

Black - Hole Scattering Eikonal Resummation Gravitational Waves



Uppsala University



NORDITA

Carlo Heissenberg



Genova
14-10-2021

BASED ON:

2008.12743 2101.05772 2104.03256

IN COLLABORATION WITH

P. Di Vecchie, R. Russo,
G. Veneziano

AND ON:

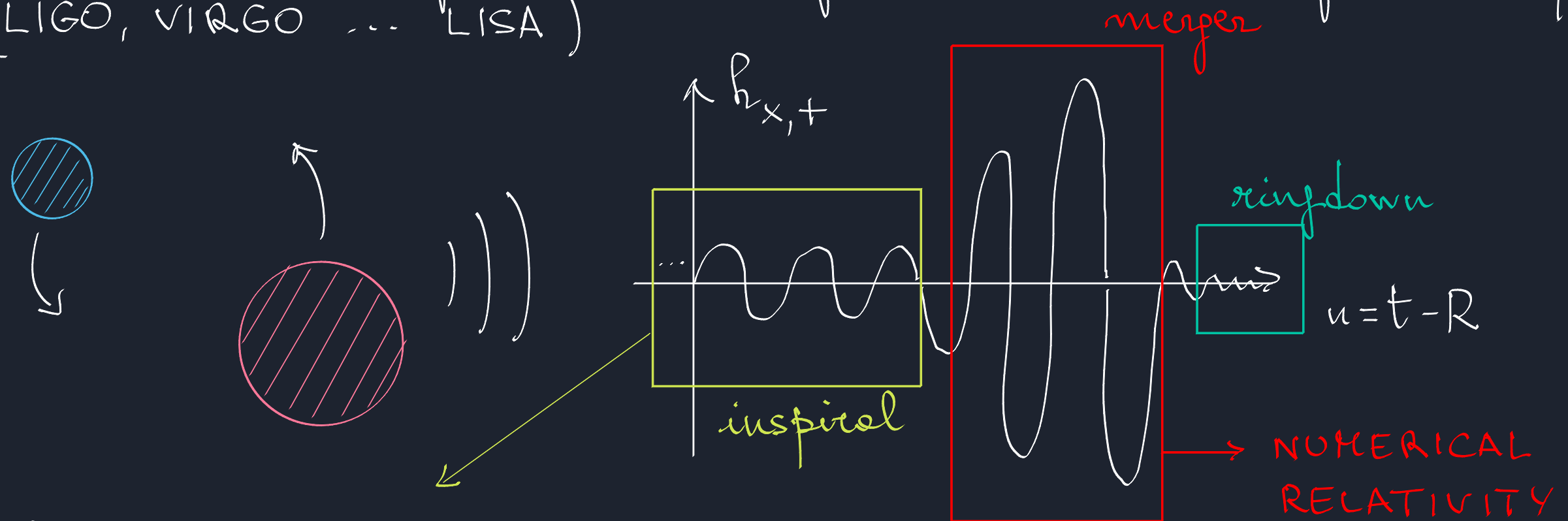
2105.04594

+ in
progress...

Introduction & Motivations

BLACK-HOLE MERGERS

- Direct detection of GW: need for accurate waveform templates (LIGO, VIRGO ... LISA)



- Analytic approximation schemes

* Post-Minkowskian: weak gravitational forces
 "small G " (v generic)

