Beauty and simplicity: from differential equations and master integrals to QCD cross sections

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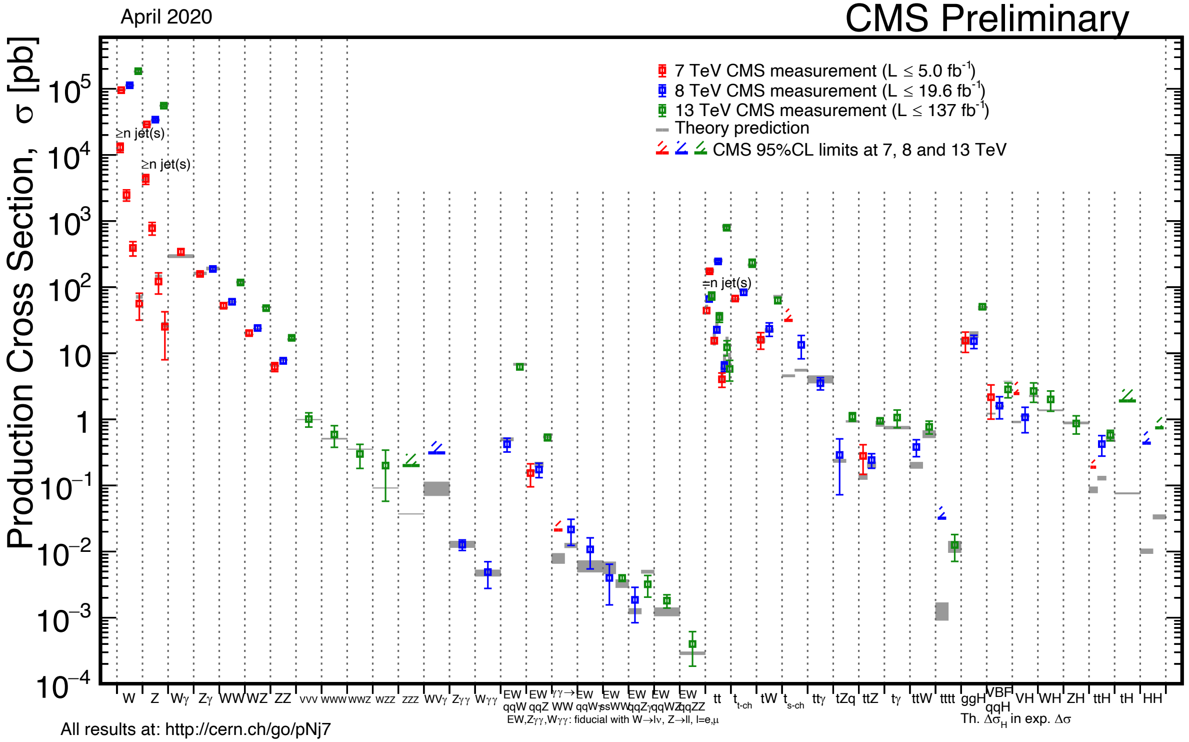
Universität Zürich

Workshop “Inspired by Precision” Ettore 80

Bologna 10.12.2021

Precision QCD physics at colliders

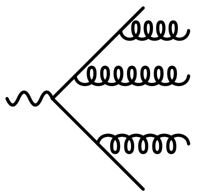
- Precision tests of the Standard Model
  - measurements of masses and couplings
- Interplay of calculations and measurements
  - experimental accuracy on many cross sections: around 5% or below
  - interpretation limited by precision on QCD predictions
- Higher order perturbative corrections
  - crucial for precision physics
  - interesting in their own right (beauty)



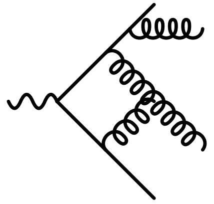
Precision QCD physics at colliders

Anatomy of a higher order QCD calculation: $e^+e^- \rightarrow$ 3 jets at NNLO

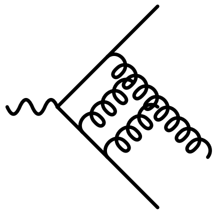
- Tree-level five-parton matrix elements
  - double real radiation



- One-loop four-parton matrix elements
  - single real radiation at one-loop



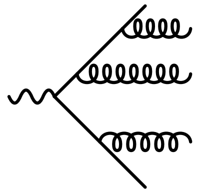
- Two-loop three-parton matrix elements
  - purely virtual correction



