# Loops and bound states

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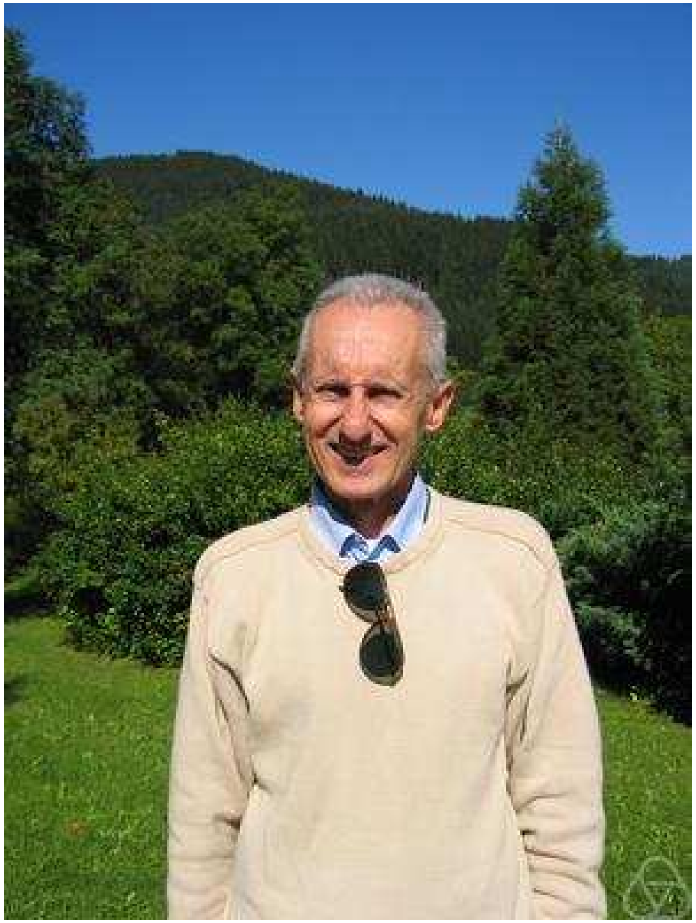
# Happy celebration, Ettore!

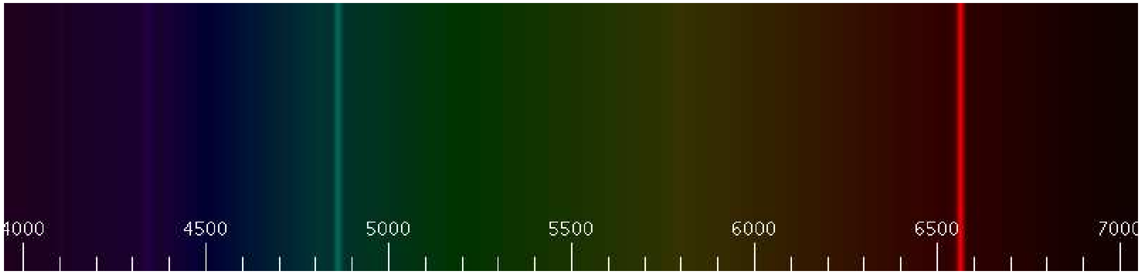


## Ettore and me: a timeline

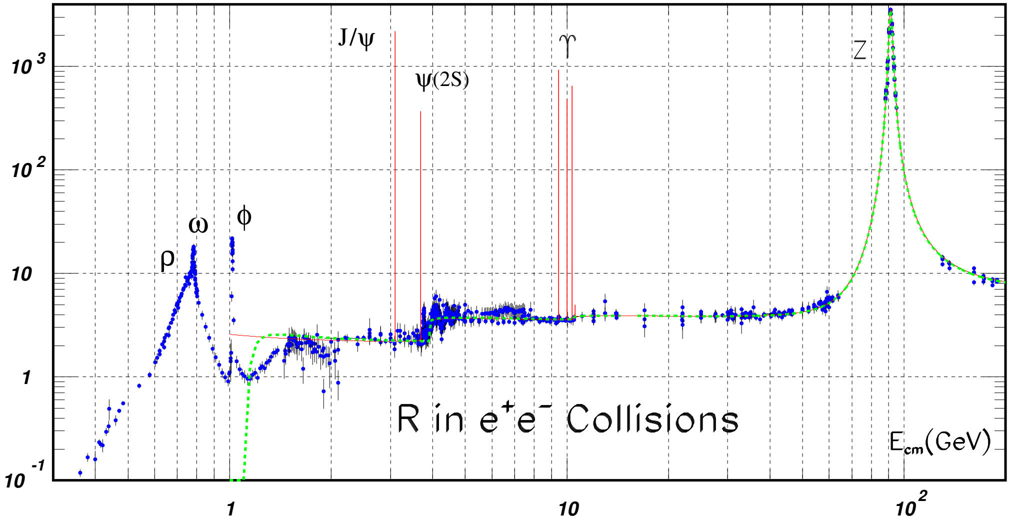
- 1989: entering the group (with Alfred Hill, Fabio Ortolani, Michele Caffo, Stefano Laporta, Sandro Turrini, ...)
- 1990: Master (QED bound states and gauge invariance)
- 1991: first conference (Elba, an exotic edition of the “Incontri di Cortona”)
- 1993: a fateful Ph.D. thesis
- 1994: first paper
- 1995: Ph.D. (Positronium)
- 1995: a long workshop in Edirne
- 1996-2003: CERN
- 2000: Heidelberg/Karlsruhe
- 2006- : Summers in Numana
- 2008- : Munich (MIAPP, ...)

