LFU ratios in B decays using Lattice QCD and Unitarity

Work in collaboration with G. Martinelli, M. Naviglio and S. Simula

[PRD ’21 (2105.02497), PRD ’21 (2105.07851), PRD ’22 (2105.08674), 2109.15248, 2202.10285, …]

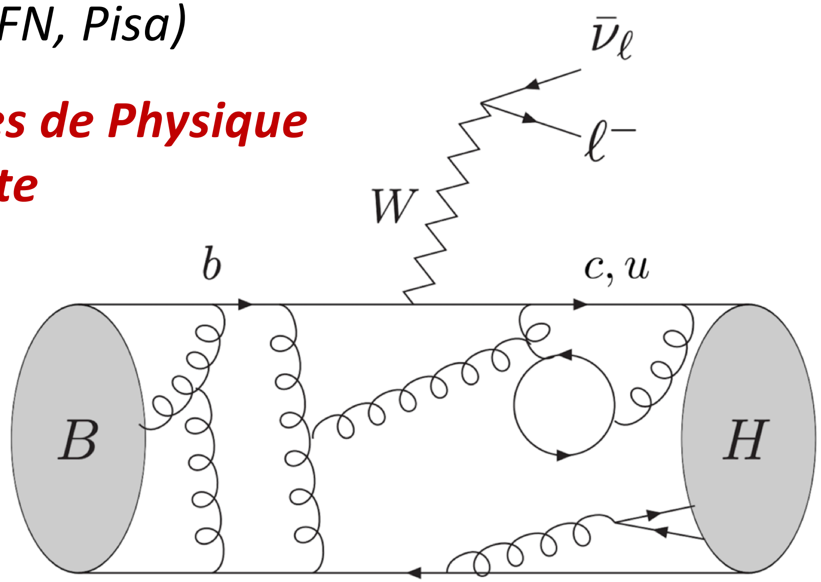
Ludovico Vittorio (SNS & INFN, Pisa)

La Thuile 2022 - Les Rencontres de Physique de la Vallée d'Aoste

SCUOLA NORMALE SUPERIORE

INFN PISA
Istituto Nazionale di Fisica Nucleare
Sezione di Pisa

MINISTERO DELL' ISTRUZIONE, DELL'UNIVERSITÀ E DELLA RICERCA
PRIN “The consequences of flavor”



(from J.Phys.G 46 (2019) 2, 023001)

State-of-the-art of the semileptonic B → D(*) decays

Two critical issues:

- $\boxed{V_{cb}}$ puzzle:

$$\boxed{\begin{array}{c} 2.8\,\sigma \\ \text{discrepancy!} \end{array}}$$

EXCLUSIVE

FLAG Review 2021 [arXiv:2111.09849]

$$|V_{cb}| \times 10^3 = 39.36(68)$$

Fit only to the B → (D, D) channels*

VS

INCLUSIVE

$$|V_{cb}| \times 10^3 = 42.00(65)$$

$$|V_{cb}| \times 10^3 = 42.16(50)$$

Bordone et al., Phys.Lett.B [2107.00604]

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3.4σ discrepancy!

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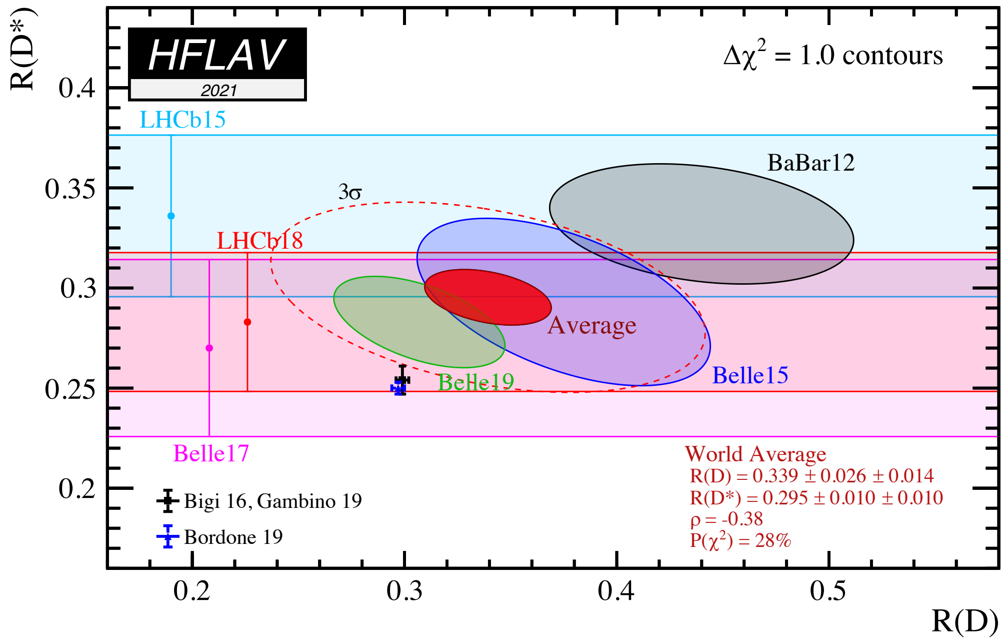

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$$R(D) = 0.298 \pm 0.003$$

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and

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The Dispersive Matrix (DM) method completely changes the picture!

$\boxed{V_{cb}}$

$\boxed{1\sigma}$ compatibility

EXCLUSIVE

$$\boxed{|V_{cb}| \times 10^3 = 41.4 \pm 0.8}$$

Combination of the $B_{(s)} \rightarrow (D, D^*, D_s, D_s^*)$ channels

VS

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$\boxed{R_{D^{(*)}}}$

The **DM method** allows to lighten both the problems!

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and

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$\boxed{1.3\sigma}$ compatibility

The central role of the Form Factors (FFs) in excl. semil. *B* decays

- Production of a pseudoscalar meson (*i.e. D*):

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{24\pi^3} \left(1 - \frac{m_\ell^2}{q^2}\right)^2 \times \left[|\vec{p}_D|^3 \left(1 + \frac{m_\ell^2}{2q^2}\right) |f^+(q^2)|^2 + m_B^2 |\vec{p}_D| \left(1 - \frac{m_D^2}{m_B^2}\right)^2 \frac{3m_\ell^2}{8q^2} |f^0(q^2)|^2 \right]$$

- Production of a vector meson (*i.e. D**):

$$\frac{d\Gamma_\tau}{dw} = \frac{d\Gamma_{\tau,1}}{dw} + \frac{d\Gamma_{\tau,2}}{dw} \quad \left\{ \begin{array}{l} \dfrac{d\Gamma_{\tau,1}}{dw} = \left(1 - \dfrac{m_\tau^2}{q(w)^2}\right)^2 \left(1 + \dfrac{m_\tau^2}{2q(w)^2}\right) \times \dfrac{d\Gamma}{dw} \qquad \left[r = m_{D^*}/m_B\right] \\ \dfrac{d\Gamma}{dw} = \dfrac{\eta_{EW}^2 G_F^2 m_{D^*}^2 |V_{cb}|^2}{48\pi^3 m_B} \sqrt{w^2-1} \left[2\, q^2(w) \left(f(w)^2 + m_B^2 m_{D^*}^2 \left(w^2-1\right) g(w)^2\right) + \mathcal{F}_1(w)^2\right] \\ \dfrac{d\Gamma_{\tau,2}}{dw} = \dfrac{\eta_{EW}^2 |V_{cb}|^2 G_F^2 m_B^5}{32\pi^3} \dfrac{m_\tau^2 (m_\tau^2 - q(w)^2)^2 r^3 (1+r)^2 (w^2-1)^{3/2} P_1(w)^2}{q(w)^6} \end{array} \right.$$

Relation between the momentum transfer and the recoil:

$$q^2 = m_B^2 + m_P^2 - 2 m_B m_P\, w$$

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Two FFs coupled to the lepton mass:

$$f_0(w)\ \text{(pseudoscalar)},\ P_1(w)\ \text{(vector)}$$

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

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The resulting description of the FFs

- **will be entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **will be independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

→ No HQET, no series expansion, no perturbative bounds with respect to the well-known other parametrizations

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How does it work?

The DM method

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \ldots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \ldots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \ldots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \ldots & \frac{1}{1-z_2 z_N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \ldots & \frac{1}{1-z_N^2} \end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \ (\text{with } i = 1, 2, \ldots N)$$

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

t: momentum transfer

The DM method

Non-perturbative values of the susceptibilities from the dispersion relations (see PRD '21 (2105.07851) and 2202.10285)

Estimates of the FFs, computed on the lattice

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One can show that

$$\det \mathbf{M} \geq 0$$

$$\Downarrow$$

$$f_{\rm lo}(z) \leq f(z) \leq f_{\rm up}(z)$$

Values of the momentum transfer @ which FFs are computed on the lattice

The “problematic” semileptonic **B → D*** channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

Two kinematical constraints (KCs):

$$\mathcal{F}_1(1) = (m_B - m_{D^*}) f(1)$$

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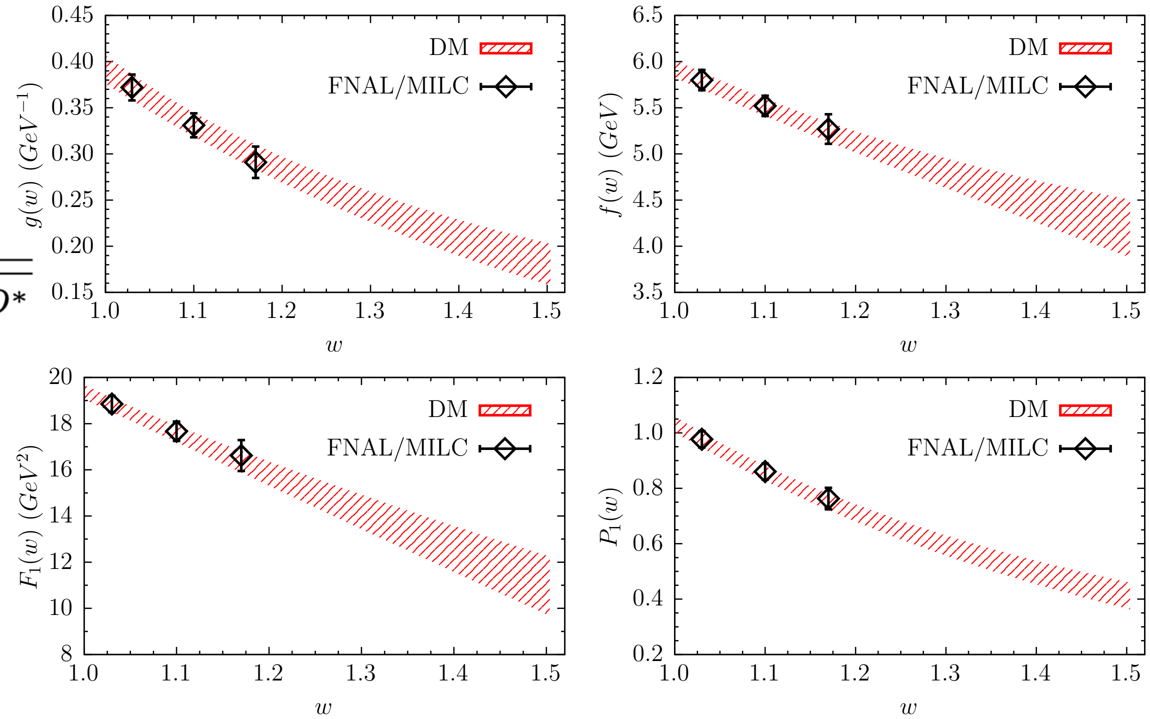