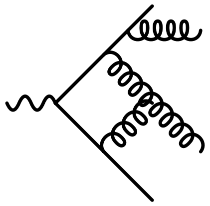
Precision QCD physics at colliders (in 1998)

Anatomy of a higher order QCD calculation: $e^+e^- \to$ 3 jets at NNLO

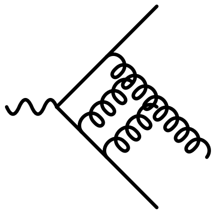
- Tree-level five-parton matrix elements ~~(beautiful)~~
  - double real radiation
  - known since 1985, must integrate over 3 jet phase space ~~(simple)~~



- One-loop four-parton matrix elements ~~(beautiful)~~
  - single real radiation at one-loop
  - known since 1995, must integrate over 3 jet phase space ~~(simple)~~



- Two-loop three-parton matrix elements
  - purely virtual correction
  - unknown in 1998, three partons = three jets



Back to 1998

Karlsruhe University
Institut für Theoretische Teilchenphysik

- Ettore arrives in fall 1998 as Humboldt Fellow
- fresh ideas: Laporta algorithm (unpublished), differential equations (for massive loop integrals)
- and codes under development (repete, solve, solve2)
- and a lot of time and patience to talk to a young postdoc, who also just arrived

- try to apply new ideas to QCD two-loop amplitudes

1998-1999: ideas to first results

Integration by parts identities (G. t'Hooft, M. Veltman; K. Chetyrkin, F. Tkachov)

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{\partial}{\partial a^\mu} \left[b^\mu f(k,l,p_i) \right] = 0 \qquad \text{with } a^\mu = k^\mu, l^\mu; b^\mu = k^\mu, l^\mu, p_i^\mu$$

- classify integrals according to
  - number of different propagators (t)
  - total number of propagators (r)
  - total number of scalar products (s)
- #equations grows faster than #unknowns
- solved using Laporta algorithm (lexicographic ordering)

$t = 7$

different $I_{t,r,s}$

r \ s	0	1	2	3	4
7	1	2	3	4	5
8	7	14	21	28	35
9	28	56	84	112	140
10	84	168	252	336	420

