Rise and shine: Special functions from Feynman integrals

Workshop “Inspired by Precision” in the honour of Prof. Ettore Remiddi on the occasion of his 80th birthday

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Content

- **E. Remiddi**, J. A. M. Vermaseren, "Harmonic Polylogarithms",
  Int. J. Mod. Phys. A15 (2000) 725,
  hep-ph/9905237

  923 citations

- M. Caffo, H. Czyz, S. Laporta, **E. Remiddi**, "The Master Differential Equations for the 2-loop Sunrise Selfmass Amplitudes",
  Nuovo Cim. A111 (1998), 365,
  hep-th/9805118

  253 citations

Section 1

Harmonic Polylogarithms

Back to the year 1999

- We knew in principle how to calculate any process to NLO, although the NLO-revolution has not yet happened.
- In preparation for LHC we were aiming at NNLO calculations:
  - Two-loop amplitudes for $pp \to 2$ jets.
  - Three-loop splitting functions.
- Not much was known what functions we should expect in two-loop Feynman integrals and in the three-loop splitting functions.
- The massless double box integral was not yet known.
- Even less was known at higher loops.

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the scalar products and masses times two transcendental functions, whose arguments are again algebraic functions of the scalar products and the masses.

The two transcendental functions are the **logarithm** and the **dilogarithm**:

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j} \\ \text{Li}_2(x) &= \sum_{j=1}^{\infty} \frac{x^j}{j^2} \end{aligned}$$

The obvious generalisation and Nielsen polylogarithmen

Classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$$

Nielsen polylogarithms:

$$S_{n,p}(x) = \frac{(-1)^{n-1+p}}{(n-1)!p!} \int_0^1 dt \frac{\ln^{n-1}(t) \ln^p(1-xt)}{t}$$

N. Nielsen, *Der Eulersche Dilogarithmus und seine Verallgemeinerungen*, 1909,

K. Kölbig, J. Mignoco, **E. Remiddi**, *On Nielsen's generalized polylogarithms and their numerical calculation*, 1969

Nielsen polylogarithms

The one-dimensional integral representation is not too enlightning, let's look at the **sum representation** of Nielsen polylogarithms:

$$S_{n,p}(x) \quad = \quad \sum_{j=1}^{\infty} \frac{x^j}{j^{n+1}} Z_{\underbrace{1\ldots1}_{p-1}}(j-1),$$

where

$$Z_{\underbrace{1\ldots1}_{k}}(n) = \sum_{j=1}^{n} \frac{1}{j} Z_{\underbrace{1\ldots1}_{k-1}}(j-1), \qquad\qquad Z_1(n) = \sum_{j=1}^{n} \frac{1}{j}.$$

Harmonic polylogarithms

Nielsen polylogarithms : $$S_{m_1-1,k}(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^{m_1}} Z_{1\dots1}\left(j-1\right),$$

Harmonic polylogarithms : $$H_{m_1 m_2 \dots m_k}(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^{m_1}} Z_{m_2\dots m_k}\left(j-1\right),$$

with the Euler-Zagier sums

$$Z_{m_1 m_2 \dots m_k}\left(n\right) = \sum_{j=1}^{n} \frac{1}{j^{m_1}} Z_{m_2\dots m_k}\left(j-1\right), \qquad Z_m\left(n\right) = \sum_{j=1}^{n} \frac{1}{j^m}.$$

Euler-Zagier sums and harmonic sums

Where does the name “harmonic” come frome?

Euler-Zagier sums $Z_{m_1 \dots m_k}(n)$ and harmonic sums $S_{m_1 \dots m_k}(n)$ are defined by

$$Z_{m_1 m_2 \dots m_k}(n) = \sum_{j=1}^{n} \frac{1}{j^{m_1}} Z_{m_2 \dots m_k}(j-1), \qquad Z_m(n) = \sum_{j=1}^{n} \frac{1}{j^m},$$

$$S_{m_1 m_2 \dots m_k}(n) = \sum_{j=1}^{n} \frac{1}{j^{m_1}} S_{m_2 \dots m_k}(j), \qquad S_m(n) = \sum_{j=1}^{n} \frac{1}{j^m}.$$

One can easily convert between the two:

$$Z_{m_1 m_2 m_3}(n) = S_{m_1 m_2 m_3}(n) - S_{(m_1+m_2) m_3}(n) - S_{m_1 (m_2+m_3)}(n) + S_{(m_1+m_2+m_3)}(n).$$

The integral representation of harmonic polylogarithms

Iterated integration with $\frac{dt}{t}$ and $-\frac{dt}{t-1}$:

$$H_{m_1 m_2 \ldots m_k}(x) = \int_0^x \frac{dt}{t} H_{(m_1-1) m_2 \ldots m_k}(t), \quad m_1 > 1,$$

$$H_{1 m_2 \ldots m_k}(x) = -\int_0^x \frac{dt}{t-1} H_{m_2 \ldots m_k}(t).$$

Denote by $w = m_1 + \cdots + m_k$ the **weight**. Then $H_{m_1 m_2 \ldots m_k}(x)$ has a w-fold integral representation, for example

$$H_{12}(x) = \int_0^x \frac{dt_1}{t_1 - 1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{t_3 - 1}.$$