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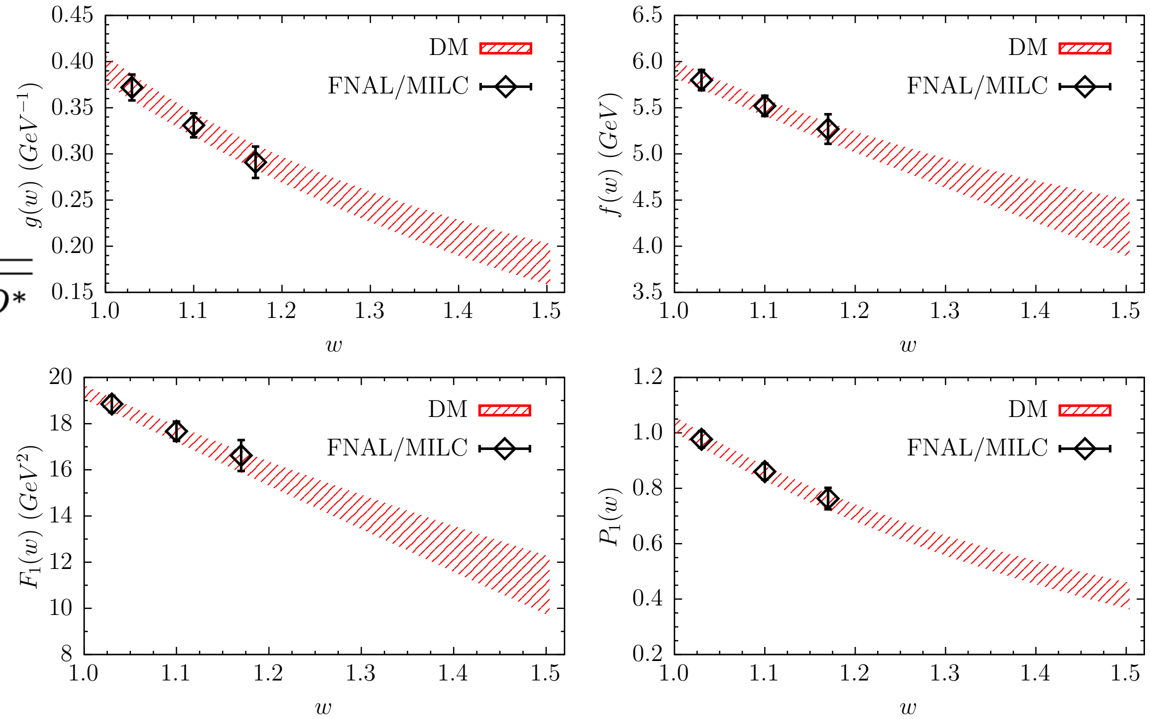

$$f(w_{max}) = 4.19 \pm 0.31 \text{ GeV} ,$$

$$g(w_{max}) = 0.180 \pm 0.023 \text{ GeV}^{-1} ,$$

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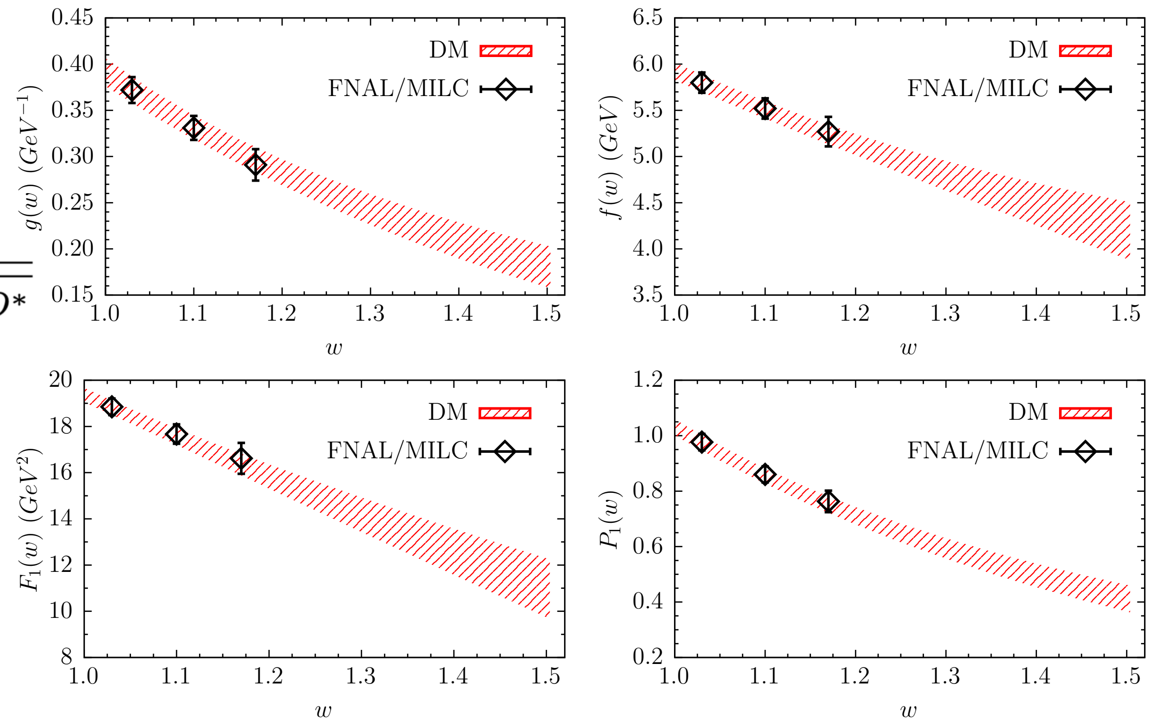

Comparison with Light Cone Sum Rules:

$$\begin{aligned} f(w_{max})|_{\rm LCSR} &= 4.37 \pm 0.66, \\ g(w_{max})|_{\rm LCSR} &= 0.19 \pm 0.04, \\ \mathcal{F}_1(w_{max})|_{\rm LCSR} &= 16.0 \pm 2.1. \end{aligned}$$

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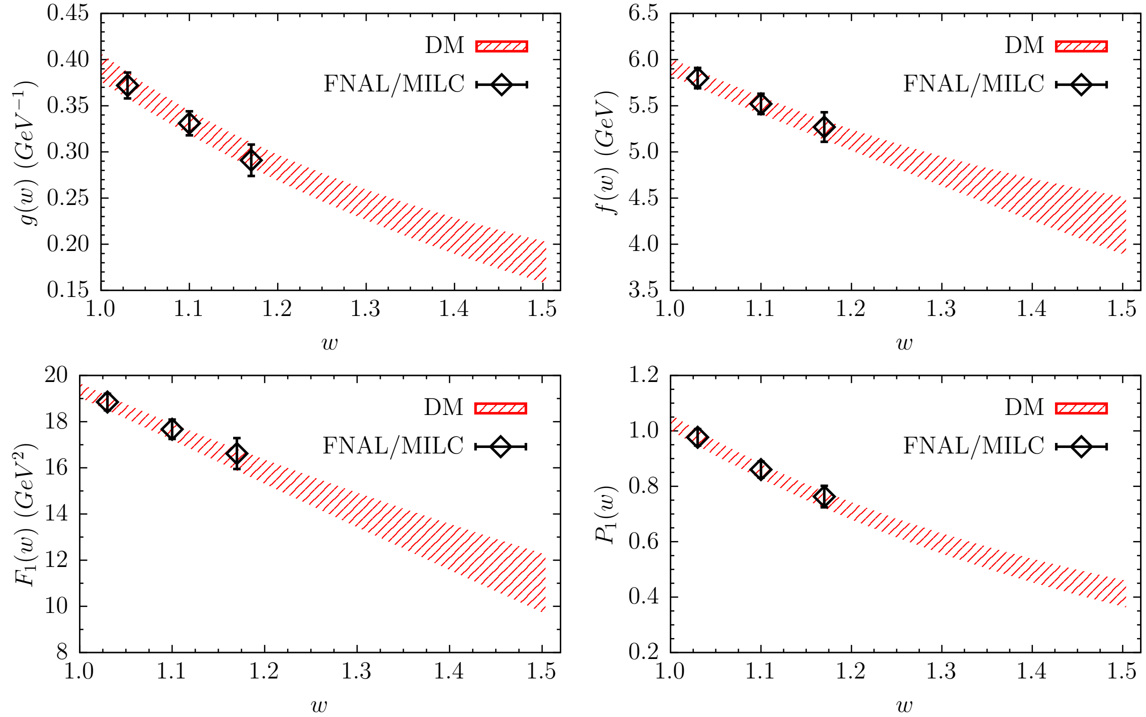
DM prediction

$$R(D^*) = 0.275 \pm 0.008$$



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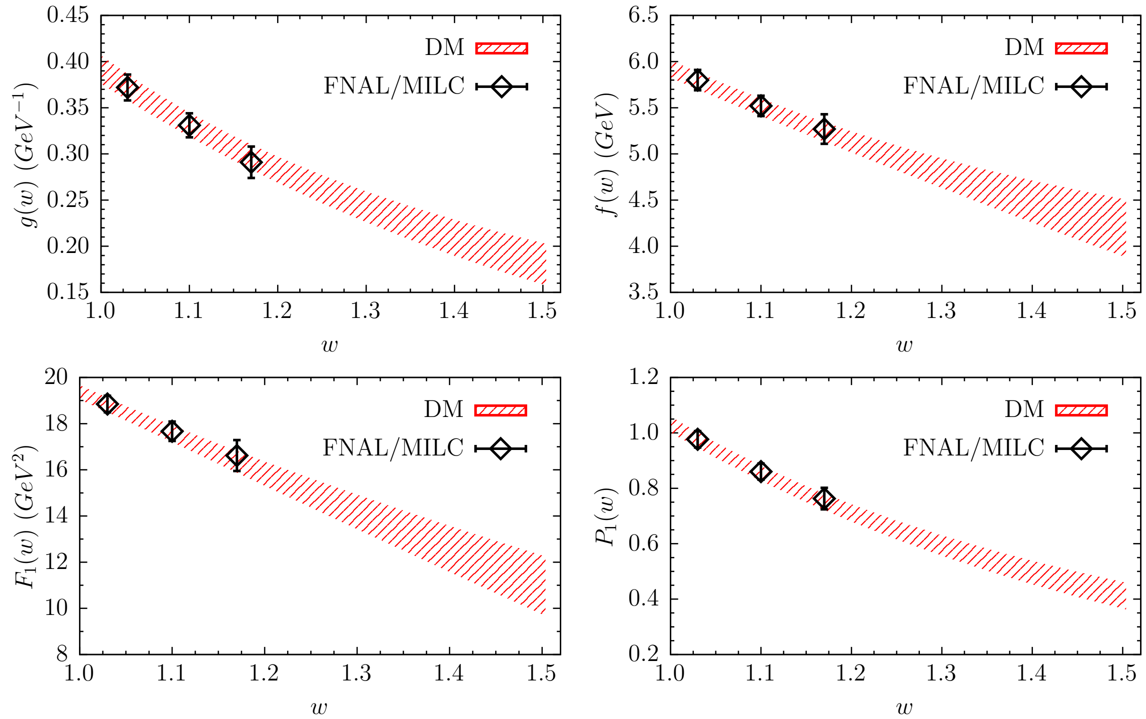

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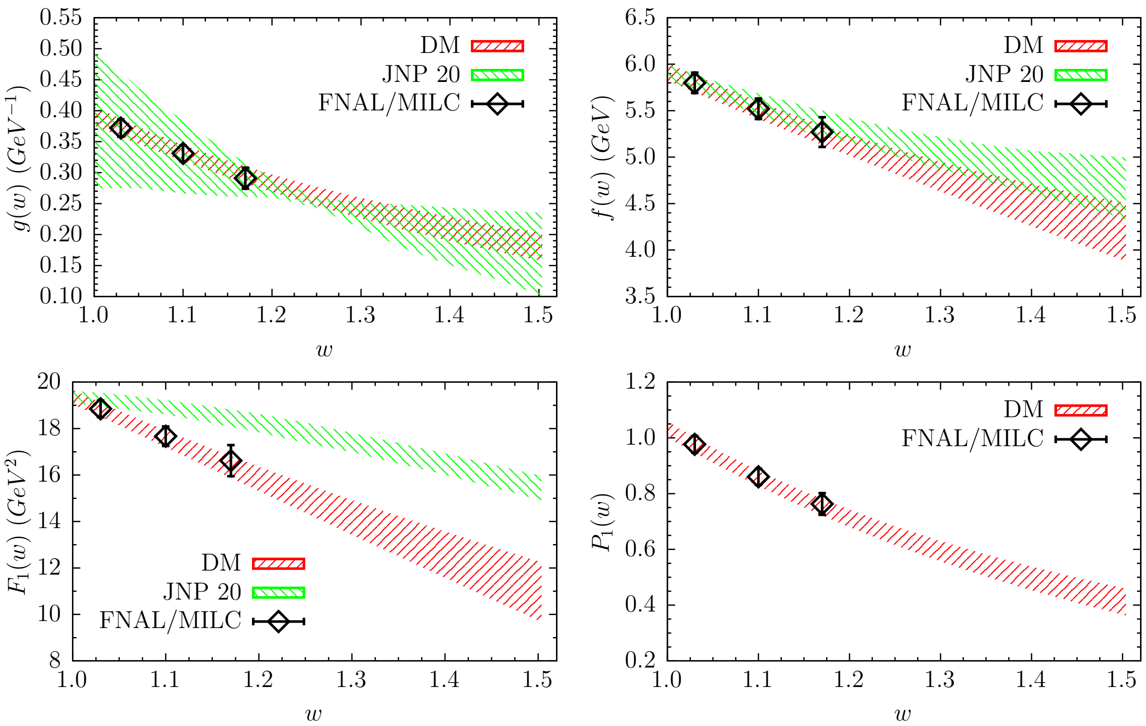