# Matter is made of bound states

- Electromagnetic bound states: atoms, molecules, ...




- Strong-interaction bound states: hadrons, nuclei, ...
  (At low $T$ and $\rho$, confinement only allows for bound states!)



## ... many of them non-relativistic

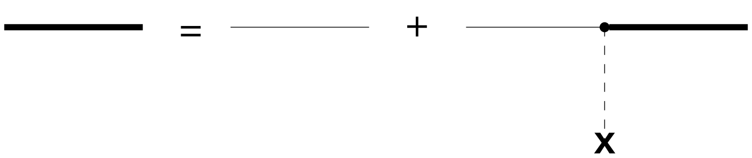
- atoms, molecules, ...
- baryonium, pionium, ...
- quarkonium (charmonium, bottomonium, top-antitop pairs, ... )

# Non-relativistic quantum theory of bound states

Non-relativistic bound states accompanied the history of the quantum theory from its inception to the establishing of the quantum theory of fields:

- 1926 Schrödinger equation: $\left(\dfrac{\mathbf{p}^2}{2m} + V\right)\phi = E\phi$

$$\begin{cases} g = g_0 + g_0(-iV)g \\ g_0 = \frac{i}{E - \mathbf{p}^2/(2m)} \end{cases}$$




- 1927 Pauli equation: $\left(\dfrac{(\mathbf{p} - e\mathbf{A})^2}{2m} + V - \dfrac{\boldsymbol{\sigma}\cdot e\mathbf{B}}{2m}\right)\phi = E\phi$

The relevant scales of the non-relativistic bound state dynamics are

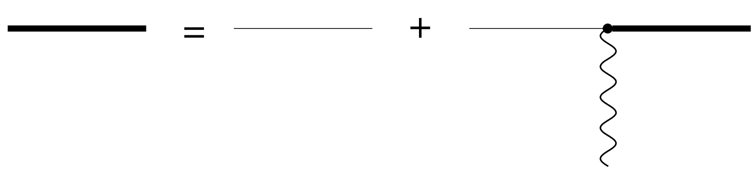
- $E \sim \dfrac{\mathbf{p}^2}{2m} \sim V \sim mv^2$,
- $p \sim 1/r \sim mv$;

a crucial observation: if $v$(elocity) $\ll 1$, then $m \gg mv \gg mv^2$.

## Relativistic quantum theory of bound states

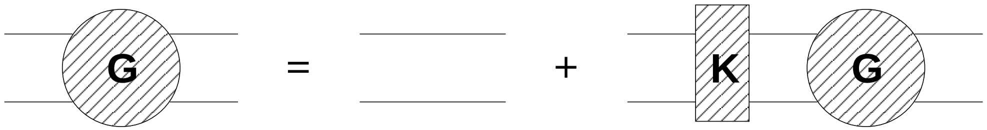
- 1928 Dirac equation: $(i\not{D} - m)\,\psi = 0$

$$\begin{cases} g^D = g_0^D + g_0^D(-ie\not{A})g^D \\ g_0^D = \frac{i}{\not{p}-m} \end{cases}$$

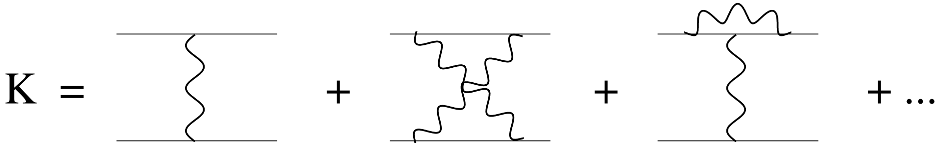



- 1951 Bethe–Salpeter equation:

$$\begin{cases} G = G_0 + G_0 K G \\ G_0 = g_0^D \otimes g_0^D \end{cases}$$




All the complexity of the field theory is in the kernel



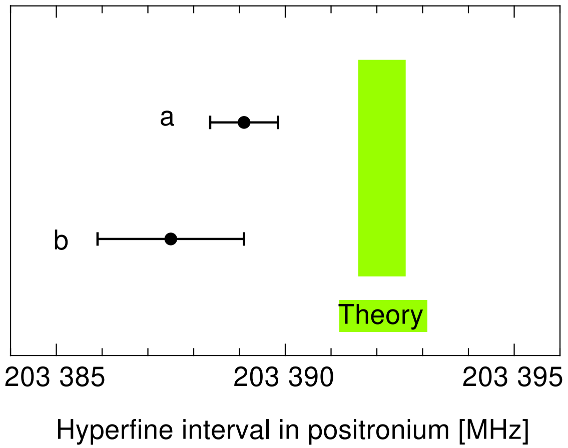

which only in the non-relativistic limit reduces to the Coulomb potential, but, in general, keeps entangled all bound-state scales.

- E.g. Barbieri Remiddi NPB 141 (1978) 413

Disentangling the bound-state scales at the Lagrangian level has advantages.

(I) It facilitates higher-order perturbative calculations.