The shuffle product

Denote differential one-forms by

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = -\frac{dt}{t-1},$$

and iterated integrals by

$$I(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_r}; x) = \int_0^x \omega_{i_1} I(\omega_{i_2}, \dots, \omega_{i_r}; t)$$

Shuffle product

$$I(\omega_{i_1}, \dots, \omega_{i_k}; x) \cdot I\left(\omega_{i_{k+1}}, \dots, \omega_{i_r}; x\right) = \sum_{\text{shuffles } \sigma} I\left(\omega_{\sigma(i_1)}, \omega_{\sigma(i_2)}, \dots, \omega_{\sigma(i_r)}; x\right)$$

where the sum runs over all permutations which preserve the relative order of $(i_1, \dots, i_k)$ *and* $(i_{k+1}, \dots, i_r)$.

The shuffle product

The shuffle product allows us to express any product of harmonic polylogarithms as a **linear combination** of harmonic polylogarithms, e.g.

$$H_{12}(x) H_1(x) = 2 H_{121}(x) + 2 H_{112}(x)$$

Multiple polylogarithms

One more generalisation:

Harmonic polylogarithms:

$$\mathrm{H}_{m_1 m_2 \dots m_k}(x) = \sum_{n_1 > n_2 > \dots > n_k > 0}^{\infty} \frac{x^{n_1}}{n_1^{m_1}} \cdot \frac{1}{n_2^{m_2}} \cdot \ldots \cdot \frac{1}{n_k^{m_k}}$$

Multiple polylogarithms:

$$\mathrm{Li}_{m_1 m_2 \dots m_k}(x_1, x_2, \dots, x_k) = \sum_{n_1 > n_2 > \dots > n_k > 0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Multiple polylogarithms

Definition based on **nested sums**:

$$\text{Li}_{m_1 m_2 \dots m_k}(x_1, x_2, \dots, x_k) = \sum_{n_1 > n_2 > \dots > n_k > 0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on **iterated integrals**:

$$G(z_1, \dots, z_k; y) = \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} \dots \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

Conversion:

$$\text{Li}_{m_1 \dots m_k}(x_1, \dots, x_k) = (-1)^k G_{m_1 \dots m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \dots, \frac{1}{x_1 \dots x_k}; 1\right)$$

Short hand notation:

$$G_{m_1 \dots m_k}(z_1, \dots, z_k; y) = G(\underbrace{0, \dots, 0}_{m_1 - 1}, z_1, \dots, z_{k-1}, \underbrace{0, \dots, 0}_{m_k - 1}, z_k; y)$$

Alphabets

- Consider $G(z_1, ..., z_k; y)$ with

$$z_j \in \mathcal{A} = \{l_1, \dots, l_N\}.$$

- $\mathcal{A}$ is called the **alphabet** and the l_j's are called **letters**.
- Up to now we considered mainly the letters 0 and 1.

Definition

The harmonic polylogarithmen have the alphabet

$$\mathcal{A} = \{-1, 0, 1\},$$

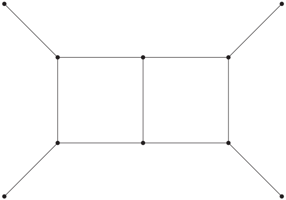
i.e. they are iterated integrals of $\frac{dt}{t+1}$, $\frac{dt}{t}$ and $\frac{dt}{t-1}$.

E. Remiddi, J. A. M. Vermaseren, 1999

Feynman integrals

The double box integral:

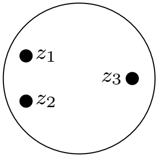
- For $pp \to 2$ jets one needs the double box integral.
- One kinematic variable $x = s/t$.
- Can be expressed to all orders in ε in terms of harmonic polylogarithms.



There are many Feynman integrals (also with more than two loops), which can be expressed in terms of multiple polylogarithms!

A little bit of mathematics

- The Riemann sphere is the complex plane plus the point at infinity: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- Mark n distinct points $(z_1, ..., z_n)$ on $\hat{\mathbb{C}}$.
- The **moduli space** $\mathcal{M}_{0,n}$ is the configuration space of n distinct marked points modulo Möbius transformations.
- $\mathcal{M}_{0,n}$ is an affine algebraic variety of dimension $(n-3)$.
- **Multiple polylogarithms** can be viewed as iterated integrals on the moduli space $\mathcal{M}_{0,n}$ with $\omega_{ij} = d \ln(z_i - z_j)$.



Section 2

The Two-Loop Sunrise

Well-studied problems in physics

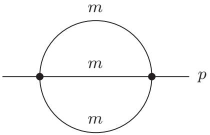
- The harmonic oscillator
- The hydrogen atom
- The sunrise integral
  The first integral which cannot be expressed in terms of multiple polylogarithmens.