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VS

HFLAV '21 SM prediction

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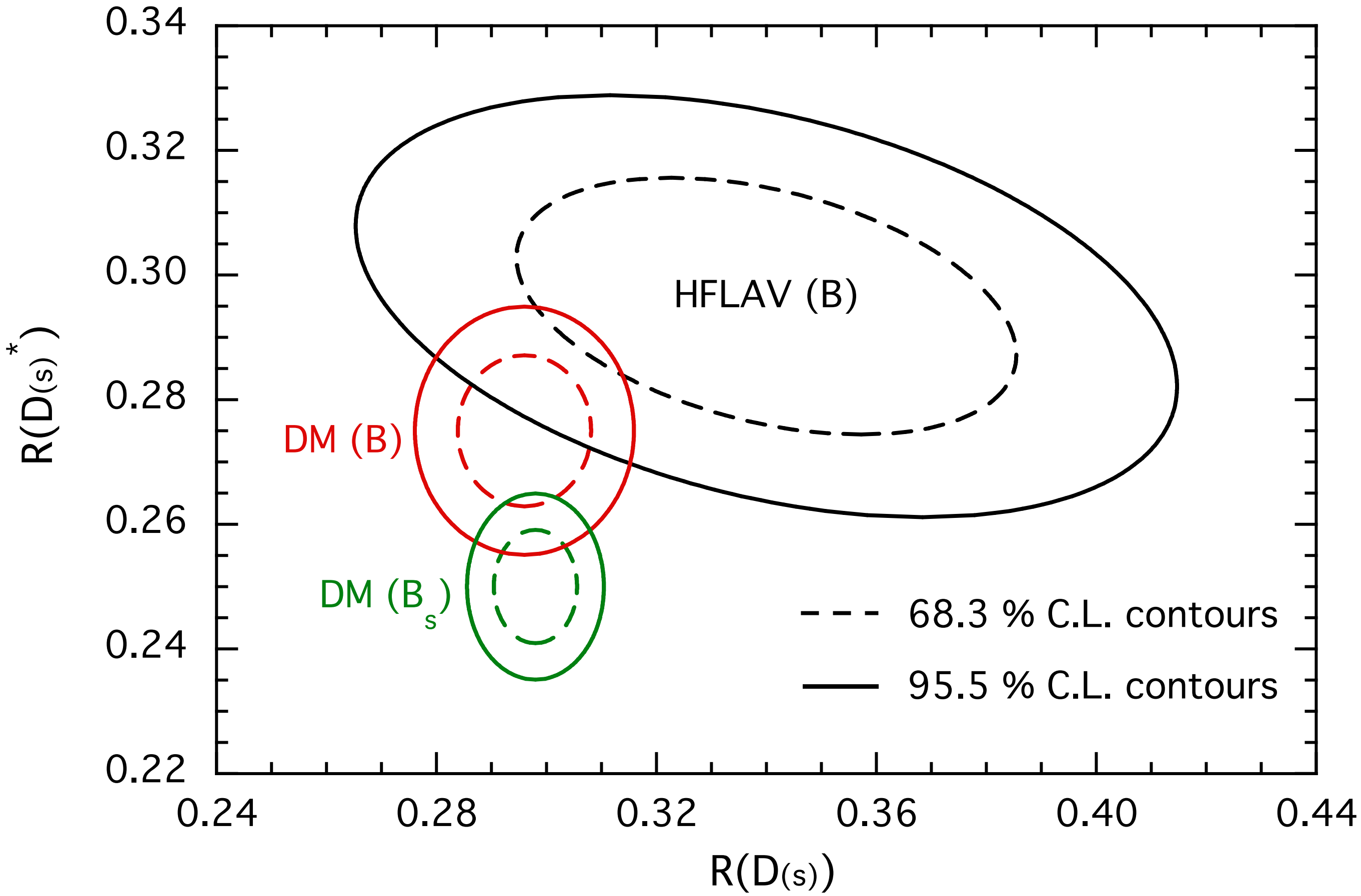
The DM expectations of *all* the LFU observables



B → D case (**PRD ’22 (2105.08674)**):

- **3 FNAL/MILC data for each FF (see PRD ‘15 (1503.07237))**

$B_s \to D_s^*$ case (***in prep.***):

- **3 data points for each FF from HPQCD fits (see PRD ‘20 (1906.00701) and arXiv:2105.11433)**

The anomalies in the B sector have been very lightened!

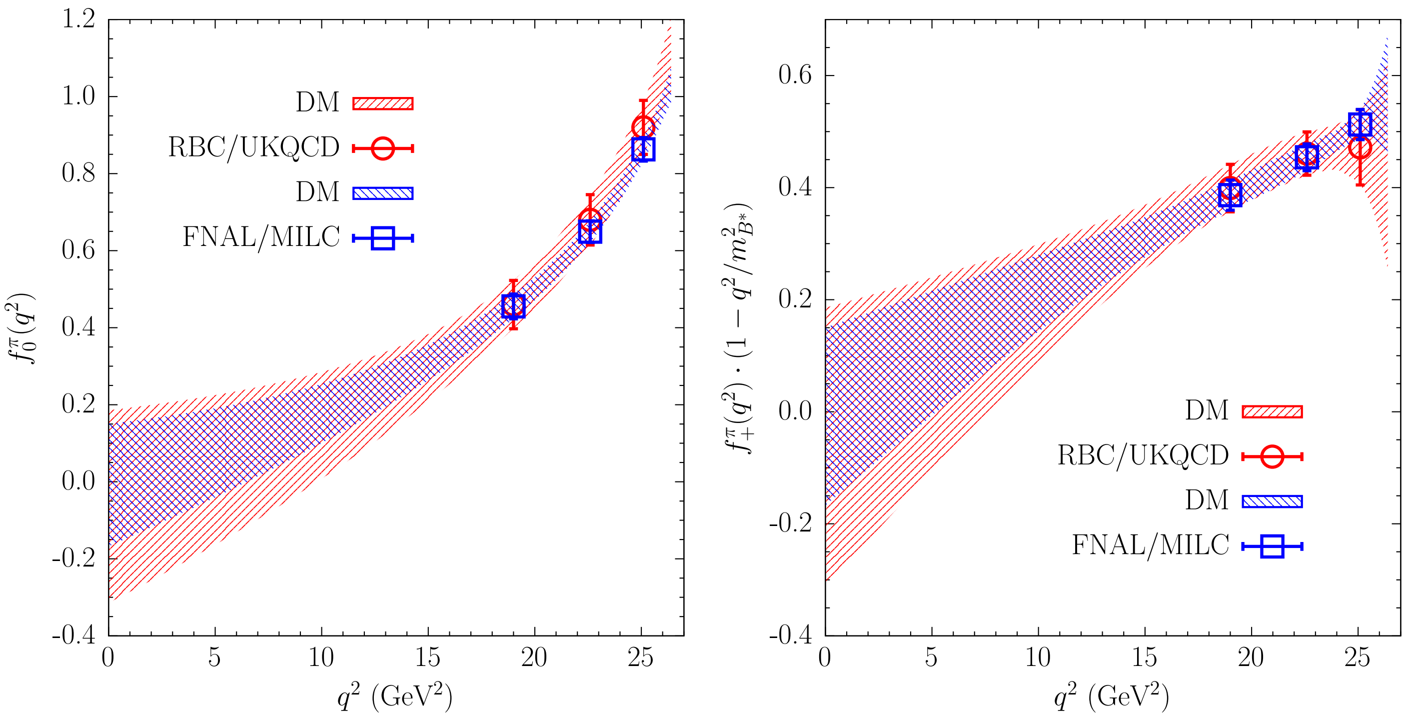
1° important issue: *avoiding the mixing between theory and exps in the description of the FFs is fundamental!*

LFU in semileptonic B → π decays

Two LQCD inputs have been used for our DM method (**arXiv:2202.10285**):

- **3 RBC/UKQCD data (points) for each FF [PRD '15 (1501.05363)]**
- **3 FNAL/MILC data (squares) for each FF [PRD '15 (1503.07839)]**

One KC: $f_0(0) = f_+(0)$



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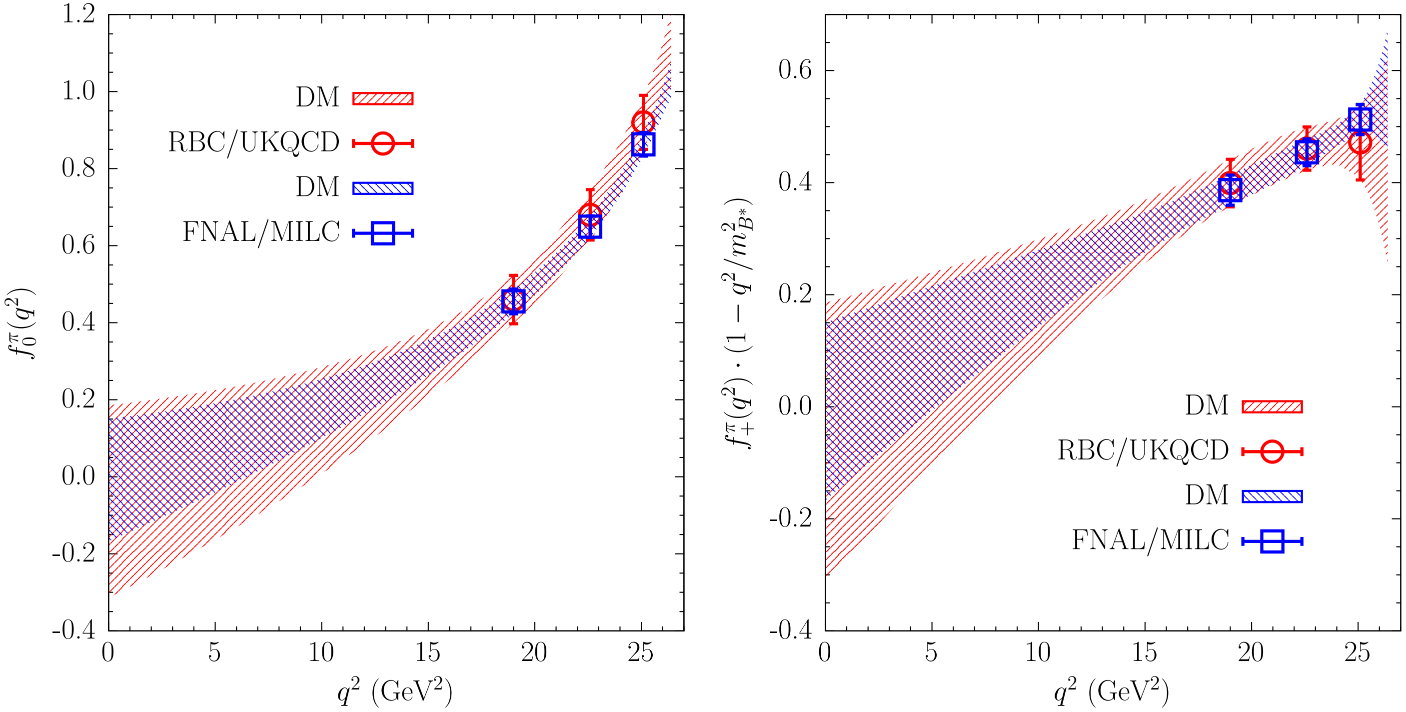
Peculiarity of B → π decays: LONG extrapolation in q^2

$$f^{\pi}(q^2=0)|_{\mathrm{RBC/UKQCD}} = -0.06 \pm 0.25$$

$$f^{\pi}(q^2=0)|_{\mathrm{FNAL/MILC}} = -0.01 \pm 0.16$$

$$f^{\pi}(q^2=0)|_{\mathrm{combined}} = -0.04 \pm 0.22$$



It seems that the mean value and the uncertainty are not stable under variation of the truncation order of a series expansion of the FFs…

The DM approach is independent of this issue!!!