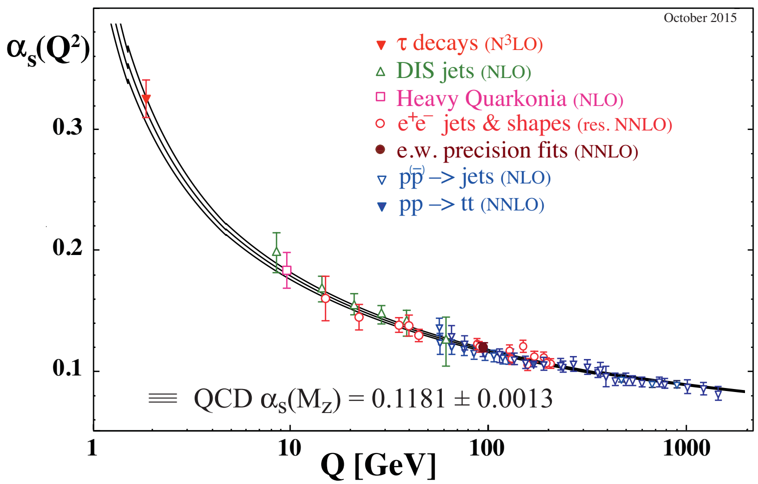
E.g. it took twenty-five years to go from the calculation of the $m\alpha^5$ correction in the hyperfine splitting of the positronium ground state to the $m\alpha^6 \ln\alpha$ term!

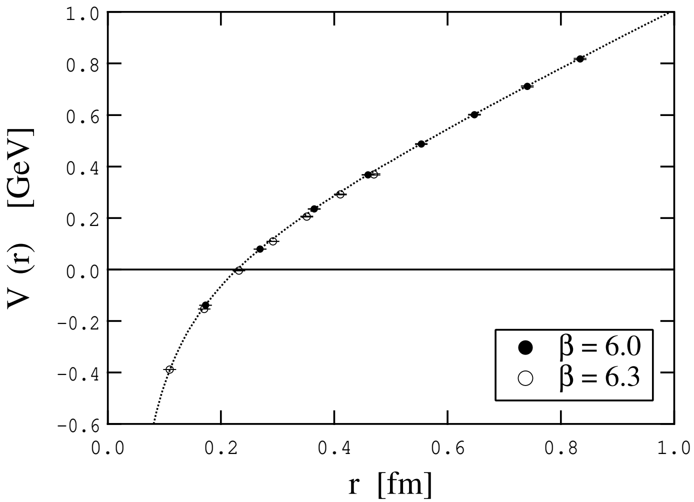



Relevant for

- atomic physics: Hydrogen atom (e.g. proton radius), positronium (e.g. width, hfs), ...
- $t\bar{t}$ threshold production, ...
- ...

(II) In QCD, it factorizes automatically high-energy (perturbative) contributions from low-energy (non-perturbative, thermal, ...) ones.



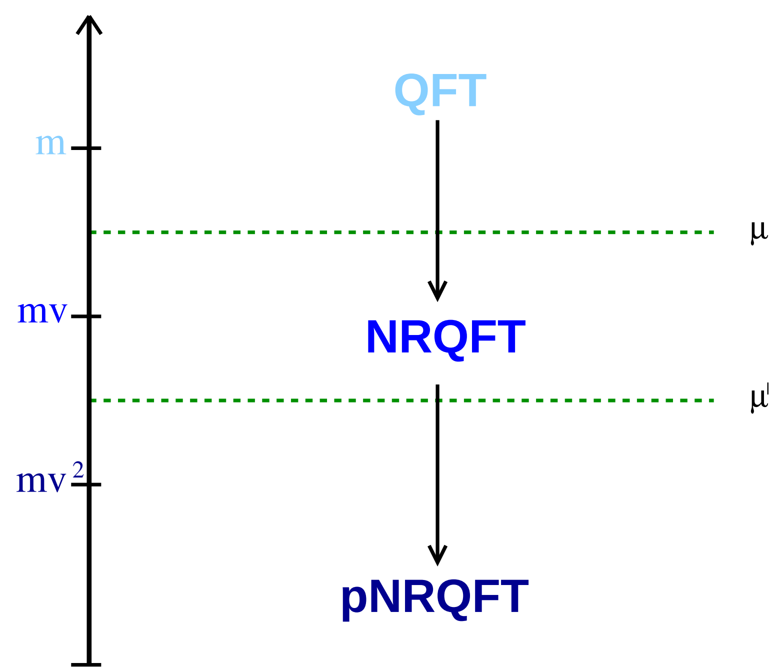

Relevant for

- pionium and precision chiral dynamics, ...
- nucleon-nucleon systems, ...
- quarkonia and new quarkonium states
- confinement and lattice calculations, ...
- quarkonium in heavy ion collisions: factorization of thermal contributions.

(III) More conceptually:
it provides a field theoretical foundation of the Schrödinger equation:

$$\mathcal{L}_{\mathrm{pNRQFT}} = \int d^3r\, \phi^\dagger \left( i\partial_0 - \frac{\mathbf{p}^2}{m} - V \right) \phi + \Delta\mathcal{L}$$

The Lagrangian $\mathcal{L}_{\mathrm{pNRQFT}}$, which separates the Schrödinger dynamics of the two-particle field $\phi$ from the low-energy dynamics encoded in $\Delta\mathcal{L}$, defines an effective field theory.