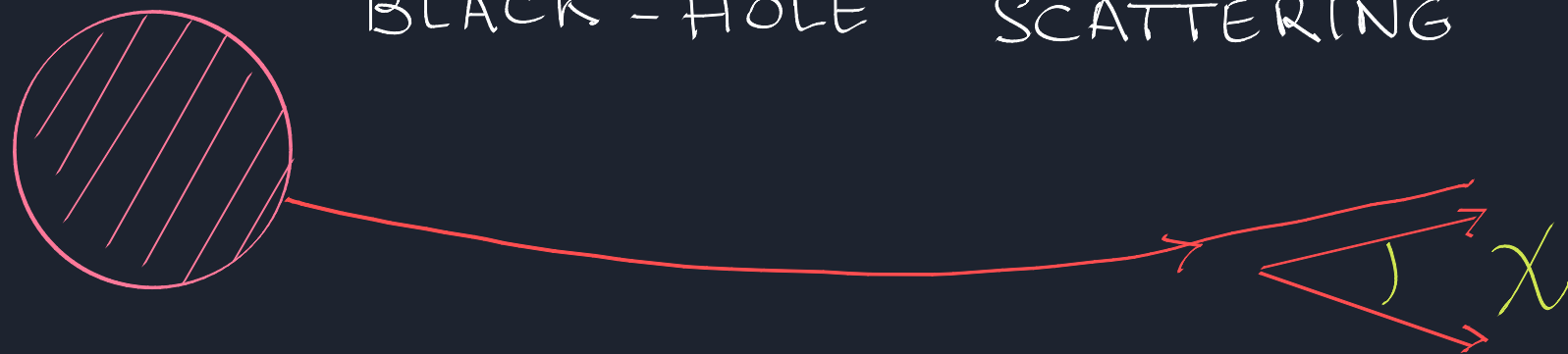
$$\frac{GM}{\pi} \ll 1$$

* Post-Newtonian: weak gravitational forces
 "small $G \sim v^2$ " and slow motions

$$\frac{GM}{\pi} \sim \frac{v^2}{2} \ll 1$$

Introduction & Motivations

BLACK-HOLE SCATTERING



- PM expansions from Scattering Amplitudes

$$A = A_0 + A_1 + \boxed{A_2} + \boxed{A_3} + \dots$$

$\mathcal{O}(G)$

TREE
LEVEL

$\mathcal{O}(G^2)$

1 LOOP

$\mathcal{O}(G^3)$

2 LOOPS

THIS
TALK

$\mathcal{O}(G^4)$

3 LOOPS

STATE OF
THE ART
(Potential
Region)

$\checkmark$ generic $\longleftrightarrow$ Lorentz
invariance

- What can we learn from scattering events?

- * Some quantities e.g. V_{eff} can be exported verbatim...

Cheung, Rothstein, Solon - 1802.02489

- * Others can be analytically continued,
e.g. $\chi \rightarrow \Delta\Phi$



Kälin, Porto -
1910.03008,
1911.09130

Introduction & Motivations

A 3PM PUZZLE

- The 1st 3PM calculation of the interaction potential $V_{\text{eff}}^{(3PM)}$ between spinless BH was achieved by Amplitude methods

Bern, Cheung, Roiban, Shen, Solon, Zeng - 1901.04424, 1908.01493

- $A_2^{\text{BCRSSZ}} \xrightarrow{(3PM)} V_{\text{eff}} \xrightarrow{\text{BCRSSZ}} \chi(E, u_1, u_2, b) \sim \left(\frac{GE}{b}\right)^3 \log\left(\frac{E^2}{u_1 u_2}\right)$
 $E \gg u_{1,2}$
 3PM ✓ ?
- Massless setup: $\chi^{\text{ACV}}(E, b) \sim \left(\frac{GE}{b}\right)^3$
 (shockwave scattering)
 Amati, Ciafaloni, Veneziano - 1990
 3PM ✓ UNIVERSAL
 e.g. valid in GR, $N=8 \dots$