LFU in semileptonic B → π decays

Two LQCD inputs have been used for our DM method (**arXiv:2202.10285**):

- **3 RBC/UKQCD data (points) for each FF [PRD '15 (1501.05363)]**
- **3 FNAL/MILC data (squares) for each FF [PRD '15 (1503.07839)]**

Peculiarity of B → π decays: LONG extrapolation in q^2

$$f^\pi(q^2=0)|_{\mathrm{RBC/UKQCD}} = -0.06 \pm 0.25$$

$$f^\pi(q^2=0)|_{\mathrm{FNAL/MILC}} = -0.01 \pm 0.16$$

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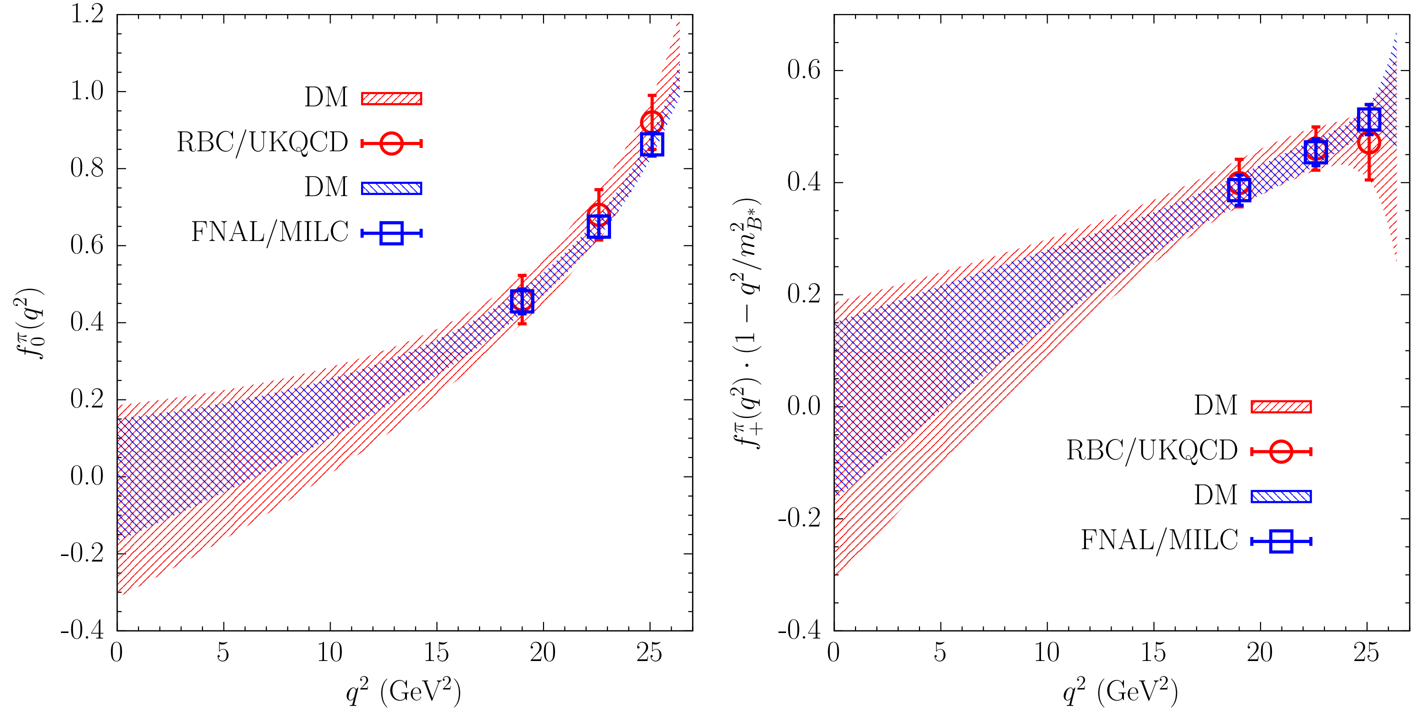

2° important issue: *the DM method equivalent to the results of **all** possible (BCL) fits which satisfy unitarity and at the same time reproduce exactly the input data*

LFU in semileptonic B → π decays

This issue is of capital importance to test LFU:

$$R_\pi^{\tau/\mu} \equiv \frac{\Gamma(B \to \pi\tau\nu_\tau)}{\Gamma(B \to \pi\mu\nu_\mu)}$$

THEORY with DM method

Input	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767(145)	0.838(75)	0.793(118)

EXPERIMENT

$$R_\pi^{\tau/\mu}|_{exp} = 1.05 \pm 0.51$$

LFU in semileptonic B → π decays

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EXPERIMENT

$$R_\pi^{\tau/\mu}|_{exp} = 1.05 \pm \cancel{0.51}$$

Expected improved precision @ Belle II (PTEP '19 (1808.10567))

$$\delta R_\pi^{\tau/\mu} \simeq 0.09$$

~80% reduction of the error!

LFU in semileptonic B → π decays

This issue is of capital importance to test LFU:

$$R_\pi^{\tau/\mu} \equiv \frac{\Gamma(B \to \pi \tau \nu_\tau)}{\Gamma(B \to \pi \mu \nu_\mu)}$$

THEORY with DM method

Input	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767($\cancel{145}$)	0.838($\cancel{75}$)	0.793($\cancel{118}$)

Expected improved precision in LQCD computations of the FFs @ high momentum transfer

Input	RBC/UKQCD	FNAL/MILC	combined
$\delta R_\pi^{\tau/\mu}$	0.73	0.38	0.59

Hypothetical 50% reduction of the error...

EXPERIMENT

$$R_\pi^{\tau/\mu}|_{exp} = 1.05 \pm \cancel{0.51}$$

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LFU in semileptonic B → π decays

This issue is of capital importance to test LFU:

$$R_\pi^{\tau/\mu} \equiv \frac{\Gamma(B \to \pi\tau\nu_\tau)}{\Gamma(B \to \pi\mu\nu_\mu)}$$

THEORY with DM method

Input	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	$0.767(\cancel{145})$	$0.838(\cancel{75})$	$0.793(\cancel{118})$

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~80% reduction of the error!

For further investigation of possible NP effects in the future, it is fundamental to extrapolate appropriately the FFs behaviour in the whole kinematical range

Conclusions

The **Dispersion Matrix approach** is an attractive tool to implement unitarity and lattice QCD calculations in the analysis of exclusive semileptonic decays of mesons and hadrons. Its most important features are the following:

- it does not rely on **any assumption about the momentum dependence of the hadronic Form Factors**
- it can be **based entirely on first principles *(i.e.* unitarity and analiticity)** using lattice determinations both of the relevant Form Factors and of the dispersive bounds (the susceptibilities)

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Two important features of the DM approach for the determination of LFU observables have been investigated in this talk:

1. **it avoids mixing among theoretical calculations and experimental data** to describe the shape of the FFs
2. **it predicts band of values that are equivalent to the infinite number of (BCL) fits** satisfying unitarity and reproducing exactly a given set of data points

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The **Dispersion Matrix approach** is an attractive tool to implement unitarity and lattice QCD calculations in the analysis of exclusive semileptonic decays of mesons and hadrons. Its most important features are the following:

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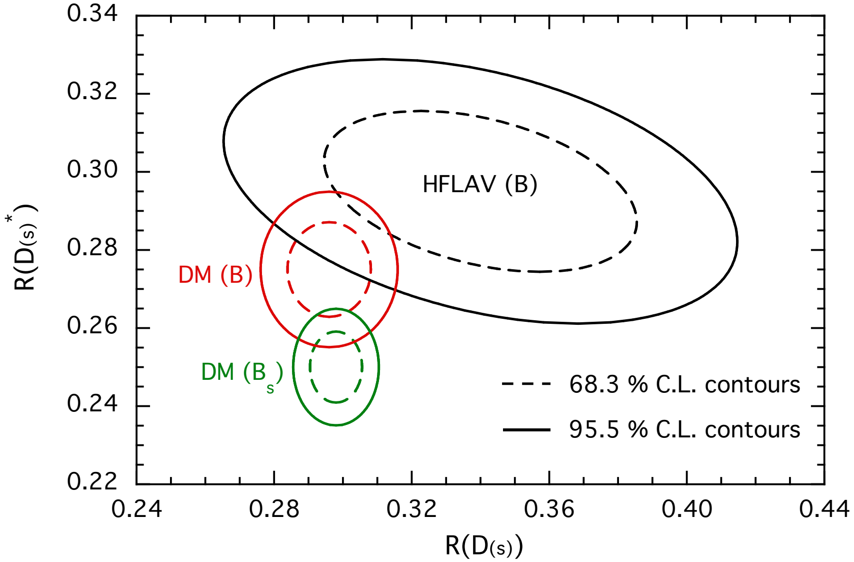
Phenomenological results for LFU:

1. **The anomalies in semileptonic (charged current) *B* decays have been lightened:** consistency between theory and experiment @ the 1.3σ level
2. **Possible SU(3)$_F$ symmetry breaking effects** in R(D*) vs R(D_s*)?
3. **Importance of DM method for B → π decays**, no LFU anomalies at present mainly due to the large experimental uncertainty



THANKS FOR

YOUR ATTENTION!

BACK-UP SLIDES

Exclusive Vcb determination through unitarity

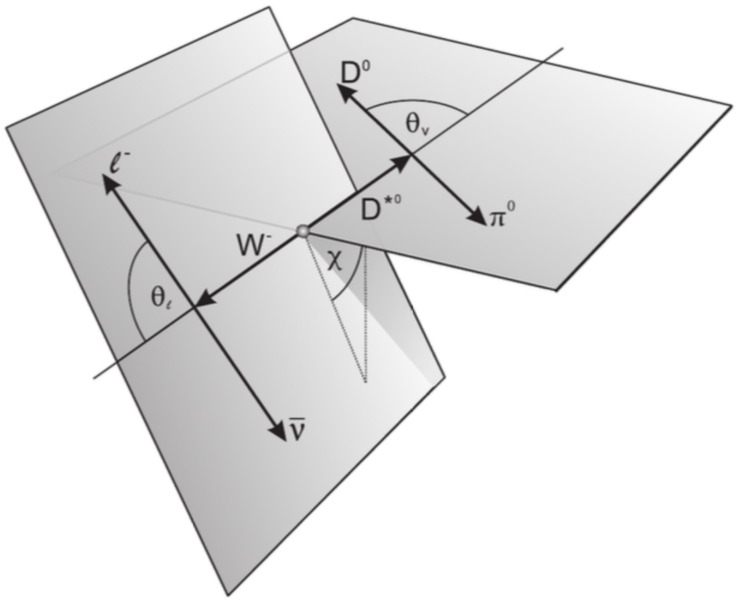
Starting from the FFs bands, we use the experimental data to compute bin-per-bin estimates of Vcb.

NB: the experimental data do NOT enter in the determination of the bands of the FFs

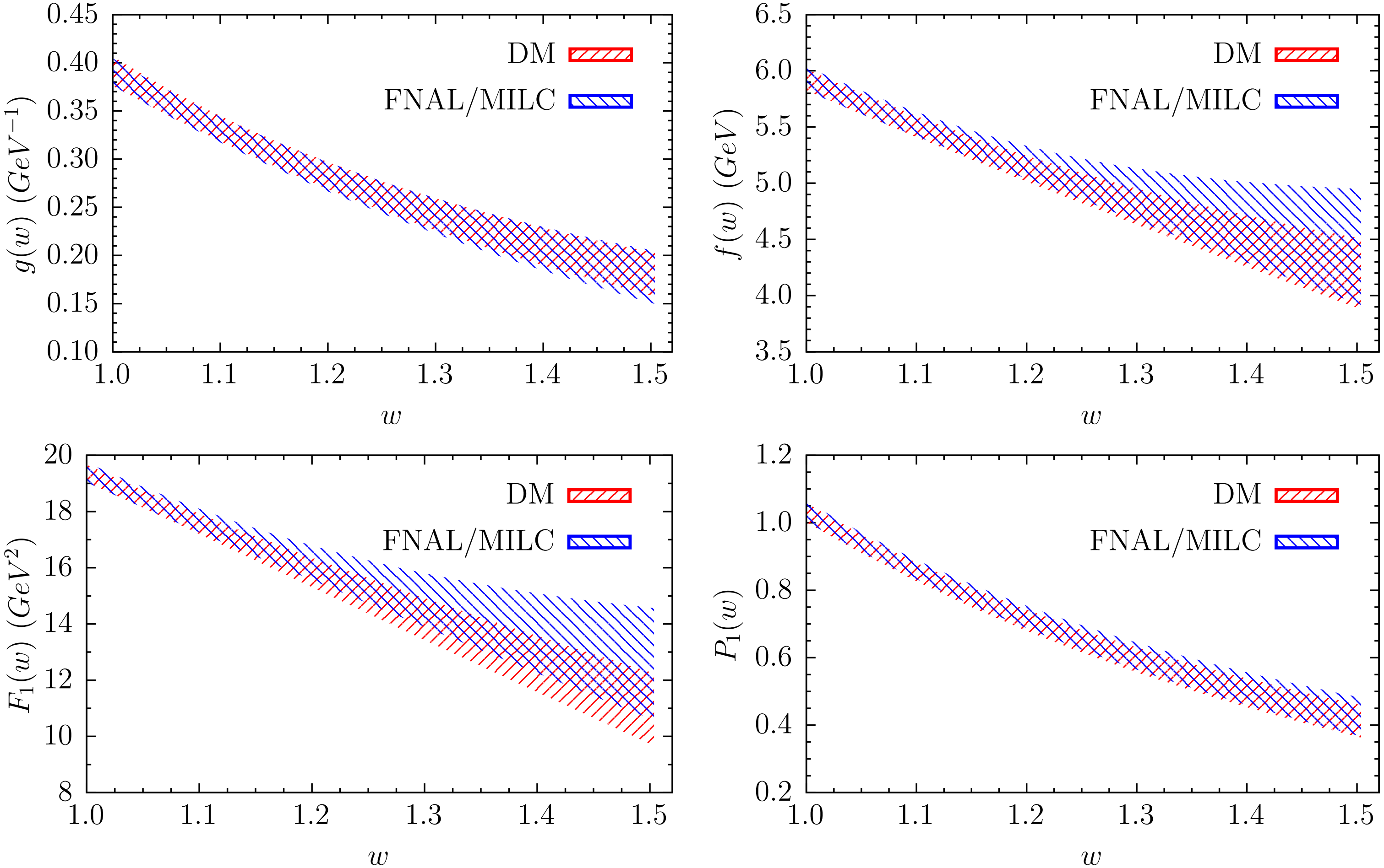
To do it, it is sufficient to compare the two sets of measurements of the differential decay widths