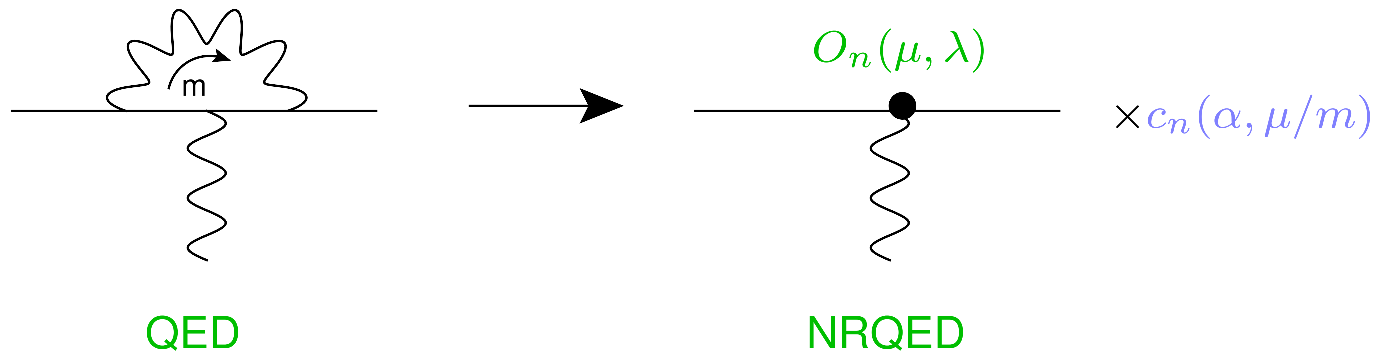


# The Lamb shift in QED

# NRQED

NRQED is obtained by integrating out modes associated with the scale $m$

E.g.



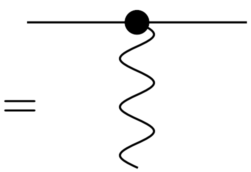

- The EFT has a cut-off $m > \mu > m\alpha$.
- The degrees of freedom are photons, electrons $\psi$, protons $N$ (static if no recoil).
- The Lagrangian is organized as an expansion in $1/m$ and $\alpha$:

$$\mathcal{L}_{\mathrm{NRQED}} = \sum_n \frac{1}{m^n} \times c_n(\alpha, \mu/m) \times O_n(\mu, \lambda)$$

## NRQED: matching

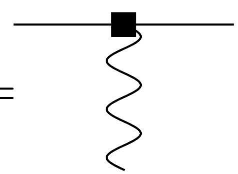
The matching at order $1/m^2$ gives:

$$c_2^{(a)}\frac{O_2^{(a)}}{m^2} =$$



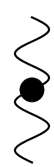
$$= \psi^\dagger e\frac{\nabla E}{8m^2}\psi \qquad \times\left(1+\frac{8}{3}\frac{\alpha}{\pi}\ln\frac{m}{\mu}+\dots\right)$$

$$c_2^{(b)}\frac{O_2^{(b)}}{m^2} =$$



$$= ie\sigma\cdot\psi^\dagger\frac{\nabla\times E - E\times\nabla}{8m^2}\psi \qquad \times(1+\dots)$$

$$c_2^{(c)}\frac{O_2^{(c)}}{m^2} =$$



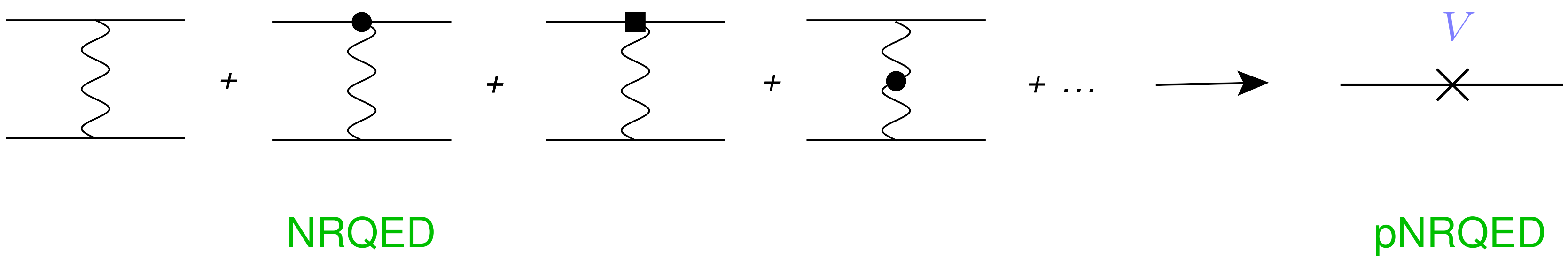
$$= \frac{F_{\mu\nu}\partial^2F^{\mu\nu}}{m^2} \qquad \times\left(\frac{\alpha}{60\pi}+\dots\right)$$

If the recoil of the proton is considered, the radius of the proton is encoded in $c_2^{(a)\,\mathrm{proton}}$.

- `Caswell Lepage PLB 167 (1986) 437`

## pNRQED

pNRQED is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim m\alpha$



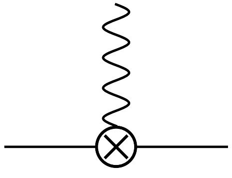

- The EFT has a cut-off $m\alpha > \mu' > m\alpha^2$.
- The degrees of freedom are photons, atoms $\phi$.
- The Lagrangian is organized as an expansion in $1/m$ , $r$, and $\alpha$.

# pNRQED: Lagrangian

$$\mathcal{L}_{\text{pNRQED}} = \int d^3r\, \phi^\dagger \left( i\partial_0 - \frac{\mathbf{p}^2}{m} + \cdots - V \right) \phi + \Delta\mathcal{L}$$

- $V = -\frac{\alpha}{r} + \pi\alpha \frac{\delta^3(r)}{m^2} \left( \frac{c_2^{(a)}}{2} - 16 c_2^{(c)} \right) + \frac{\alpha}{4m^2} \frac{\mathbf{L}\cdot\boldsymbol{\sigma}}{r^3} c_2^{(b)} + \cdots$
- At leading order the bound-state field $\phi$ just satisfies the Schrödinger equation.
- $\Delta\mathcal{L}$ contains terms suppressed by powers of $r$:

$$\Delta\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \int d^3r\, \phi^\dagger \mathbf{r} \cdot e\mathbf{E} \phi + \cdots$$