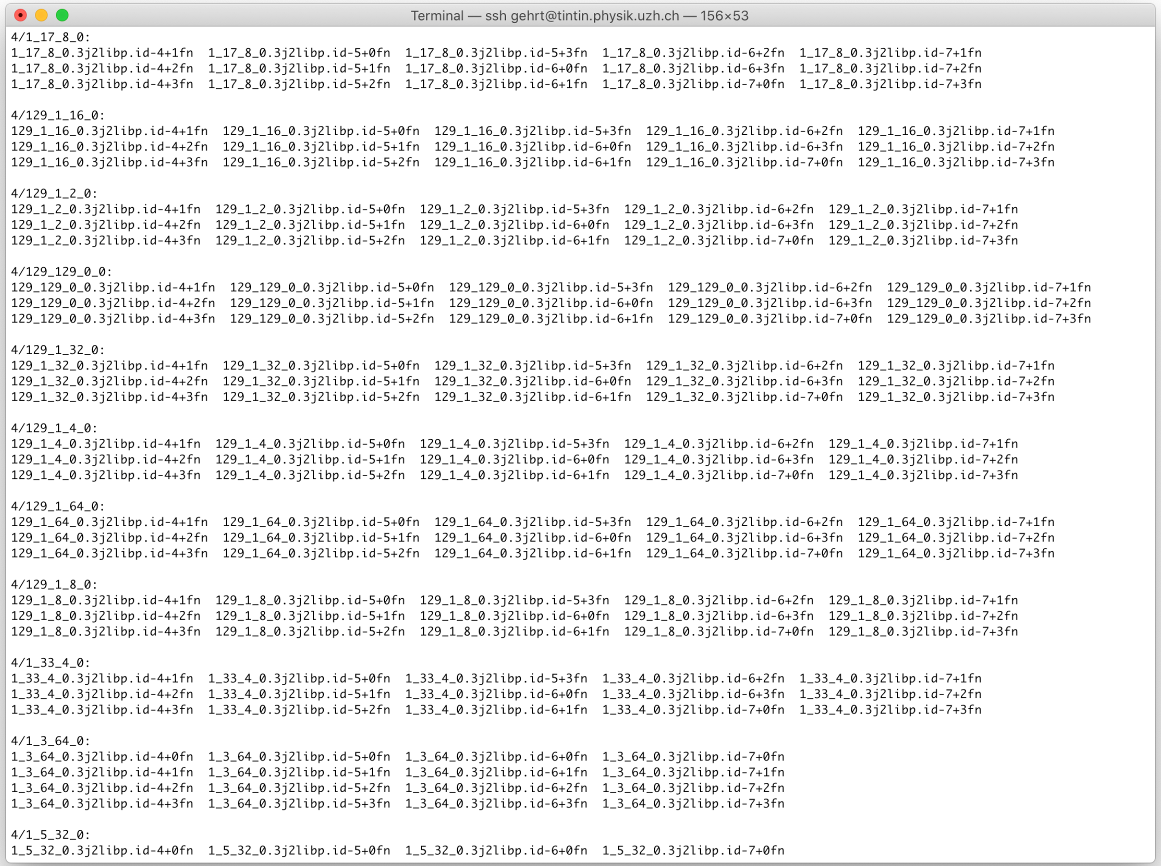
accumulated equations / unknowns

r \ s	0	1	2	3
7	13 22	39 45	78 76	130 115
8	104 106	312 213	624 354	1040 535
9	468 358	1404 717	2808 1196	4680 1795

1998-1999: ideas to first results

Implementation of the Laporta algorithm

- Codes developed by Ettore: repete, solve2
- C driver code for FORM and MAPLE
- Ettore's clear vision of the scope and complexity of the problem
- reduce all Feynman integrals to a few master integrals (here: 35'000 ⟶ 19)
- in parallel to Java development work



1998-1999: ideas to first results

Implementation of the Laporta algorithm

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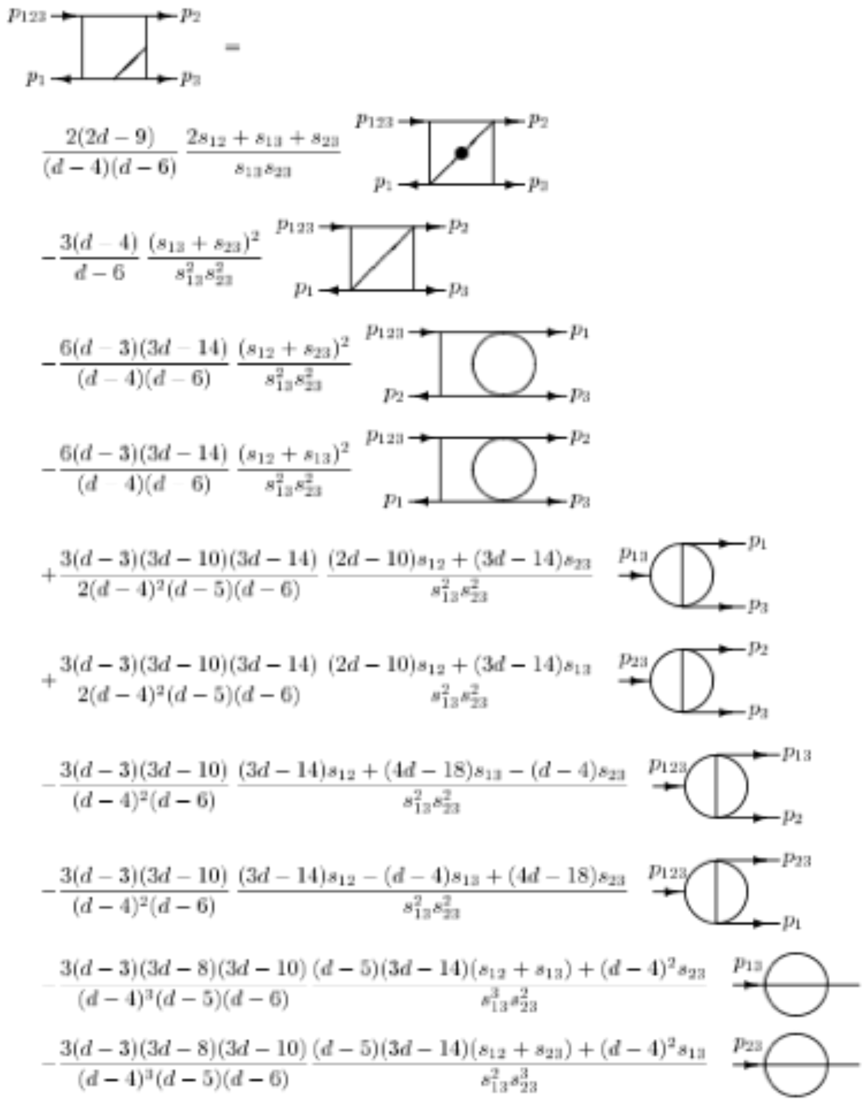