$$d\Gamma/dx, \quad x = w, \cos\theta_l, \cos\theta_v, \chi$$

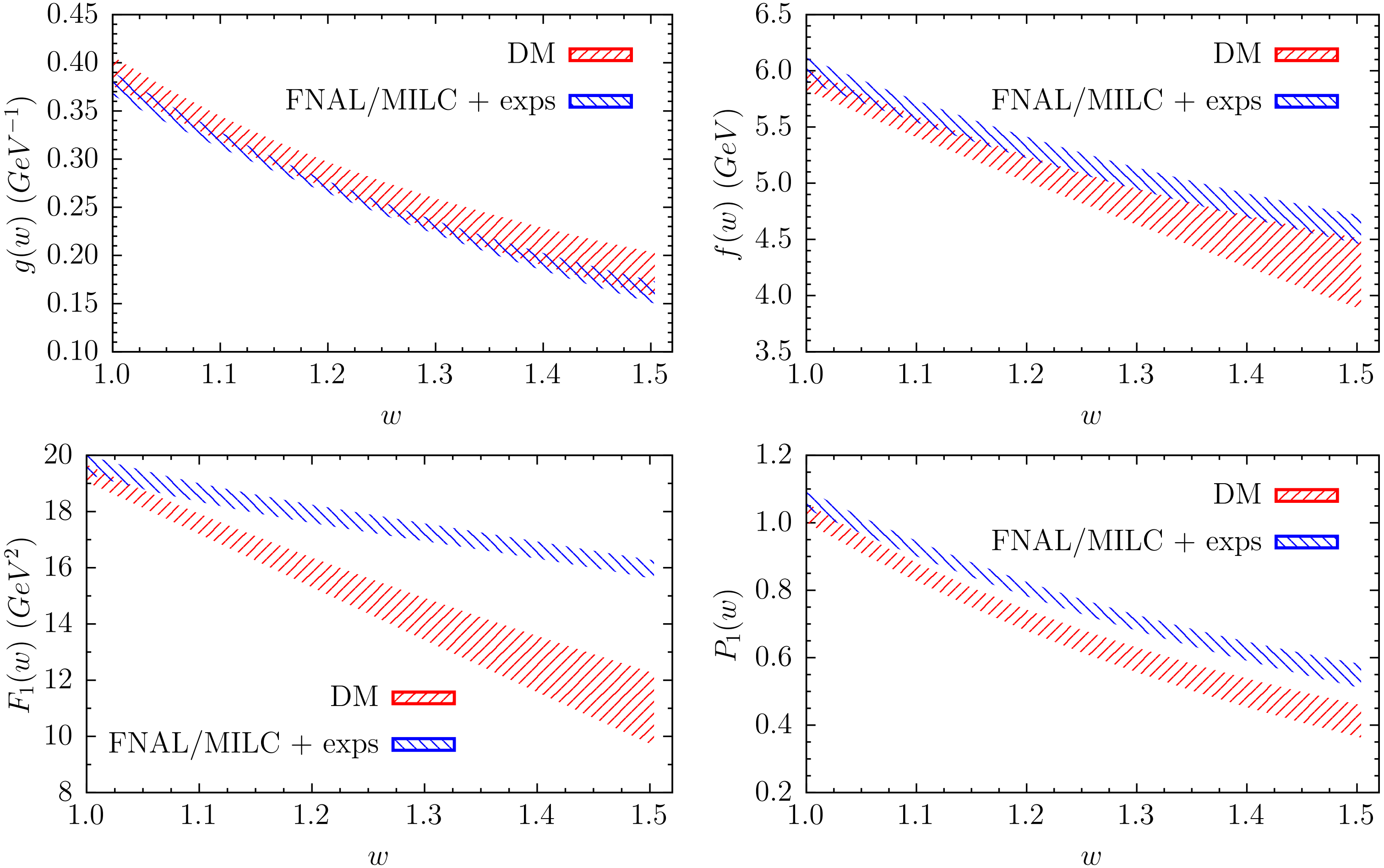
by the Belle Collaboration **(arXiv:1702.01521, arXiv:1809.03290)** with their *theoretical estimate*, computed through the unitarity bands shown before.



$$\frac{d\Gamma(B \to D^*(\to D\pi)\ell\nu)}{dw\,d\cos\theta_\ell\,d\cos\theta_v\,d\chi} = \frac{G_F^2|V_{cb}|^2\eta_{EW}^2}{4(4\pi)^4} 3m_B m_{D^*}^2\sqrt{w^2-1}$$
$$\times B(D^* \to D\pi)\{(1-\cos\theta_\ell)^2\sin^2\theta_v|H_+|^2$$
$$+(1+\cos\theta_\ell)^2\sin^2\theta_v|H_-|^2 + 4\sin^2\theta_\ell\cos^2\theta_v|H_0|^2$$
$$-2\sin^2\theta_\ell\sin^2\theta_v\cos 2\chi H_+H_-$$
$$-4\sin\theta_\ell(1-\cos\theta_\ell)\sin\theta_v\cos\theta_v\cos\chi H_+H_0$$
$$+4\sin\theta_\ell(1+\cos\theta_\ell)\sin\theta_v\cos\theta_v\cos\chi H_-H_0\},$$

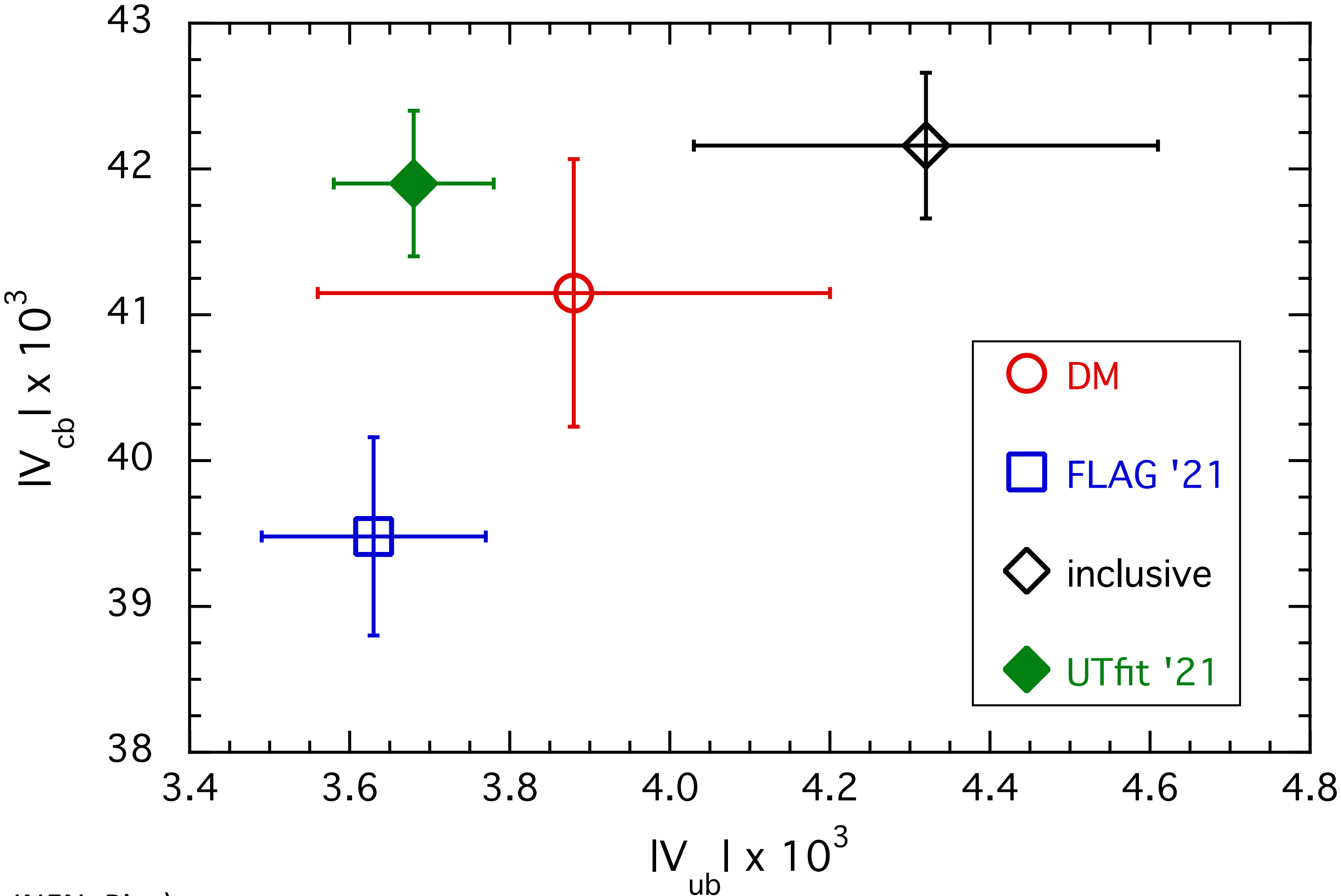


DM
FNAL/MILC
$g(w)$ (GeV^{-1})
w
$f(w)$ (GeV)
$F_1(w)$ (GeV^2)
$P_1(w)$



DM
FNAL/MILC + exps
$g(w)$ (GeV^{-1})
$f(w)$ (GeV)
$F_1(w)$ (GeV^2)
$P_1(w)$
w

CKM matrix elements



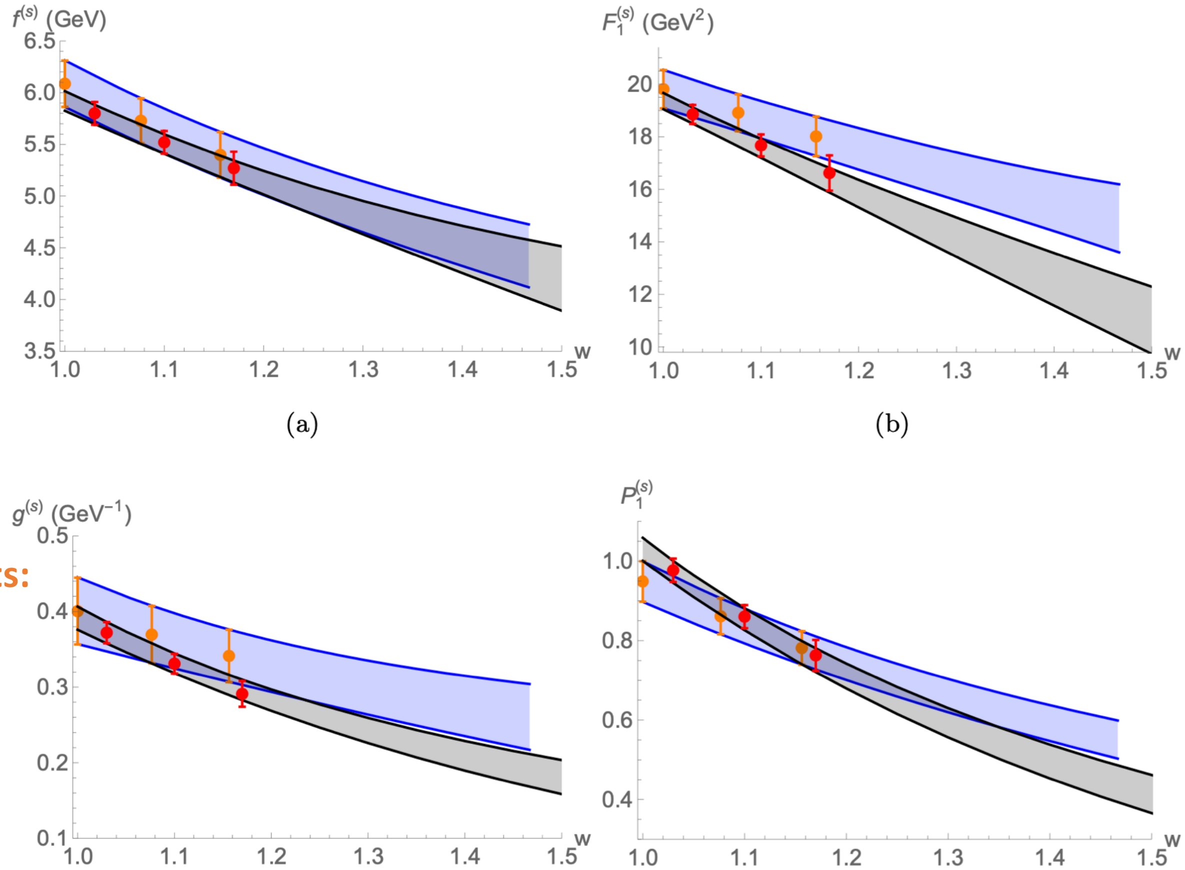
L. Vittorio (SNS & INFN, Pisa)

Semileptonic $B_s \to D_s^*$ decays *(in prep.)*

$SU(3)_F$ symmetry breaking effects?

