w^2) = 4\int_0^\infty I_0(wx)K_0^3(x)x\mathrm{d}x,$$

with w^2 the norm of the external momentum. The elliptic **periods**

$$\begin{aligned} f &= \frac{4\sqrt{3}}{\mathbf{agm}\left(\sqrt{(w+3)(w-1)^3}, \sqrt{16w}\right)} = \frac{\sqrt{3}\Im I(w^2 + \mathrm{i}0)}{\pi^2} \text{ for } w > 3, \\ 2zf &= \frac{4\sqrt{-3}}{\mathbf{agm}\left(\sqrt{(w+3)(w-1)^3}, \sqrt{(w-3)(w+1)^3}\right)}, \end{aligned}$$

yield the **nome** $q = \exp(2\pi\mathrm{i}z)$. Their **Wronskian** determines

$$g = \frac{w^2(w^2-1)(w^2-9)f^3}{2^6 3^4} = \frac{\eta_2^5\eta_3^4\eta_6}{\eta_1^4} = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3} = \sum_{n=1}^\infty \frac{n^2(q^n - q^{5n})}{1 - q^{6n}}.$$

With $\chi_6(n) = \chi_2(n)\chi_3(n)$ and $\chi_2(n) = (1 - (-1)^n)/2$, the **solution**

$$\frac{I(w^2)}{f} = \frac{\pi\log(-1/q)}{\sqrt{3}} - 3\sum_{n=1}^\infty \frac{\chi_6(n)}{n^2}\frac{1+q^n}{1-q^n}$$

was obtained by **Bloch** and **Vanhove**, using the finiteness of $I(1) = \pi^2/4$.

4.6 Atkin-Lehner transformations and optimal nomes:

$$z_2 = \frac{2z-1}{6z-2}, \quad z_3 = \frac{3z-2}{6z-3}, \quad z_6 = \frac{-1}{6z}, \quad q_k = \exp(2\pi \mathrm{i} z_k),$$

$$f_2(z) = \frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2}, \quad f_3(z) = \frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3}, \quad f_6(z) = \frac{\eta_1\eta_6^6}{\eta_2^2\eta_3^3},$$

$$-\left(q_k\frac{\mathrm{d}}{\mathrm{d}q_k}\right)^2 \frac{I(w^2)}{6f_k(z_k)} = g_k(z_k),$$

$$g_2(z) = \frac{\eta_1^5\eta_3\eta_6^4}{\eta_2^4}, \quad g_3(z) = \frac{\eta_1^4\eta_2\eta_6^5}{\eta_3^4}, \quad g_6(z) = \frac{\eta_1\eta_2^4\eta_3^5}{\eta_6^4},$$

from which I obtain **alternative expansions**

$$\begin{aligned}
\frac{I(w^2)}{f_2(z_2)} &= I(0) - \sum_{n=1}^{\infty}\frac{6\chi_3(n)}{n^2}\frac{q_2^n}{1+q_2^{2n}},\\
\frac{I(w^2)}{f_3(z_3)} &= I(1) - \sum_{n=1}^{\infty}\frac{6\chi_2(n)}{n^2}\frac{q_3^n}{1+q_3^n+q_3^{2n}},\\
\frac{I(w^2)}{f_6(z_6)} &= -3\log^2(-q_6) + \sum_{n=1}^{\infty}\frac{6}{n^2}\frac{q_6^n}{1-q_6^n+q_6^{2n}}.
\end{aligned}$$

For a given w^2 we choose the **smallest** of the **4 nomes**, for best convergence.

4.7 Three-loop sunrise: In 2008, **Bailey, Borwein,** Broadhurst and **Glasser** used the third-order differential equation to develop the **momentum expansion**

$$J(t) = 8\int_0^\infty I_0(\sqrt{t}x)K_0^4(x)x\mathrm{d}x = 7\zeta_3 + (7\zeta_3 - 6)\frac{t}{16} + (49\zeta_3 - 54)\frac{t^2}{1024} + \ldots \quad (3)$$

about $t = 0$. A neat q-expansion comes from the transformation

$$t = 10 - w^2 - \frac{9}{w^2} = -64\left(\frac{\eta_2\eta_6}{\eta_1\eta_3}\right)^6 = -64q + O(q^2)$$

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3 \frac{J(t)}{(wf/3)^2} = 24h = 24\sum_{n=1}^\infty \frac{n^3(q^n - 8q^{3n} + q^{5n})}{1 - q^{6n}}$$

$$\frac{J(t)}{(wf/3)^2} = 7\zeta_3 + 24\sum_{n=1}^\infty \frac{\phi(n)}{n^3}\frac{q^n}{1 - q^n} \quad (4)$$

with $\phi(n) = 0, \mathbf{1}, 0, -\mathbf{8}, 0, \mathbf{1}$, for $n = 0, 1, 2, 3, 4, 5 \bmod 6$. This expansion works well for $|t| < 8$, where $|q| < \exp(-\pi\sqrt{2}/3) < 0.22742$. On-shell, at $t = 1$, we have **good convergence** with $-q = \exp(-\pi\sqrt{5/3}) < 0.017322$.