Differential equations for master integrals

Feynman integrals fulfil differential equations in internal masses and external Mandelstam invariants

$$s_{12}\frac{\partial}{\partial s_{12}} = \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu} + p_2^\mu\frac{\partial}{\partial p_2^\mu} - p_3^\mu\frac{\partial}{\partial p_3^\mu}\right)$$

$$s_{13}\frac{\partial}{\partial s_{13}} = \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu} - p_2^\mu\frac{\partial}{\partial p_2^\mu} + p_3^\mu\frac{\partial}{\partial p_3^\mu}\right)$$

$$s_{23}\frac{\partial}{\partial s_{23}} = \frac{1}{2}\left(-p_1^\mu\frac{\partial}{\partial p_1^\mu} + p_2^\mu\frac{\partial}{\partial p_2^\mu} + p_3^\mu\frac{\partial}{\partial p_3^\mu}\right)$$

IL NUOVO CIMENTO VOL. 110 A, N. 12 Dicembre 1997

Differential equations for Feynman graph amplitudes

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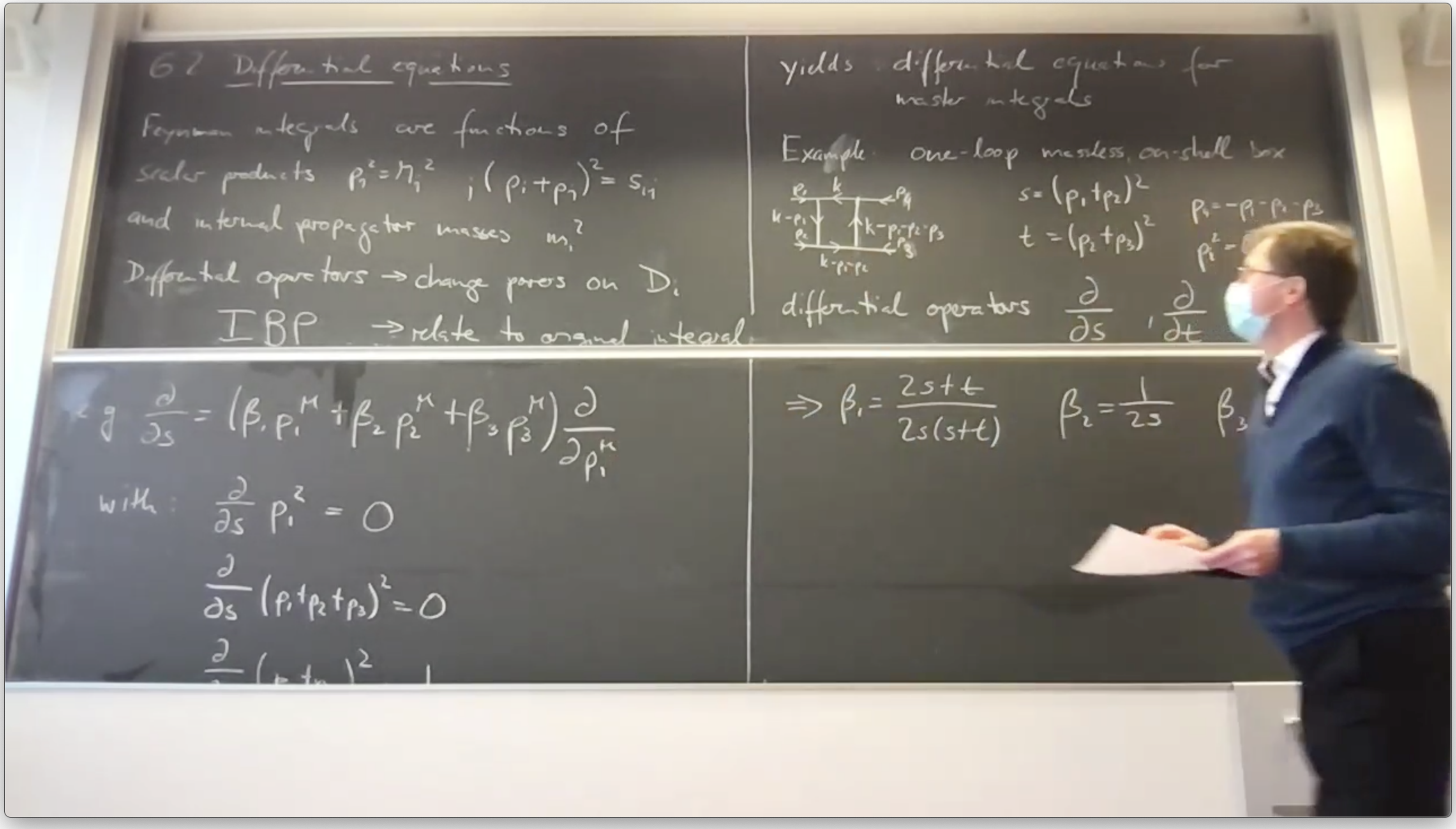
INFN, Sezione di Bologna - Bologna, Italy