Black bands:
$B \to D^*$
Blue bands:
$B_s \to D_s^*$

Orange points:
HPQCD
Red points:
FNAL/MILC



(a) (b) (c) (d)

$$f(w_{max}) = 4.19 \pm 0.31 \text{ GeV} ,$$

$$g(w_{max}) = 0.180 \pm 0.023 \text{ GeV}^{-1} ,$$

$$\mathcal{F}_1(w_{max}) = 11.0 \pm 1.3 \text{ GeV}^2 ,$$

$$P_1(w_{max}) = 0.411 \pm 0.048 .$$

$$f^s(w_{max}) = 4.42 \pm 0.30 \,\text{GeV},$$

$$g^s(w_{max}) = 0.261 \pm 0.044 \,\text{GeV}^{-1},$$

$$\mathcal{F}_1^s(w_{max}) = 14.9 \pm 1.3 \,\text{GeV}^2,$$

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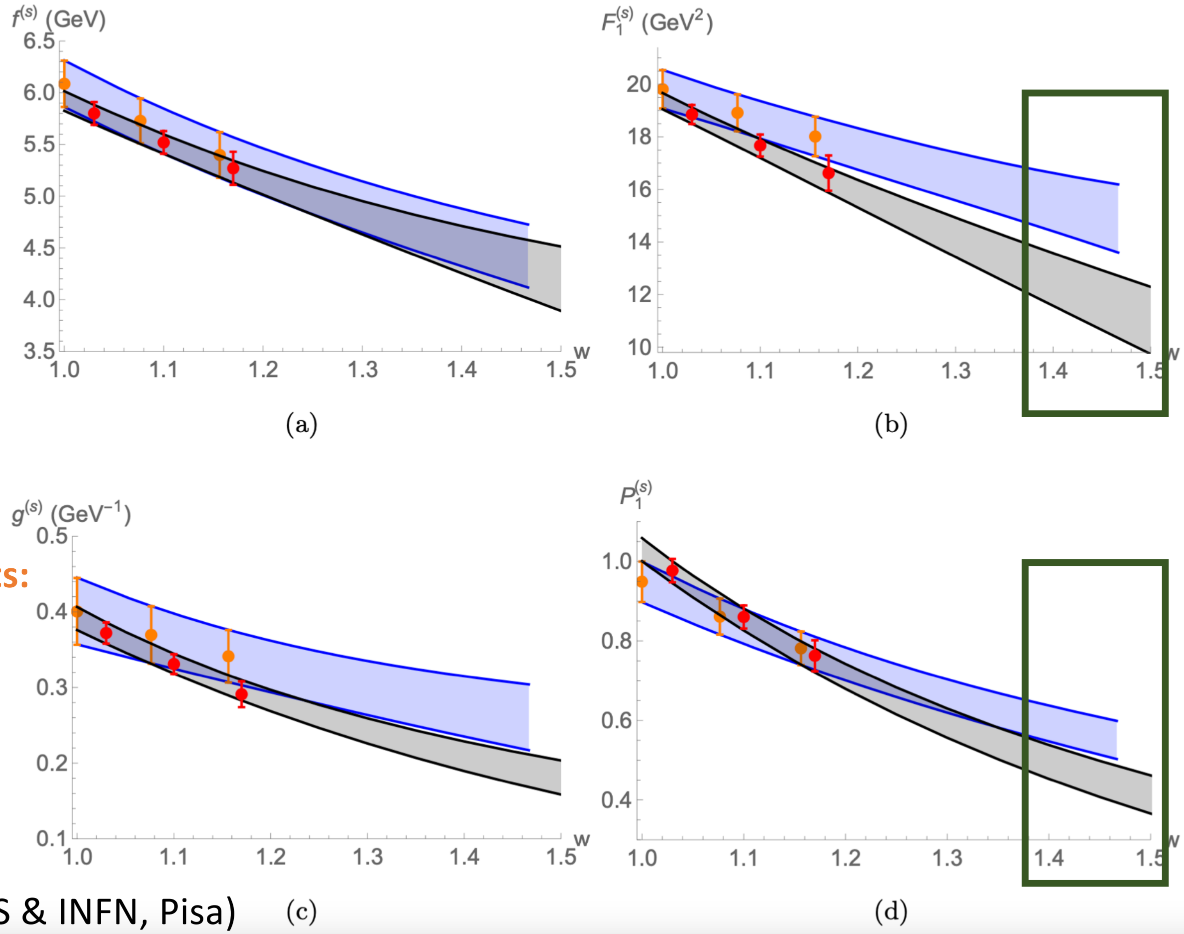

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A methodological break: comparison with BGL/BCL

What is the **main improvement** with respect to BGL/BCL parametrization?

Boyd, Grinstein and Lebed, Phys. Lett. B353, 306 (1995)
Boyd, Grinstein and Lebed, Nucl. Phys. B461, 493 (1996)
Boyd, Grinstein and Lebed, Phys. Rev. D 56, 6895 (1997)

Basics of BGL: the hadronic FFs corresponding to definite spin-parity can be represented as an expansion, originating from unitarity, analyticity and crossing symmetry, in terms of the conformal variable *z*, for instance

$$g(z) = \frac{1}{\sqrt{\chi_{1^-}(q_0^2)}} \frac{1}{\phi_g(z, q_0^2)\, P_{1^-}(z)} \sum_{n=0}^{\infty} a_n\, z^n$$

Unitarity:

$$\sum_{n=0}^{\infty} a_n^2 \leq 1$$

Basics of BCL: similar to BGL, the expansion series has a simpler form, for instance

$$f_+(z) = \frac{1}{1 - q^2/m_{B^*}^2} \sum_{n=0}^{N_z - 1} a_k \left[z^n - (-1)^{n-N_z} \frac{n}{N_z} z^{N_z} \right],$$

$$f_0(z) = \sum_{n=0}^{N_z-1} b_k z^k .$$

Bourrely, Caprini and Lellouch, Phys. Rev. D 79, 013008 (2009)

Unitarity:

$$\sum_{i,j=0}^{N_z} B_{mn}^{+} a_m a_n \leq 1, \qquad \sum_{i,j=0}^{N_z} B_{mn}^{0} b_m b_n \leq 1$$

LFU in semileptonic B → π decays

Fit	$N_z = 3$	$N_z = 4$	$N_z = 5$
χ^2/dof	2.5	0.64	0.73
dof	6	4	2
p	0.02	0.63	0.48
$\sum B^+_{mn} b^+_m b^+_n$	0.11(2)	0.016(5)	1.0(2.3)
$\sum B^0_{mn} b^0_m b^0_n$	0.33(8)	2.8(1.7)	8(19)
$f(0)$	0.00(4)	0.20(14)	0.36(27)
b^+_0	0.395(15)	0.407(15)	0.408(15)
b^+_1	−0.93(11)	−0.65(16)	−0.60(21)
b^+_2	−1.6(1)	−0.5(9)	−0.2(1.4)
b^+_3		0.4(1.3)	3(4)
b^+_4			5(5)
b^0_0	0.515(19)	0.507(22)	0.511(24)
b^0_1	−1.84(10)	−1.77(18)	−1.69(22)
b^0_2	−0.14(25)	1.3(8)	2(1)
b^0_3		4(1)	7(5)
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Table XIII
*of **arXiv:1503.07839***
(FNAL/MILC Coll.)

$$f^\pi(q^2=0)|_{\rm RBC/UKQCD} = -0.06 \pm 0.25$$

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Table XIX
*of **arXiv:1501.05363***
(RBC/UKQCD Coll.)

			$f_+^{B\pi}$						$f_0^{B\pi}$					
K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	$f(q^2=0)$	χ^2/dof	p
1	0.447(36)				0.00394(63)							0.447(36)	4.02	2%
2	0.410(39)	-1.30(52)			0.0120(59)							0.241(83)	0.30	58%
3	0.420(43)	-1.46(59)	-4.7(7.2)		0.15(42)							0.07(32)		
						1	0.460(61)				0.0225(60)	0.460(61)	90.1	0%
						2	0.516(61)	-4.09(55)			0.408(63)	-0.074(73)	0.03	87%
						3	0.516(61)	-3.94(97)	0.7(3.8)		0.32(41)	-0.02(28)		
2	0.366(37)	-2.79(54)			0.0337(85)	2	0.587(58)	-3.33(38)			0.346(55)	0.040(65)	6.18	0%
3	0.427(40)	-1.62(46)	-7.7(1.5)		0.38(15)	2	0.521(60)	-4.03(52)			0.404(62)	-0.066(70)	0.10	91%
2	0.410(39)	-1.24(51)			0.0113(56)	3	0.520(60)	-3.12(42)	4.5(1.3)		0.41(17)	0.248(82)	0.58	56%
3	0.424(41)	-1.50(57)	-6.0(5.0)		0.24(38)	3	0.519(60)	-3.81(81)	1.2(3.4)		0.27(25)	0.01(24)	0.07	79%

LFU in semileptonic B → π decays

Same considerations developed for the FNAL/MILC case…

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$$f^{\pi}(q^2=0)|_{\mathrm{FNAL/MILC}} = -0.01 \pm 0.16$$

$$f^{\pi}(q^2=0)|_{\mathrm{combined}} = -0.04 \pm 0.22$$

$f_+^{B\pi}$						$f_0^{B\pi}$								
K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	$f(q^2=0)$	χ^2/dof	p
1	0.447(36)				0.00394(63)							0.447(36)	4.02	2%
2	0.410(39)	-1.30(52)			0.0120(59)							0.241(83)	0.30	58%
3	0.420(43)	-1.46(59)	-4.7(7.2)		0.15(42)							0.07(32)		
						1	0.460(61)				0.0225(60)	0.460(61)	90.1	0%
						2	0.516(61)	-4.09(55)			0.408(63)	-0.074(73)	0.03	87%
						3	0.516(61)	-3.94(97)	0.7(3.8)		0.32(41)	-0.02(28)		
2	0.366(37)	-2.79(54)			0.0337(85)	2	0.587(58)	-3.33(38)			0.346(55)	0.040(65)	6.18	0%
3	0.427(40)	-1.62(46)	-7.7(1.5)		0.38(15)	2	0.521(60)	-4.03(52)			0.404(62)	-0.066(70)	0.10	91%
2	0.410(39)	-1.24(51)			0.0113(56)	3	0.520(60)	-3.12(42)	4.5(1.3)		0.41(17)	0.248(82)	0.58	56%
3	0.424(41)	-1.50(57)	-6.0(5.0)		0.24(38)	3	0.519(60)	-3.81(81)	1.2(3.4)		0.27(25)	0.01(24)	0.07	79%

2° important issue: *the DM method equivalent to the results of* ***all*** *possible fits which satisfy unitarity and at the same time reproduce exactly the input data*

The Dispersive Matrix (DM) method

Let us examine the case of the production of a pseudoscalar meson (as for the $B \to D$ case). Supposing to have *n* LQCD data for the FFs at the quadratic momenta $\{t_1, \cdots, t_n\}$ (hereafter $t \equiv q^2$), we define

$$
\mathbf{M} = \begin{pmatrix}
\langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\
\langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\
\langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle
\end{pmatrix}
$$

$$
\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)
$$

$$
g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}
$$

CENTRAL REQUIREMENT:

$$
\boxed{\det \mathbf{M} \geq 0}
$$

The conformal variable z is related to the momentum transfer as:

$$
z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}
$$

$$
t_\pm \equiv (m_B \pm m_D)^2
$$

$\Rightarrow$

Two advantages:

1. *z is real*
2. *1-to-1 correspondence:*

[0, t_{max}=t_-] ⇨ [z_{max}, 0]

A lot of work in the past:

L. Lellouch, NPB, 479 (1996), p. 353-391
C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp. 157 - 181
E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

The DM method

$$\mathbf{M} = \begin{pmatrix} \boxed{\langle \phi f | \phi f \rangle} & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

We also have to define the kinematical functions

$$\phi_0(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{3t_+t_-}{4\pi}} \frac{1}{t_+ - t_-} \frac{1+z}{(1-z)^{5/2}} \left(\beta(0) + \frac{1+z}{1-z}\right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-2},$$

$$\phi_+(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{1}{\pi(t_+ - t_-)} \frac{(1+z)^2}{(1-z)^{9/2}}} \left(\beta(0) + \frac{1+z}{1-z}\right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-3}, \quad \beta(t) \equiv \sqrt{\frac{t_+ - t}{t_+ - t_-}}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ $\{t_1,\ldots,t_n\}$: from Cauchy's theorem (for generic *m*)

$$\boxed{\langle g_{t_m} | \phi f \rangle} = \phi(t_m, Q^2) f(t_m) \qquad \boxed{\langle g_{t_m} | g_{t_l} \rangle} = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

LQCD data!

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \boxed{\langle \phi f | \phi f \rangle}$$

Since the susceptibilities are computed on the lattice, we can in principle use whatever value of Q^2 !