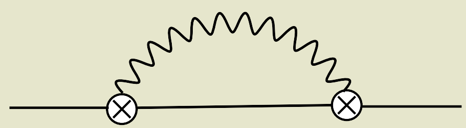


dipole interaction

# The Hydrogen spectrum at $O(m\alpha^5)$

The Hydrogen spectrum at order $m\alpha^5$ (responsible for the Lamb shift) reads

$$E_n = \langle n | \frac{\mathbf{p}^2}{m} + V | n \rangle + \Delta E_n$$

$\Delta E_n = \langle n|$  $|n\rangle$

$$= -\frac{\alpha}{3\pi}\left(\frac{2\pi\alpha}{m^2}\right)|\phi_n(0)|^2\left(-\frac{5}{3} + 2\ln 2 + \ln\frac{m^2}{\mu^2}\right)$$
$$-\frac{\alpha}{3\pi}\sum_i \left|\langle n|\frac{\mathbf{p}}{m}|i\rangle\right|^2 (E_n - E_i)\ln\left(\frac{E_n - E_i}{m}\right)^2$$

- The $\mu$ dependence cancels against the $\mu$ dependence of $c_2^{(a)}$ in the potential.
- The Bethe logarithm $\sim \alpha^5 \ln\alpha, \alpha^5$ follows from the one-loop diagram in the EFT.

○ Pineda Soto PLB 420 (1998) 391

# The static Lamb shift of QCD

## pNRQCD (in weak coupling)

pNRQCD is the EFT for nonrelativistic quark-antiquark pairs ($Q\bar{Q}$) near threshold.

- It is obtained by integrating out hard and soft gluons with $p$ or $E$ scaling like $m$, $mv$.
- The d.o.f. are $Q\bar{Q}$ pairs (sometimes cast in color singlet $S$ and color octet $O$) and ultrasoft modes (e.g. light quarks, low-energy gluons): $\phi = S$
- The Lagrangian is organized as an expansion in $1/m$ and $r$.
- The power counting is ($v \sim \alpha_{\rm s}$)
  $\rightarrow p \sim 1/r \sim mv$ (soft scale),
  $\rightarrow E \sim \mathbf{p}^2/2m \sim V^{(0)} \sim \mathbf{P}_{\rm cm} \sim 1/\mathbf{R}_{\rm cm} \sim mv^2$ (ultrasoft scale),
  $\rightarrow$ operators in $\Delta\mathcal{L}$ scale like $(mv^2)^{\rm dimension}$.

∘ Brambilla Pineda Soto Vairo RMP 77 (2005) 1423

## pNRQCD: Lagrangian

$$\mathcal{L}_{\mathrm{pNRQCD}} = \int d^3r\, S^\dagger \left( i\partial_0 - \frac{\mathbf{p}^2}{m} + \cdots - V \right) S + \Delta\mathcal{L}$$

$$\Delta\mathcal{L} = \int d^3r\ \mathrm{Tr}\left\{ O^\dagger \left( iD_0 - \frac{\mathbf{p}^2}{m} + \cdots - V_o \right) O \right.$$
$$\left. V_A O^\dagger \mathbf{r}\cdot g\mathbf{E}\, S + \mathrm{H.c.} + \frac{V_B}{2} O^\dagger \mathbf{r}\cdot g\mathbf{E}\, O + \mathrm{c.c.} \right\} + \cdots$$
$$-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \sum_{i=1}^{n_f} \bar{q}_i\, i\not{D} q_i$$

The (weak coupling) matching coefficients are the Coulomb potential:

$$V(r) = -C_F \frac{\alpha_{\mathrm{s}}}{r} + \ldots, \qquad V_o(r) = \frac{1}{2N} \frac{\alpha_{\mathrm{s}}}{r} + \ldots, \qquad N = 3,\ C_F = \frac{4}{3}$$

and $V_A = 1 + \mathcal{O}(\alpha_{\mathrm{s}}^2)$, $V_B = 1 + \mathcal{O}(\alpha_{\mathrm{s}}^2)$.

- Pineda Soto NP PS 64 (1998) 428
  Brambilla Pineda Soto Vairo NPB 566 (2000) 275

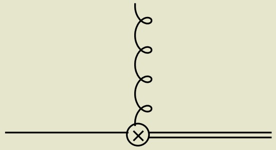
# Feynman rules



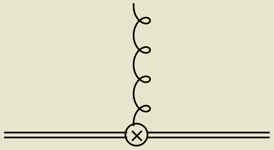
$$= \theta(t)\, e^{-it(\mathbf{p}^2/m+V)}$$



$$= \theta(t)\, e^{-it(\mathbf{p}^2/m+V_o)} \left(e^{-i\int dt\, A^{\rm adj}}\right)$$



$$= \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{S}$$

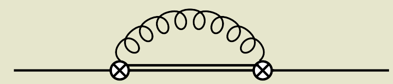


$$= \mathrm{O}^\dagger \{\mathbf{r}\cdot g\mathbf{E},\, \mathrm{O}\}$$

- Brambilla Pineda Soto Vairo NPB 566 (2000) 275, RMP 77 (2005) 1423

## Static potential (in weak coupling)

$$V^{(0)}(r,\mu') = \lim_{T\to\infty}\frac{i}{T}\ln\left\langle \Box \right\rangle -$$



$$+\cdots$$

$$= E_0(r) + \frac{i}{N}\int_0^\infty dt\, e^{-it(V_o-V)}\langle \mathrm{Tr}\,\mathbf{r}\cdot g\mathbf{E}(t)\,\mathbf{r}\cdot g\mathbf{E}(0)\rangle(\mu') + \cdots$$