(ricevuto il 27 Novembre 1997; approvato il 18 Dicembre 1997)

Summary. — It is by now well established that, by means of the integration by part identities, all the integrals occurring in the evaluation of a Feynman graph of given topology can be expressed in terms of a few independent master integrals. It is shown in this paper that the integration by part identities can be further used for obtaining a linear system of first-order differential equations for the master integrals themselves. The equations can then be used for the numerical evaluation of the amplitudes as well as for investigating their analytic properties, such as the asymptotic and threshold behaviours and the corresponding expansions (and for analytic integration purposes, when possible). The new method is illustrated through its somewhat detailed application to the case of the one-loop self-mass amplitude, by explicitly working out expansions and quadrature formulas, both in arbitrary continuous dimension n and in the $n \to 4$ limit. It is then shortly discussed which features of the new method are expected to work in the more general case of multi-point, multi-loop amplitudes.

PACS 11.10 – Field theory.
PACS 11.10.Kk – Field theories in dimensions other than four.
PACS 11.15.Bt – General properties of perturbation theory.

Differential equations for master integrals



Differential equations for master integrals

Playing with differential operators in external momenta

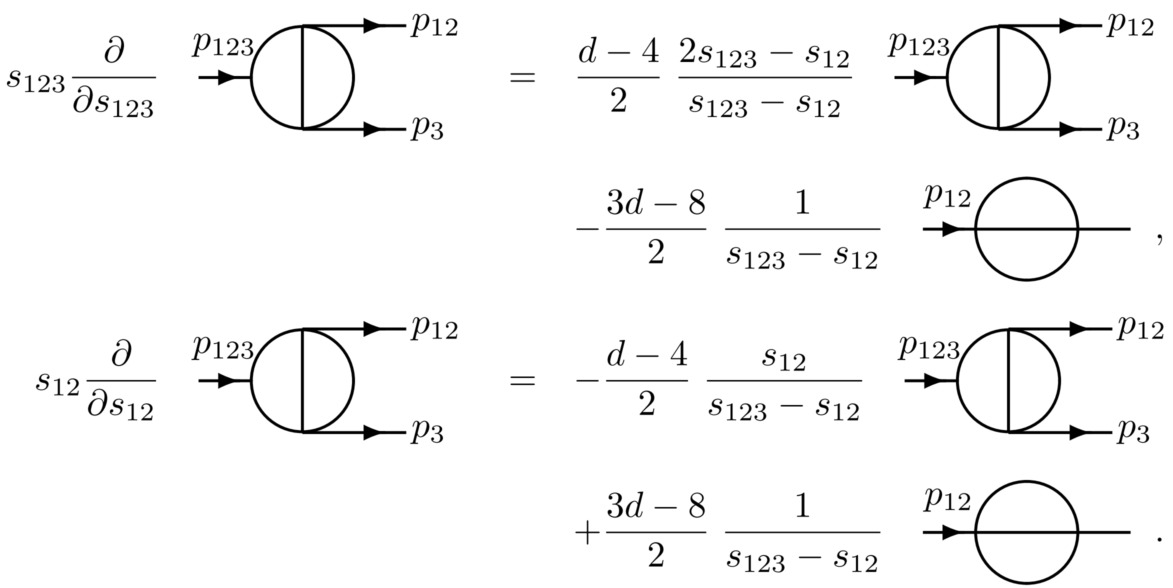
- 9 operators, 3 on-shell constraints, 3 invariants (= differential equations)
- ‘discovered’ new Lorentz-invariance (LI) identities