Literature

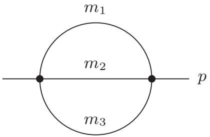
A selection:

- Caffo, Czyz, Laporta, **Remiddi**, *The Master Differential Equations for the 2-loop Sunrise Selfmass Amplitudes*, 1998
- Caffo, Czyz, **Remiddi**, *The Pseudothreshold expansion of the two loop sunrise selfmass master amplitudes*, 1999
- Caffo, Czyz, **Remiddi**, *The Threshold expansion of the two loop sunrise selfmass master amplitudes*, 2001
- Laporta, **Remiddi**, *Analytic treatment of the two loop equal mass sunrise graph*, 2004
- Caffo, Czyz, Gunia, **Remiddi**, *BOKASUN: A Fast and precise numerical program to calculate the Master Integrals of the two-loop sunrise diagrams*, 2008
- **Remiddi**, Tancredi, *Schouten identities for Feynman graph amplitudes; The Master Integrals for the two-loop massive sunrise graph*, 2013
- **Remiddi**, Tancredi, *Differential equations and dispersion relations for Feynman amplitudes. The two-loop massive sunrise and the kite integral*, 2016
- **Remiddi**, Tancredi, *An Elliptic Generalization of Multiple Polylogarithms*, 2017

The sunrise integral



- Two-loop equal mass sunrise
  - 3 master integrals
  - 1 kinematic variable



- Two-loop unequal mass sunrise
  - 7 master integrals
  - 3 kinematic variable

Geometry

From the point of view of algebraic geometry there are two objects of interest:

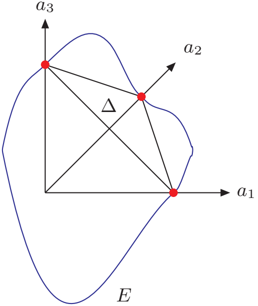
- the zero set E of $\mathcal{F} = 0$:

$$E \quad : \quad -a_1 a_2 a_3 p^2 + \left(a_1 m_1^2 + a_2 m_2^2 + a_3 m_3^2\right)\left(a_1 a_2 + a_2 a_3 + a_3 a_1\right) \; = \; 0$$

  E is an **elliptic curve**.

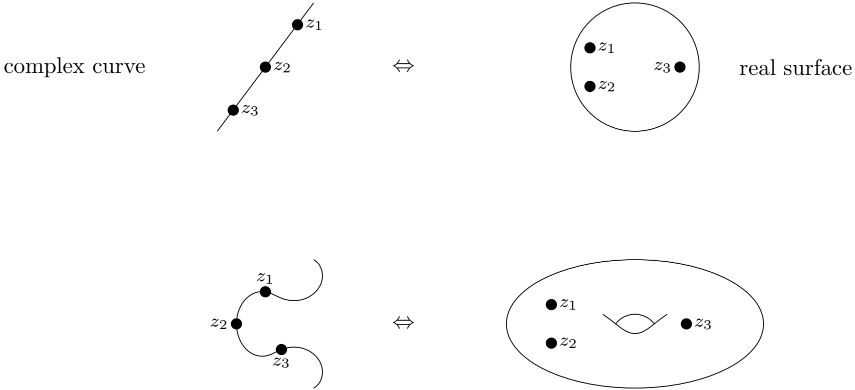
- the domain of integration Δ.

E and Δ intersect at three points, this marks three points on the elliptic curve.



Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(z_1, \ldots, z_{n-3})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, z_1, \ldots, z_{n-1})$

Differential one-forms on $\mathcal{M}_{1,n}$

1. *From modular forms* ($f_k(\tau)$ *modular form*):

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

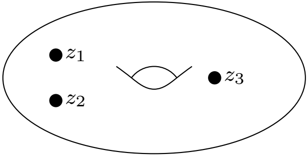
2. *From the Kronecker function* ($g^{(k)}(z,\tau)$ *coefficient of Kronecker function*):

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(L(z),\tau) \, dL(z) + (k-1) \, g^{(k)}(L(z),\tau) \frac{d\tau}{2\pi i} \right],$$

$$L(z) = \sum_{j=1}^{n-1} \alpha_j z_j + \beta.$$

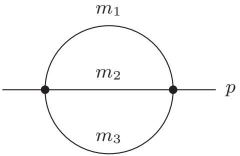
The sunrise integral

- The sunrise integral can be expressed to any order in the dimensional regularisation parameter ε as iterated integrals on $\mathcal{M}_{1,3}$ with a finite number of ω's.



Iterated integrals on $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$

- Iterated integrals on $\mathcal{M}_{0,n}$ with at most simple poles are **multiple polylogarithms**.
  Most of the known Feynman integrals fall into this category.
- Iterated integrals on $\mathcal{M}_{1,n}$ are **iterated integrals of modular forms** and **elliptic multiple polylogarithms** (and mixtures thereof).
  The simplest example is the two-loop sunrise integral with non-zero masses.

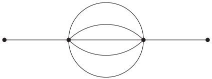


Section 3

Outlook

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of higher genus g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and higher dimensional objects: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

The next generation

- Solve integration-by-parts identities with the help of an ordering criteria.

  S. Laporta

- Finite field methods.

  T. Peraro

- Maximal cuts are solutions of the homogeneous differential equation.

  A. Primo, L. Tancredi

- Scalar product on the vector space of integrands.

  P. Mastrolia, S. Mizera