- Brambilla Pineda Soto Vairo PRD 60 (1999) 091502

The static energy $E_0(r)$ is known at three loops:

$$E_0(r) = \Lambda_s - \frac{C_F\alpha_s}{r}(1+\#\alpha_s+\#\alpha_s^2+\#\alpha_s^3+\#\alpha_s^3\ln\alpha_s+\#\alpha_s^4\ln^2\alpha_s+\#\alpha_s^4\ln\alpha_s+\dots)$$

- Anzai Kiyo Sumino PRL 104 (2010) 112003
  A.Smirnov V.Smirnov Steinhauser PRL 104 (2010) 112002

## Infrared logarithms

$\ln \alpha_s$ in $E_0$ signals the cancellation of contributions coming from soft and ultrasoft gluons:

$$\ln \alpha_s = \ln \frac{\mu'}{1/r} + \ln \frac{\alpha_s/r}{\mu'}$$

The appearance of a static Lamb shift in QCD relies on the fact that a static quark-antiquark pair may change its color configuration from singlet to octet and in so doing it generates the scale $V - V_o$. Such a scale does not exist in QED and the corresponding diagram vanishes in dimensional regularization.

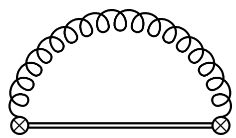
IR logarithms in the potential may be more easily computed in the EFT. Because of the cancellation mechanism between soft and ultrasoft contributions, they may be computed from the UV divergences of the chromoelectric correlator. Three (four) loop logarithms only require the computation of the one (two) loop chromoelectric correlator.

The cancellation between soft and ultrasoft divergences in the static energy solved the long standing ADM problem.

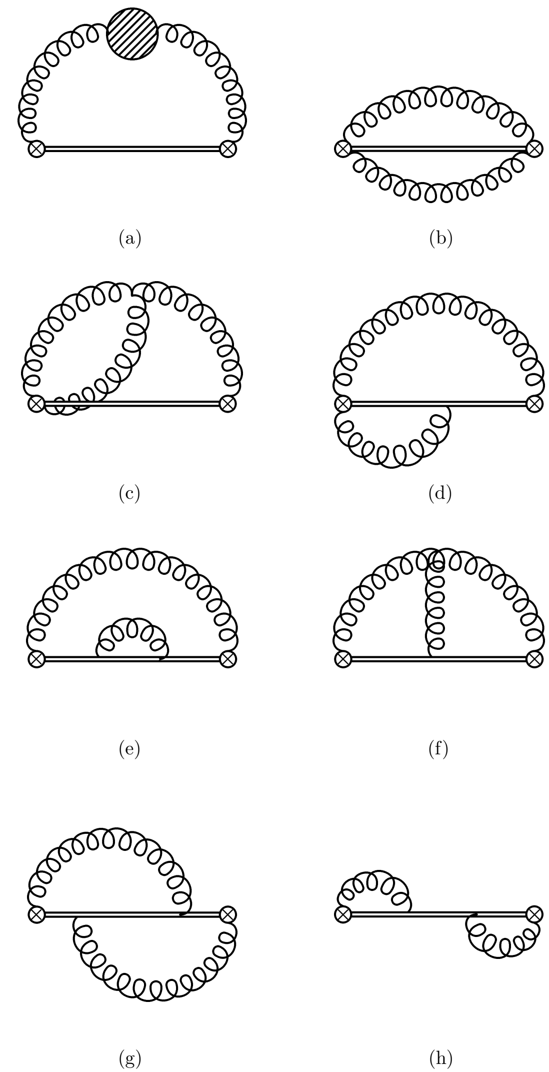

○ Appelquist Dine Muzinich PRD 17 (1978) 2074

# Chromoelectric field correlator: $\langle E(t)E(0)\rangle$

It is known at two loops.



LO




NLO

- Eidemüller Jamin PLB 416 (1998) 415

## Static singlet potential at N$^4$LO

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_{\rm s}(1/r)}{r} \Bigg\{ 1 + \frac{\alpha_{\rm s}(1/r)}{4\pi} a_1 + \left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^2 a_2 \\
& + \left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^3 \left[\frac{16\,\pi^2}{3} N^3 \,\ln r\mu + a_3\right] \\
& + \left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^4 \left[a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\, N^3 \beta_0(-5 + 6\ln 2)\right) \ln r\mu + \dots\right] \\
& + \qquad \cdots \qquad \Bigg\}
\end{aligned}
$$

- Brambilla Pineda Soto Vairo PRD 60 (1999) 091502
  Brambilla Garcia Soto Vairo PLB 647 (2007) 185

# Resummation of logarithms

The potential satisfies renormalization group equations

$$\begin{cases} \mu' \dfrac{d}{d\mu'} V^{(0)} = -\dfrac{2}{3} C_F \dfrac{\alpha_{\rm s}}{\pi} r^2 \left[V_o^{(0)} - V^{(0)}\right]^3 \left(1 + \dfrac{\alpha_{\rm s}}{\pi}\, c\right) \\ \mu' \dfrac{d}{d\mu'} V_o^{(0)} = \dfrac{1}{N} \dfrac{\alpha_{\rm s}}{\pi} r^2 \left[V_o^{(0)} - V^{(0)}\right]^3 \left(1 + \dfrac{\alpha_{\rm s}}{\pi}\, c\right) \\ \mu' \dfrac{d}{d\mu'} \alpha_{\rm s} = \alpha_{\rm s} \beta(\alpha_{\rm s}); \qquad\qquad c = \dfrac{-5n_f + N(6\pi^2 + 47)}{108} \end{cases}$$