The DM method

$$\mathbf{M} = \begin{pmatrix} \boxed{\langle\phi f|\phi f\rangle} & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle \end{pmatrix}$$

In the presence of **poles @** $t_{P1}, t_{P2}, \cdots \ldots, t_{PN}$ **:**

$$\phi(z,q^2) \to \phi_P(z,q^2) \equiv \phi(z,q^2) \times \frac{z - z(t_{P1})}{1-\bar{z}(t_{P1})z} \times \cdots \times \frac{z - z(t_{PN})}{1-\bar{z}(t_{PN})z}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ *{t_1,…,t_n}*: from Cauchy's theorem (for generic *m*)

$$\boxed{\langle g_{t_m}|\phi f\rangle} \quad = \quad \phi(t_m, Q^2) f(t_m)$$

LQCD data!

$$\boxed{\langle g_{t_m}|g_{t_l}\rangle} = \frac{1}{1-\bar{z}(t_l)z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \boxed{\langle\phi f|\phi f\rangle}$$

The DM method

At this point, the form of the matrix is much simpler:

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \dots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \dots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \dots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \dots & \frac{1}{1-z_2 z_N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \dots & \frac{1}{1-z_N^2} \end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \;\; (\text{with } i = 1, 2, \dots N)$$

The DM method

The positivity of the original inner products guarantee that $\det \mathbf{M} \geq 0$: the solution of this inequality can be computed analitically, bringing to

LOWER bound $$\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: unitarity is satisfied if γ is semipositive definite, namely if

$$\chi \geq \sum_{i,j=1} N f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

This is a **parametrization-independent unitarity test** of the LQCD input data

The DM method

The positivity of the original inner products guarantee that $\det \mathbf{M} \geq 0$: the solution of this inequality can be computed analitically, bringing to

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

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How do we treat the uncertainties?

Statistical and systematic uncertainties

How can we finally combine all the N_U lower and upper bounds of both the FFs??

One bootstrap event case:

after a single extraction, we have one value of the lower bound f_L and one value of the upper one f_U for each FF. Assuming that the true value of each FF can be **everywhere inside the range (f_U - f_L) with equal probability**, we associate to the FFs a *flat* distribution

$$P(f_{0(+)}) = \frac{1}{f_{U,0(+)} - f_{L,0(+)}} \Theta(f_{0(+)} - f_{L,0(+)}) \Theta(f_{U,0(+)} - f_{0(+)})$$

Many bootstrap events case:

how to mediate over the whole set of bootstrap events? Since the lower and the upper bounds of a generic FF are deeply correlated, we will assume a multivariate Gaussian distribution:

$$P(f_L, f_U) = \frac{\sqrt{\det \rho}}{2\pi} \exp\left[- \frac{\rho_{up,up}(f_U - \langle f_U \rangle)^2 + \rho_{lo,lo}(f_L - \langle f_L \rangle)^2 + 2\rho_{lo,up}(f_U - \langle f_U \rangle)(f_L - \langle f_L \rangle)}{2} \right]$$

In conclusion, we can combine the bounds of each FF in a final mean value and a final standard deviation, defined as

$$\langle f \rangle = \frac{\langle f_L \rangle + \langle f_U \rangle}{2},$$

$$\sigma_f = \frac{1}{12}(\langle f_U \rangle - \langle f_L \rangle)^2 + \frac{1}{3}(\sigma_{f_{lo}}^2 + \sigma_{f_{up}}^2 + \rho_{lo,up}\sigma_{f_{lo}}\sigma_{f_{up}})$$

NO PARAMETRIZATION ADOPTED!!!

Kinematical Constraints (KCs)

REMINDER: after the unitarity filter we were left with $N_U < N$ *survived events!!!*

Let us focus on the pseudoscalar case. Since by construction the following *kinematical constraint* holds

$$\boxed{f_0(0) = f_+(0)}$$

we will filter only the $N_{KC} < N_U$ events for which the two bands of the FFs intersect each other @ $t = 0$.
Namely, for each of these events we also define

$$\phi_{lo} = \max[F_{+,lo}(t=0), F_{0,lo}(t=0)]$$

$$\phi_{up} = \min[F_{+,up}(t=0), F_{0,up}(t=0)]$$

From WE theorem

$$\langle D(p_D)|\, V^\mu \,|B(p_B)\rangle \quad = \quad f_+(p_B + p_D)^\mu + f_-(p_B - p_D)^\mu$$

One then defines

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_B^2 - m_D^2} f_-(q^2)$$

$$\boxed{\langle D(p_D)\,|\,V^\mu\,|\,B(p_B)\rangle = f^+(q^2)\left(p_B^\mu + p_D^\mu - \frac{m_B^2 - m_D^2}{q^2} q^\mu\right) + f^0(q^2)\frac{m_B^2 - m_D^2}{q^2} q^\mu}$$

Kinematical Constraints (KCs)

We then consider a **modified matrix**

$$\mathbf{M_C} = \begin{pmatrix} \phi f|\phi f\rangle & \langle \phi f|g_t\rangle & \langle \phi f|g_{t_1}\rangle & \cdots & \langle \phi f|g_{t_n}\rangle & \langle \phi f|g_{t_{n+1}}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle & \langle g_t|g_{t_{n+1}}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle & \langle g_{t_1}|g_{t_{n+1}}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle & \langle g_{t_n}|g_{t_{n+1}}\rangle \\ \langle g_{t_{n+1}}|\phi f\rangle & \langle g_{t_{n+1}}|g_t\rangle & \langle g_{t_{n+1}}|g_{t_1}\rangle & \cdots & \langle g_{t_{n+1}}|g_{t_n}\rangle & \langle g_{t_{n+1}}|g_{t_{n+1}}\rangle \end{pmatrix}$$

with $t_{n+1} = 0$. Hence, we compute the new lower and upper bounds of the FFs in this way. For each of the N_{KC} events, we extract $N_{KC,2}$ values of $f_0(0) = f_+(0) \equiv f(0)$ with uniform distribution defined in the range $[\phi_{lo}, \phi_{up}]$. Thus, for both the FFs and for each of the N_{KC} events we define

$$\begin{aligned} F_{lo}(t) &= \min[F_{lo}^1(t), F_{lo}^2(t), \cdots, F_{lo}^{N_{KC,2}}(t)], \\ F_{up}(t) &= \max[F_{up}^1(t), F_{up}^2(t), \cdots, F_{up}^{N_{KC,2}}(t)] \end{aligned}$$

Non-perturbative computation of the susceptibilities

In **arXiv:2105.07851**, we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to c$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

How are they defined? The starting point is the HVP tensor:

$$\Pi^V_{\mu\nu}(Q) = \int d^4x\; e^{-iQ\cdot x}\langle 0|T\left[\bar{b}(x)\gamma^E_\mu c(x)\;\bar{c}(0)\gamma^E_\nu b(0)\right]|0\rangle$$
$$= -Q_\mu Q_\nu \Pi_{0^+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi_{1^-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;,\qquad C_{0^+}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0 c(x)\;\bar{c}(0)\gamma_0 b(0)\right]|0\rangle\;,$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4\frac{j_1(Qt)}{Qt}\;C_{1^-}(t)\qquad C_{1^-}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j c(x)\;\bar{c}(0)\gamma_j b(0)\right]|0\rangle\;,$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^-}(t)\;,\qquad C_{0^-}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0\gamma_5 c(x)\;\bar{c}(0)\gamma_0\gamma_5 b(0)\right]|0\rangle\;,$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4\frac{j_1(Qt)}{Qt}\;C_{1^+}(t)\qquad C_{1^+}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j\gamma_5 c(x)\;\bar{c}(0)\gamma_j\gamma_5 b(0)\right]|0\rangle$$

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To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^-}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^+}(t)$$

Non-perturbative computation of the susceptibilities

The possibility to compute the χs on the lattice allows us to choose *whatever value of Q^2 !!!!* (i.e. near the region of production of the resonances)

NOT POSSIBLE IN PERTURBATION THEORY!!!

$$(m_b + m_c)\Lambda_{QCD} << (m_b + m_c)^2 - q^2$$

POSSIBLE IMPROVEMENT IN THE STUDY OF THE FFs through our method!

Work in progress...

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

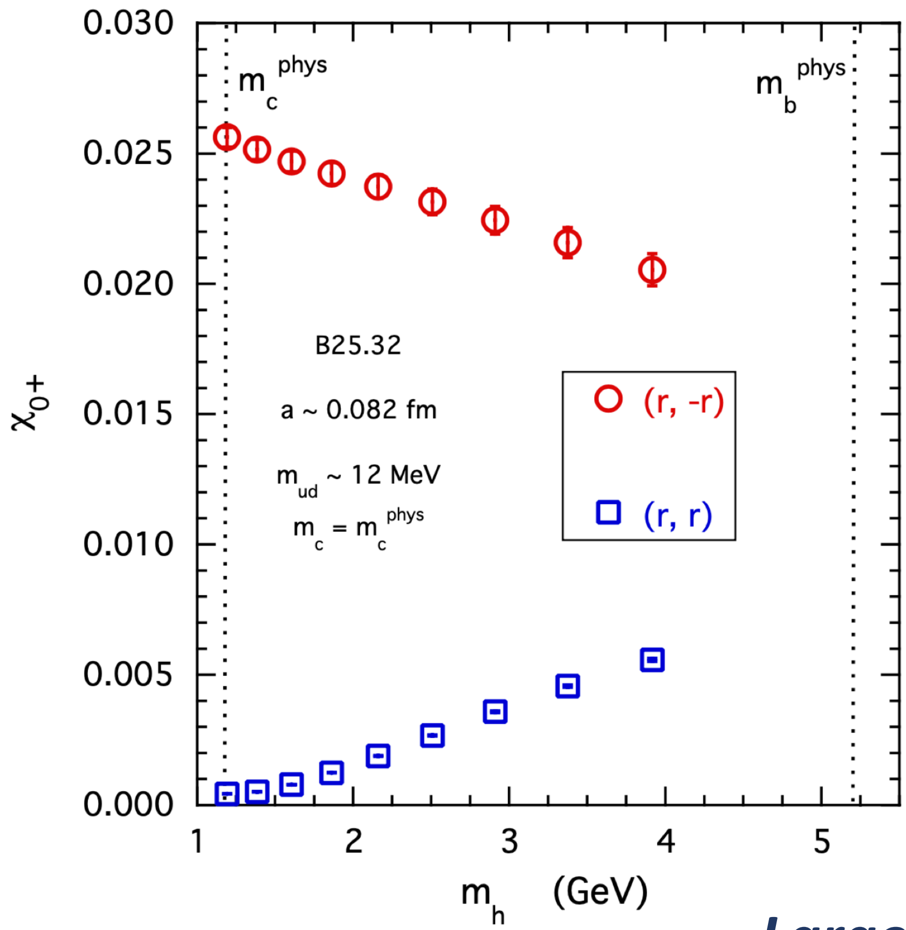
$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^+}(t)\ , \xrightarrow{W.\ I.} \frac{1}{4}\int_0^\infty dt'\ t'^4\ \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^-}(t)\ , \xrightarrow{W.\ I.} \frac{1}{4}\int_0^\infty dt'\ t'^4\ \frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^+}(t)$$

Non-perturbative computation of the susceptibilities



Following set of masses:

$$m_h(n) = \lambda^{n-1}\, m_c^{phys} \qquad \text{for } n = 1, 2, \ldots$$

$$m_h = a\mu_h/(Z_P a)$$

$$\lambda \equiv [m_b^{phys}/m_c^{phys}]^{1/10} = [5.198/1.176]^{1/10} \simeq 1.1602$$

Nine masses values!

$$m_h(1) = m_c^{phys}$$

$$m_h(9) \simeq 3.9 \text{ GeV} \simeq 0.75\; m_b^{phys}$$

***r*: Wilson parameter**

Large discretisation effects and contact terms

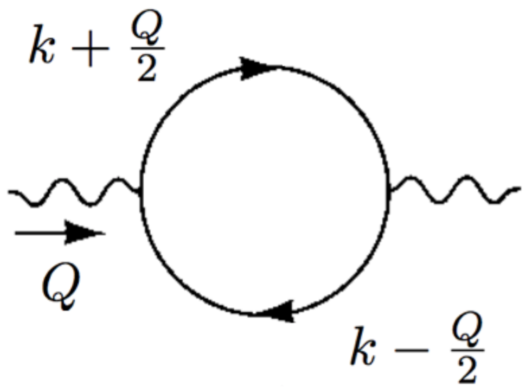
Contact terms & perturbative subtraction

In **twisted mass LQCD:**

$$\boxed{\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\Big[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\Big],}$$

$$G_i(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + \mathcal{M}_i(p) - i r_i \mu_{q,i}\gamma_5}{\mathring{p}_\mu^2 + \mathcal{M}_i^2(p) + \mu_{q,i}^2}$$

$$\mathring{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu), \qquad \mathcal{M}_i(p) \equiv m_i + \frac{r_i}{2} a\hat{p}_\mu^2, \qquad \hat{p} \equiv \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right).$$



$$\begin{aligned}\Pi_V^{\alpha\beta} &= a^{-2}(Z_1^I + (r_1^2 - r_2^2)Z_2^I + (r_1^2 - r_2^2)(r_1^2 + r_2^2)Z_3^I)g^{\alpha\beta}\\ &+ (\mu_1^2 Z^{\mu_1^2} + \mu_2^2 Z^{\mu_2^2} + \mu_1\mu_2 Z^{\mu_1\mu_2})g^{\alpha\beta} + (Z_1^{Q^2} + (r_1^2 - r_2^2)Z_2^{Q^2})Q\cdot Q g^{\alpha\beta}\\ &+ (Z_1^{Q^\alpha Q^\beta} + (r_1^2 - r_2^2)Z_2^{Q^\alpha Q^\beta})Q^\alpha Q^\beta + \underline{r_1 r_2(a^{-2}Z_1^{r_1 r_2}g^{\alpha\beta} + (Z_2^{r_1 r_2} + (r_1^2 + r_2^2)Z_3^{r_1 r_2}}\\ &\underline{+ (r_1^4 + r_2^4)Z_4^{r_1 r_2})\boxed{Q\cdot Q}g^{\alpha\beta} + (\mu_1^2 Z_5^{r_1 r_2} + \mu_2^2 Z_6^{r_1 r_2})g^{\alpha\beta})} + O(a^2),\end{aligned}$$

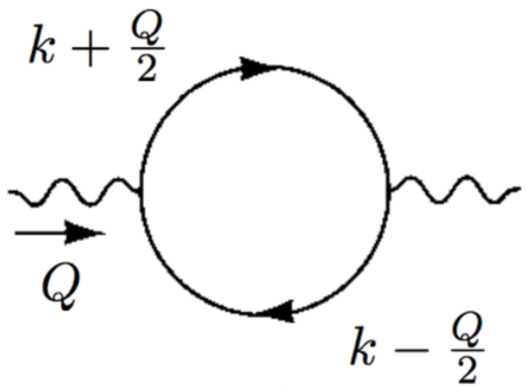
CONTACT TERMS!!!