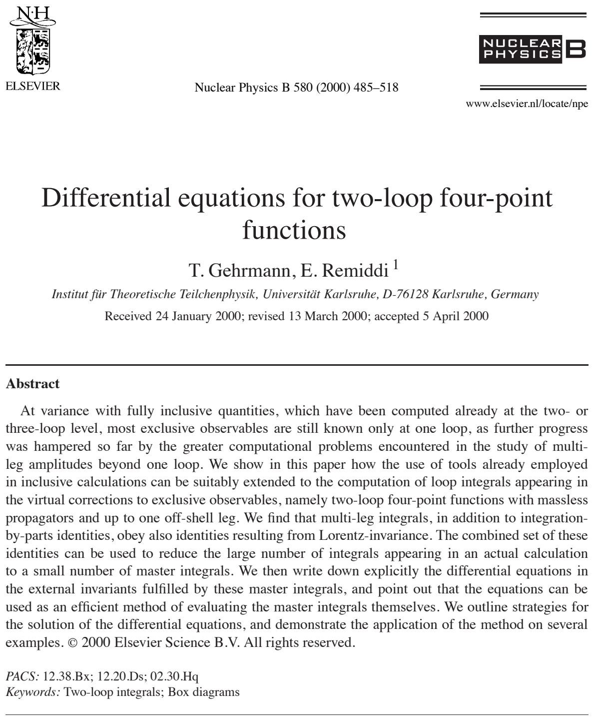
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \delta\varepsilon^\mu_\nu \left(\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) f(k, l, p_i) = 0$$

- Taken together, IBP and LI identities follow from the Poincare-invariance of Feynman integrals in dimensional regularization

Differential equations for master integrals

Beautiful and simple example:





ELSEVIER
Nuclear Physics B 580 (2000) 485–518
NUCLEAR PHYSICS B
www.elsevier.nl/locate/npe

Differential equations for two-loop four-point functions

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Received 24 January 2000; revised 13 March 2000; accepted 5 April 2000

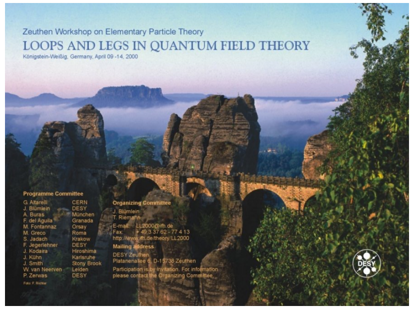
Abstract

At variance with fully inclusive quantities, which have been computed already at the two- or three-loop level, most exclusive observables are still known only at one loop, as further progress was hampered so far by the greater computational problems encountered in the study of multi-leg amplitudes beyond one loop. We show in this paper how the use of tools already employed in inclusive calculations can be suitably extended to the computation of loop integrals appearing in the virtual corrections to exclusive observables, namely two-loop four-point functions with massless propagators and up to one off-shell leg. We find that multi-leg integrals, in addition to integration-by-parts identities, obey also identities resulting from Lorentz-invariance. The combined set of these identities can be used to reduce the large number of integrals appearing in an actual calculation to a small number of master integrals. We then write down explicitly the differential equations in the external invariants fulfilled by these master integrals, and point out that the equations can be used as an efficient method of evaluating the master integrals themselves. We outline strategies for the solution of the differential equations, and demonstrate the application of the method on several examples. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 12.38.Bx; 12.20.Ds; 02.30.Hq
Keywords: Two-loop integrals; Box diagrams

Solve differential equations and match to boundary conditions

ECFA Study
Physics and Detectors for a Linear Collider



Solving differential equations

Closed-form solutions: increasingly complicated hypergeometric functions

- expansion in dimensional regularization: polylogarithmic functions

Harmonic polylogarithms (E.Remiddi, J.Vermaseren, 1999)

$$H(1;x) = -\ln(1-x)$$
$$H(0;x) = \ln x$$
$$H(-1;x) = \ln(1+x)$$

$$f(1;x) = \frac{1}{1-x}$$
$$f(0;x) = \frac{1}{x}$$
$$f(-1;x) = \frac{1}{1+x}$$

$$H(a,\vec{b};x) = \int_0^x \mathrm{d}x\, f(a;x) H(\vec{b};x)$$

- extend to generic rational factors: generalized polylogarithms
  (E. Kummer, 1840; A. Goncharov, 2000)