4.8 Atkin-Lehner transformation at three loops: For $|t|>8$, it is better to use the more ornate Fourier expansion of

$$h_6(z)=\frac{-64h}{t}=1+2h-30\sum_{n=1}^{\infty}\frac{n^3(q^{2n}+q^{4n}-8q^{6n})}{1-q^{6n}}.$$

Using the **alternative nome** $q_6=\exp(2\pi\mathrm{i}z_6)$, with $z_6=-1/(6z)$, we obtain

$$\left(q_6\frac{\mathrm{d}}{\mathrm{d}q_6}\right)^3\frac{J(t)}{(wf_6(z_6))^2}=-24h_6(z_6)$$

$$\frac{J(t)}{(wf_6(z_6))^2}=-4\log^3(q_6)+24\sum_{n=1}^{\infty}\frac{15\phi(n+3)-\phi(n)}{n^3}\frac{1+q_6^n}{1-q_6^n}. \qquad (5)$$

Bloch, Kerr and **Vanhove** obtained this **alternative solution**, neglecting issues of analytic continuation or efficient convergence, I resolved both issue at Les Houches in June 2014. This simple Pari-GP procedure

```
Z(t)={local(x=2/(sqrt(4-t)+sqrt(16-t)),a=sqrt((1-x)^3*(1+3*x)));
I/2*agm(a,4*x*sqrt(x))/agm(a,sqrt((1+x)^3*(1-3*x)));}
```

determines z and hence the nomes $q=\exp(2\pi\mathrm{i}z)$ and $q_6=\exp(-\pi\mathrm{i}/(3z))$.

4.9 On-shell evaluation at the 15th singular value: As **Laporta** and I discovered at Bielefeld in July 2007, the **on-shell** integral

$$J(1) = \frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30\sqrt{5}} \tag{6}$$

has a stunningly simple evaluation. Its **proof** by Bloch, Kerr and Vanhove, elucidated by **Detchat Samart** in 2016, was **complicated** by expansion in complex q_6. By **contrast**, expansion in real q reduces the burden of proof to showing that

$$\sum_{n=1}^{\infty}\frac{\phi(n)}{n^3}\frac{1-\exp(-\pi\sqrt{5/3}n)}{1+\exp(-\pi\sqrt{5/3}n)} = \frac{\pi^3}{12\sqrt{15}}$$

which results from an **Eichler integral** that is **rational**:

$$\int_0^{\sqrt{1/15}} h\left(\frac{1+\mathrm{i}y}{2}\right) y\mathrm{d}y = \frac{1}{120}, \quad h(z) = \frac{\eta_2^{16}}{\eta_1^8} - 9\frac{\eta_6^{16}}{\eta_3^8},$$

somewhat to the surprise of **Bruce Berndt**.

4.10 Theta series at the 15th singular value: With $b > a \geq 0$ and $c \geq 0$, I define **Bessel moments**

$$M(a,b,c) = \int_0^\infty I_0^a(x) K_0^b(x) x^c \mathrm{d}x.$$

Then the 5-Bessel **matrix**

$$\left[\begin{array}{cc} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{array}\right] = \left[\begin{array}{cc} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{array}\right]$$

has a **determinant** $2\pi^3/\sqrt{3^3 5^5}$ that is **free** of the 3-loop constant C in

$$J(1) = 8\pi^2 C = \frac{64\pi^3(1 - 1/\sqrt{5})}{\mathbf{agm}^2(8\sqrt{2}, \sqrt{15} + 3\sqrt{5} + \sqrt{3} - 1)} \tag{7}$$

$$= \frac{\pi^3}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\left(1 + 2\sum_{n=1}^\infty q_{\mathbf{tiny}}^{n^2}\right)^4, \tag{8}$$

$$= 8.570280443374461268149695824246537286105438217336449736197549\ldots$$

where $q_{\mathbf{tiny}} = \exp(-\pi\sqrt{\mathbf{15}}) < 0.0000051975$ is the **tiny nome of Bologna**.

4.11 Critical L-value: The **L-series** for 5 Bessel functions comes from a **modular form** of weight **3** and level **15**:

$$f_{3,15}(z) = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A(n)q^n,$$

$$L(s) = \sum_{n>0} \frac{A(n)}{n^s}, \quad \text{for } s > 2,$$

$$J(1) = \frac{8\pi^2}{5}L(1) = \frac{8\pi^2}{5}\sum_{n>0}\frac{A(n)}{n}\left(2+\frac{\sqrt{\mathbf{15}}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{\mathbf{15}}}\right) \tag{9}$$

as **Francis Brown** and I discovered at Les Houches in June 2010.

4.12 Finite part in 4 dimensions: With the measure $d^Dk/(\pi^{D/2}\Gamma(1+\varepsilon))$ for a loop momentum k, in $D = 4 - 2\varepsilon$ dimensions, the Laurent expansion of the on-shell 3-loop sunrise diagram is

$$\frac{2}{\varepsilon^3} + \frac{22}{3\varepsilon^2} + \frac{577}{36\varepsilon} + \frac{6191}{216} - \frac{\pi^2}{3}\left(4C + \frac{7}{40C}\right) + O(\varepsilon)$$

as I discovered using PSLQ in 2007.

Summary: I have given **9 ways** of evaluating a **3-loop** diagram, using

1. a **Bessel moment**, in **co-ordinate space**,
2. a **dispersion** integral over **squares of elliptic integrals**,
3. a **momentum** expansion from an inhomogeneous **differential equation**,
4. an expansion in powers of a **nome**, using **eta quotients**,
5. an **alternative** expansion obtained by an **Atkin-Lehner transformation**,
6. a product of **Gamma values** at the **15th singular value**,
7. the **inverse square** of an **arithmetic-geometric mean**,
8. the fourth power of a **theta series**, for the **fastest** method,
9. a **critical value** of an L-series of a **modular form** of weight 3.

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