F. Burger et al., ETM Coll., JHEP '15 [arXiv:1412.0546]

Contact terms & perturbative subtraction

In **twisted mass LQCD** (tmLQCD):

$$\boxed{\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\Big[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\Big],}$$



Thus, by separating the *longitudinal* and the *transverse* contributions, we can compute the susceptibilities for all the spin-parity quantum numbers **in the free theory on the lattice**, *i.e.* at order $\mathcal{O}(\alpha_s^0)$ using twisted-mass fermions!

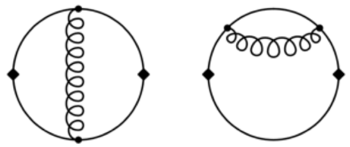
$$\chi_j^{free} = \boxed{\chi_j^{LO}} + \boxed{\chi_j^{discr}}$$

LO term of PT @ $\mathcal{O}(\alpha_s^0)$ contact terms and discretization effects @ $\mathcal{O}(\alpha_s^0 a^m)$ with $m \geq 0$

Perturbative subtraction:

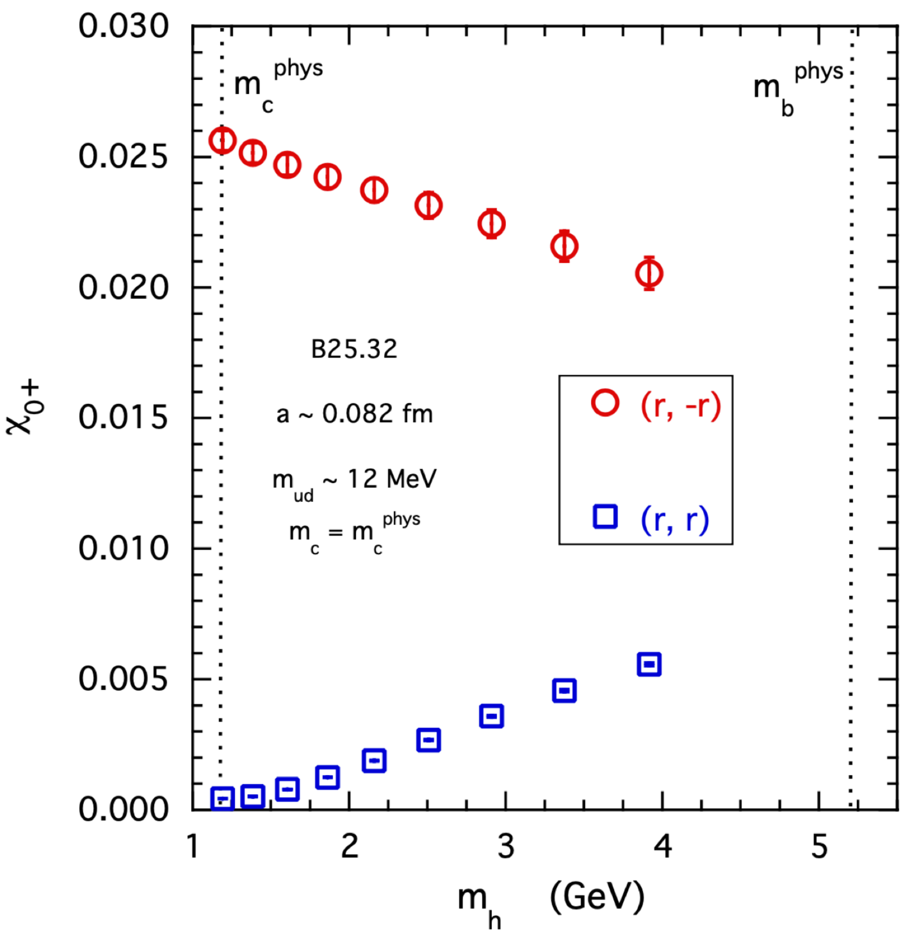
$$\chi_j \to \chi_j - \left[\chi_j^{free} - \chi_j^{LO}\right]$$

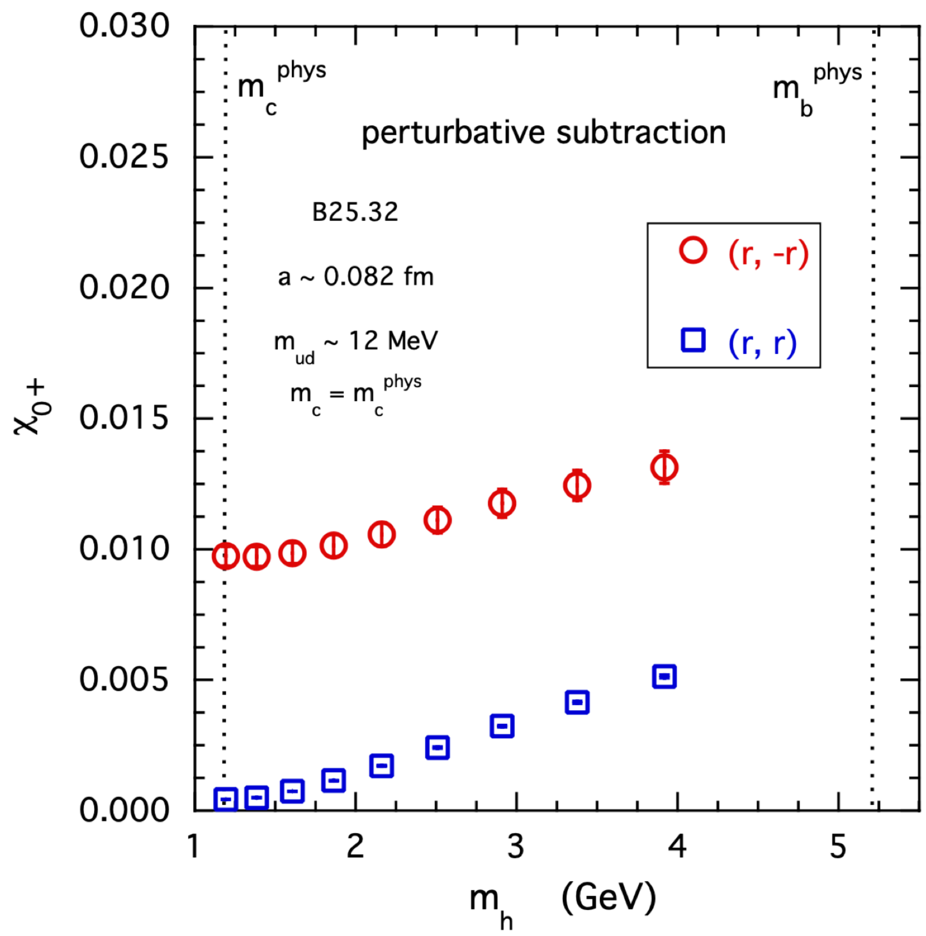
Higher order corrections?



Work in progress...

Contact terms & perturbative subtraction

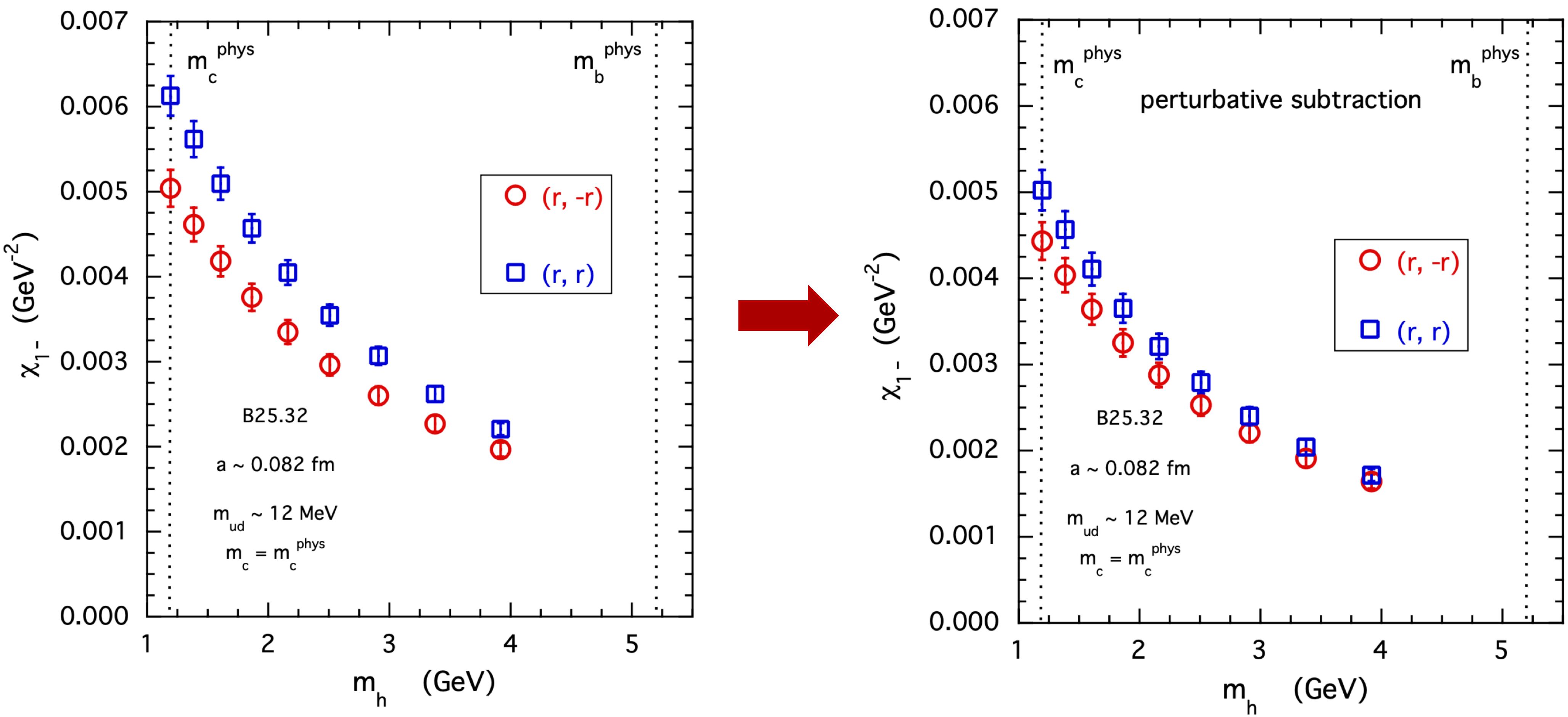




NOT ENOUGH…

L. Vittorio (SNS & INFN, Pisa)

Contact terms & perturbative subtraction



OK

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \boxed{\frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}}$$

to ensure that $\lim_{n\to\infty} R_j(n) = 1$

$$\boxed{\begin{array}{l}\rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1\ , \\ \rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2\end{array}}$$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light, *in prep.*) transition current densities:**

b → c | ***b → u***

	Perturbative	With subtraction	Non-perturbative	With subtraction	Non-perturbative	With subtraction
$\chi_{V_L}\,[10^{-3}]$	6.204(81)	—	7.58(59)	—	2.04(20)	—
$\chi_{A_L}\,[10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)	2.34(13)	—
$\chi_{V_T}\,[10^{-4}\ \text{GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)	4.88(1.16)	4.45(1.16)
$\chi_{A_T}\,[10^{-4}\ \text{GeV}^{-2}]$	3.894	—	4.69(30)	—	4.65(1.02)	—

Bigi, Gambino PRD '16
Bigi, Gambino, Schacht PLB '17
Bigi, Gambino, Schacht JHEP '17

ETMC ratio method & final results

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b → u

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2.04(20)	—
2.34(13)	—
4.88(1.16)	4.45(1.16)
4.65(1.02)	—

Differences with PT? **~4% for 1^-, ~7% for 0^-, ~20 % for 0^+ and 1^+**