$$G(w_1,\ldots,w_n;z) = \int_0^z \frac{dt}{t-w_1} G(w_2,\ldots,w_n;t)$$

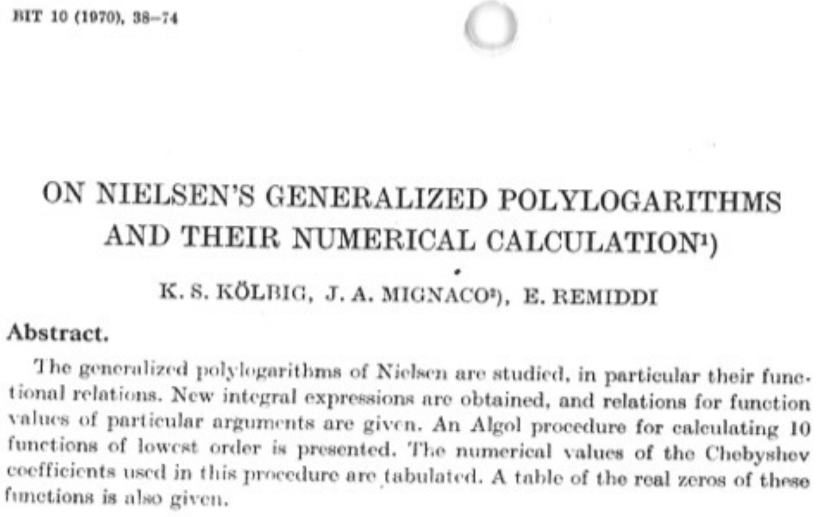
Harmonic Polylogarithms

Two-loop four-point master integrals for $e^+e^- \rightarrow$ 3 jets

- completed in summer 2000, shortly before parting Karlsruhe for Bologna & CERN
- compact ε-expansions, expressed in HPLs, 2dHPLs (beautiful and simple): 1 page each
- analytical results: no hidden zeros and convergent series representation

Numerical implementation of HPLs and 2dHPLs

- argument transformation and series expansion
- strongly inspired by earlier work


BIT 10 (1970), 38–74

ON NIELSEN'S GENERALIZED POLYLOGARITHMS AND THEIR NUMERICAL CALCULATION[1])

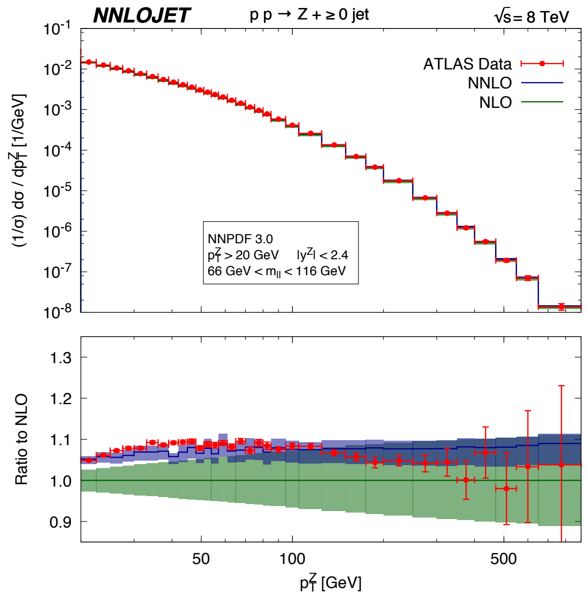
K. S. KÖLBIG, J. A. MIGNACO[2]), E. REMIDDI

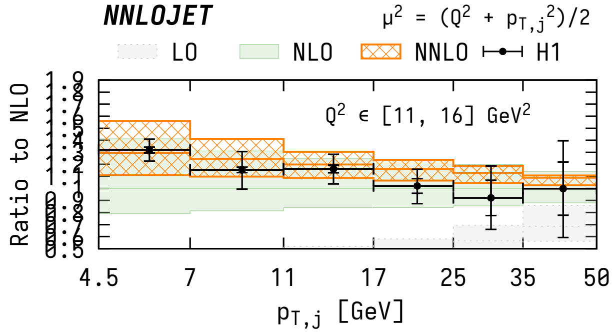
Abstract.