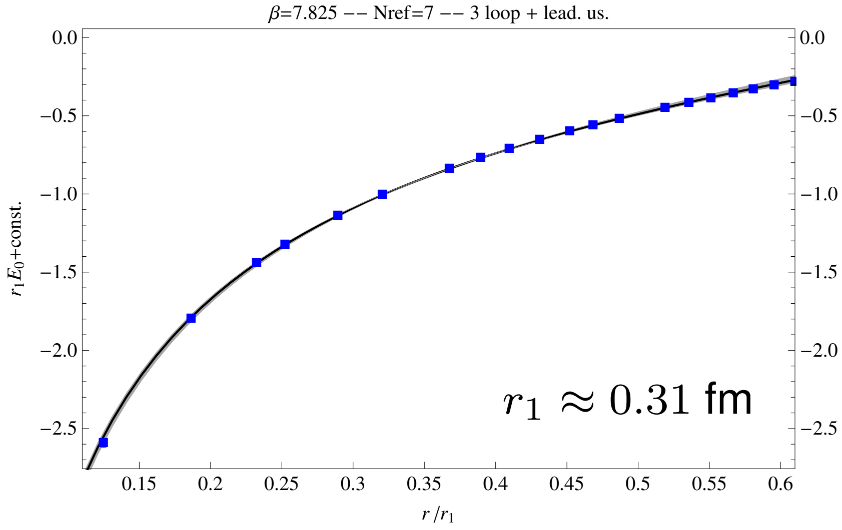
whose solution provides $V^{(0)}$ with N$^3$LL accuracy:

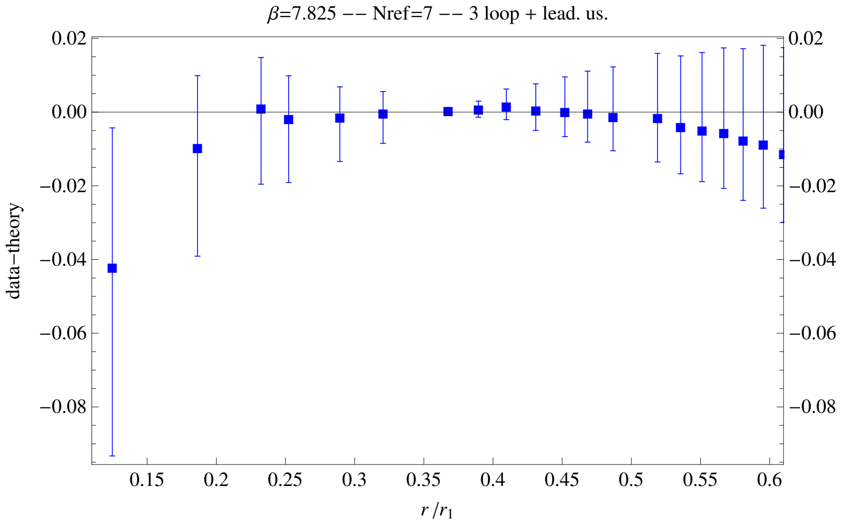
$$V^{(0)}(r,\mu') = V^{(0)}(r,1/r) - \frac{C_F N^3}{6\beta_0} \frac{\alpha_{\rm s}^3(1/r)}{r} \left\{ \left(1 + \frac{3}{4} \frac{\alpha_{\rm s}(1/r)}{\pi} a_1\right) \ln \frac{\alpha_{\rm s}(1/r)}{\alpha_{\rm s}(\mu')} \right.$$
$$\left. \left(\frac{\beta_1}{4\beta_0} - 6c\right) \left[\frac{\alpha_{\rm s}(\mu')}{\pi} - \frac{\alpha_{\rm s}(1/r)}{\pi}\right] \right\}$$


- Pineda Soto PLB 495 (2000) 323
  Brambilla Garcia Soto Vairo PRD 80 (2009) 034016

# Static energy vs lattice data

A typical application is the comparison of the perturbative expression of the QCD static energy with lattice data at short distances and the extraction of $\Lambda_{\overline{\rm MS}}$ or equivalently $\alpha_s$:

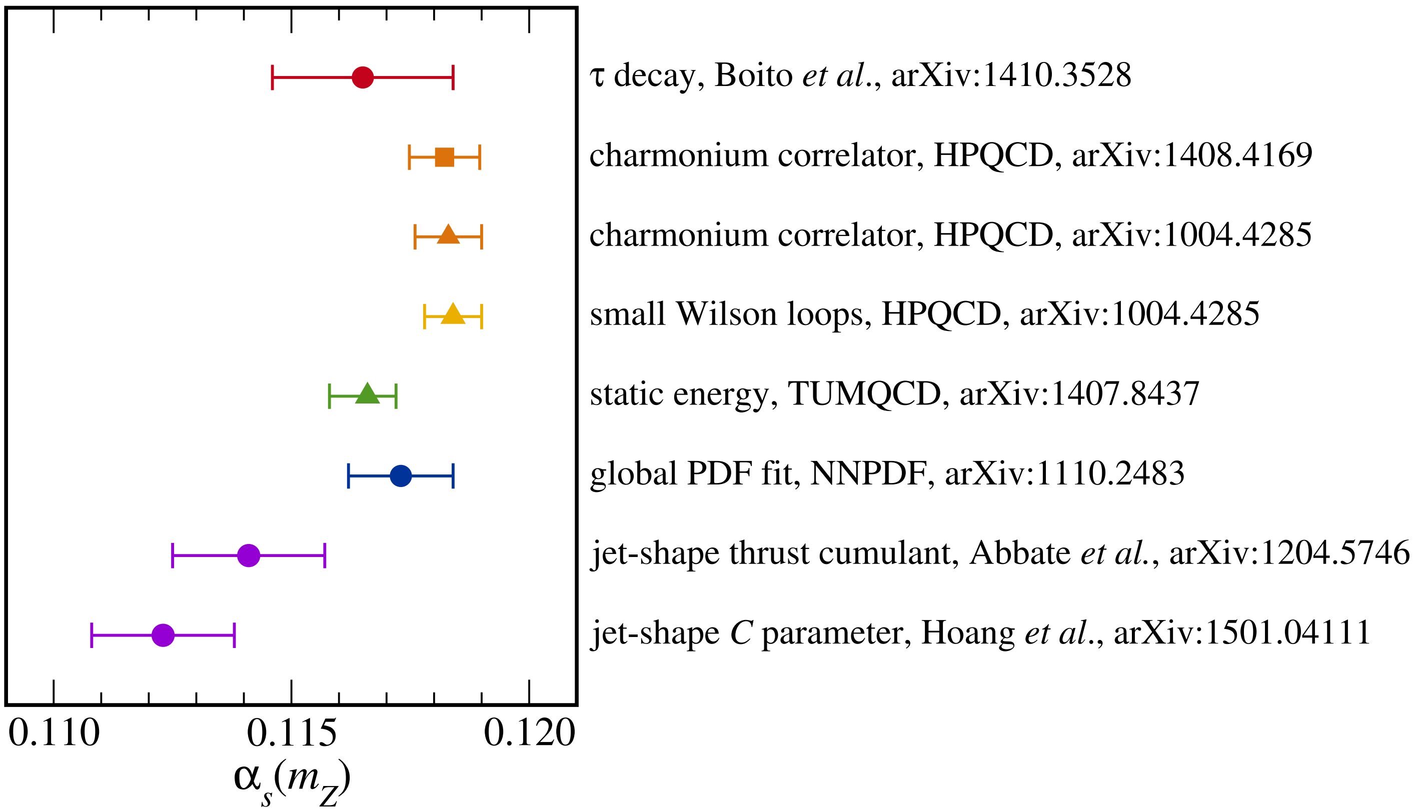



$$\Lambda_{\overline{\rm MS}} = 315^{+18}_{-12}\ \text{MeV} \quad \text{or} \quad \alpha_s(M_Z) = 0.1166^{+0.0012}_{-0.0008}$$

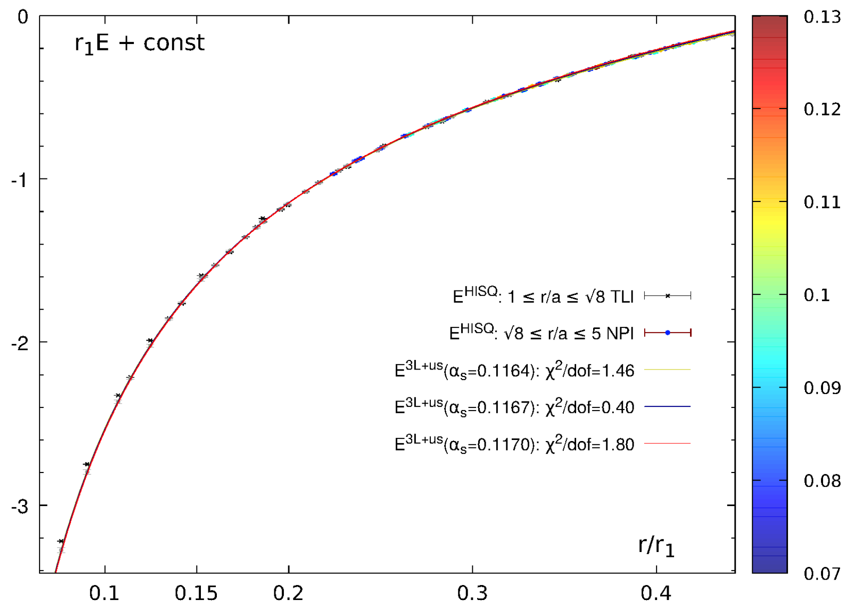
- Bazavov Brambilla Garcia Petreczky Soto Vairo PRD 90 (2014) 074038

# Comparison with other determinations




- Kronfeld 2016

# Static energy vs lattice data @ very short distances




$$\Lambda_{\overline{\rm MS}} = 314.0^{+15.5}_{-8}\ \text{MeV} \quad \text{or} \quad \alpha_{\rm s}(M_Z) = 0.11660^{+0.00110}_{-0.00056}$$

- TUMQCD coll PRD 100 (2019) 114511

## This branch of the family: some of the grandchildren

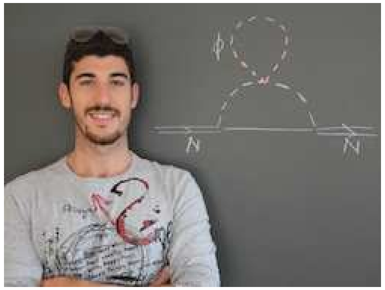
Emanuele Mereghetti @LANL, Los Alamos

Research: Low energy hadron and nuclear physics
neutrinoless double beta decay, chiral EFT, Higgs-top

Jacopo Ghiglieri @CNRS, SUBATECH, Nantes

Research: thermal field theory, sterile neutrinos
thermal leptogenesis, gravitational waves in plasma

# This branch of the family: some of the grandchildren



Simone Biondini @University of Basel

Research: thermal field theory, thermal leptogenesis
dark matter

Vladyslav Shtabovenko @KIT, Karlsruhe

Research: Multi loop/legs, FeynCalc, B physics

## Il cavaliere inesistente