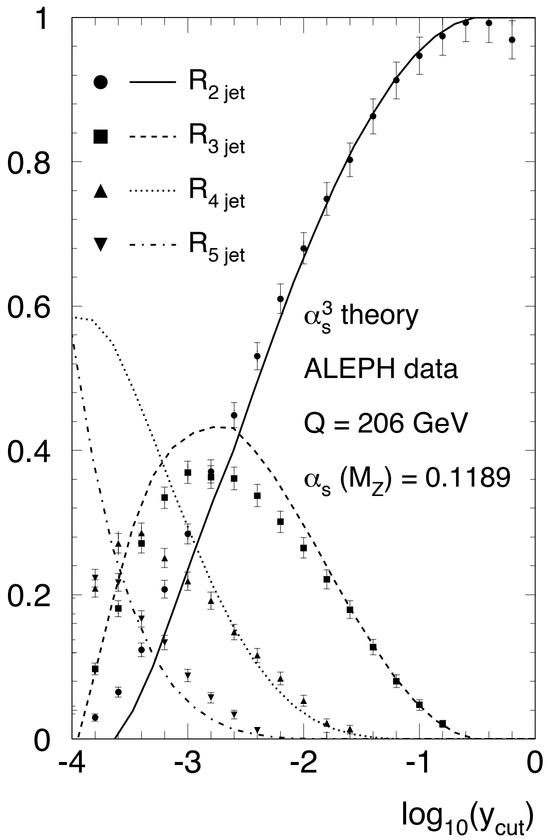
The generalized polylogarithms of Nielsen are studied, in particular their functional relations. New integral expressions are obtained, and relations for function values of particular arguments are given. An Algol procedure for calculating 10 functions of lowest order is presented. The numerical values of the Chebyshev coefficients used in this procedure are tabulated. A table of the real zeros of these functions is also given.

Two-loop matrix element for $e^+e^- \to$ 3 jets

- completed in 2002
- reasonably compact expression for matrix element (15 pages)
- analytic continuation to ep → (2+1) jets and pp → V+jet (extensively documented)
- basis for a multi-year research program in precision phenomenology: EERAD3 and NNLOJET
- analytical follow-ups: two-loop soft current and splitting amplitudes



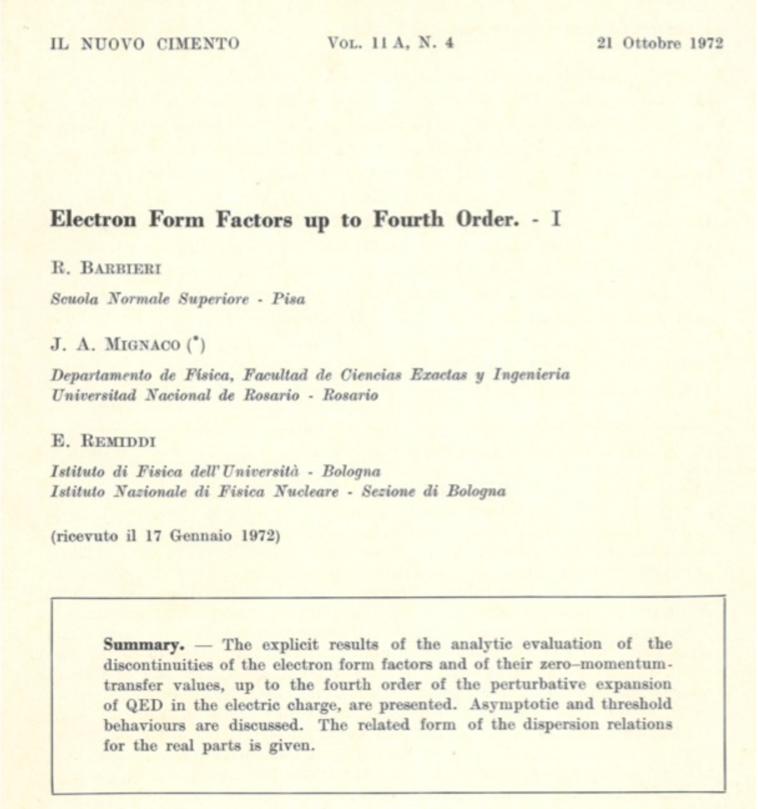




Following up

Back to massive loop integrals: two-loop form factors in QED and QCD

- taking up a decades-old challenge
- with Bologna team (R. Bonciani, P. Mastrolia)
- completed in 2005, many applications
  - e.g. $A_{FB}(b)$, linear collider physics, $(g\text{-}2)_t$
- prepare territory for all-massive sunrise graphs
  - entirely new and beautiful analytical structures: elliptic integrals and their iterations (L. Tancredi)



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Electron Form Factors up to Fourth Order. - I

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Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

(ricevuto il 17 Gennaio 1972)

Summary. — The explicit results of the analytic evaluation of the discontinuities of the electron form factors and of their zero-momentum-transfer values, up to the fourth order of the perturbative expansion of QED in the electric charge, are presented. Asymptotic and threshold behaviours are discussed. The related form of the dispersion relations for the real parts is given.

This is not a summary

Many, many thanks to Ettore for sharing your unique experience and insights with me, well beyond physics

Image credit: Oberwolfach 2005

Truly “Inspired by Precision”
(and inspiring a whole research community)

“Take one step at a time”

Wishing you many happy returns !