Outline

- Classical Gravity from Scattering Amplitudes

EIKONAL EXPONENTIATION

- $N=8$ as a Theoretical Laboratory

POTENTIAL + RADIATION REACTION @ 3PM

- Radiative Effects in General Relativity

3-PARTICLE CUT & RADIATION

Outline

- Classical Gravity from Scattering Amplitudes

EIKONAL EXPONENTIATION

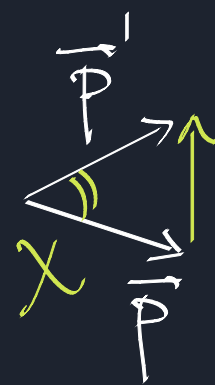
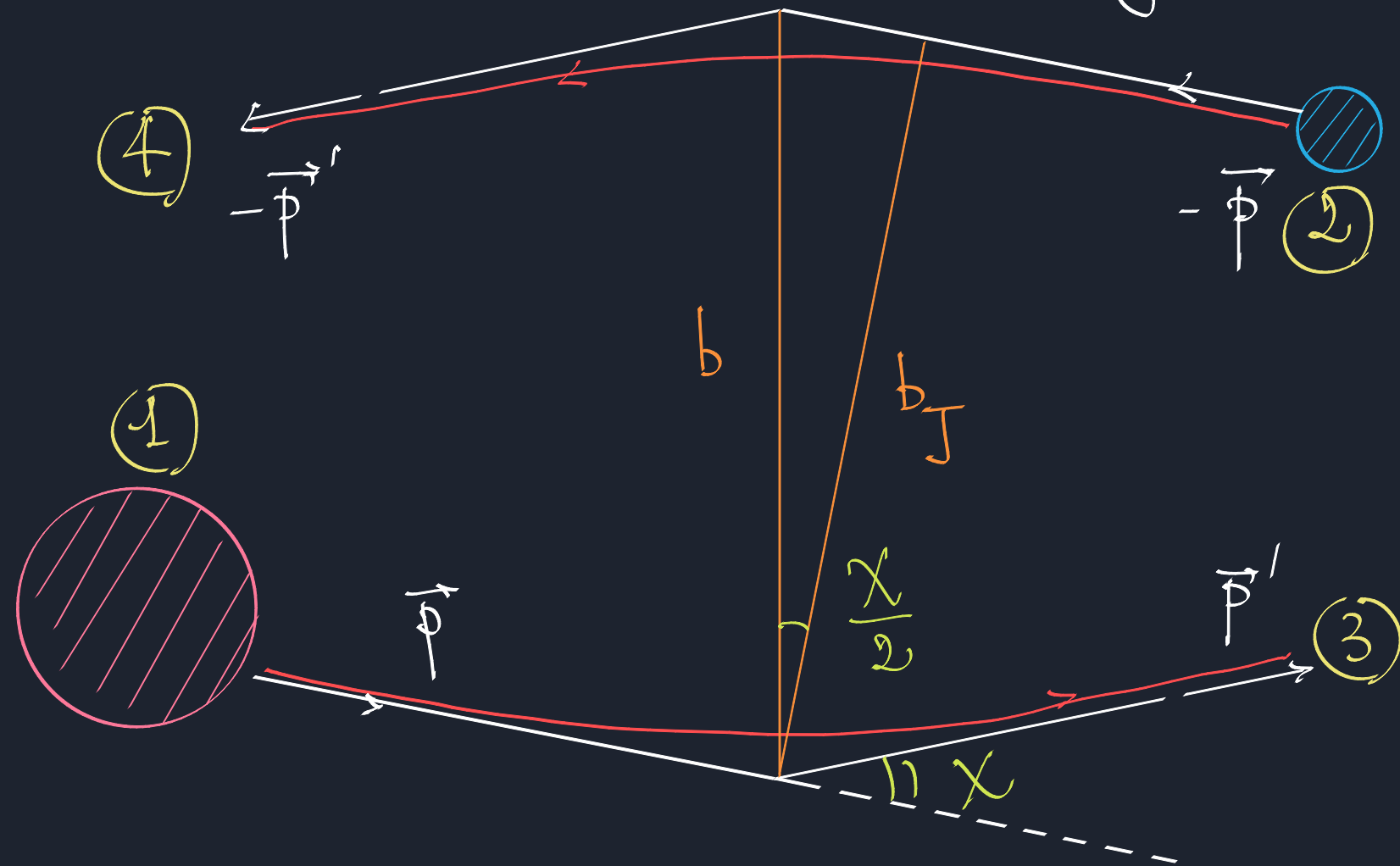
- $N=8$ as a Theoretical Laboratory

POTENTIAL + RADIATION REACTION @ 3PM

- Radiative Effects in General Relativity

3-PARTICLE CUT & RADIATION

Black-Hole Scattering



$\vec{Q}$ CLASSICAL
MOMENTUM
TRANSFER

$$|\vec{Q}| = 2|\vec{p}| \sin \frac{\chi}{2}$$

$$b_J |\vec{p}| = J \text{ ANGULAR MOMENTUM}$$

$$b_J = b \cos \frac{\chi}{2}$$

Hierarchy of length scales:

Quantum wavelength
 $\frac{\hbar}{m}$ or $\frac{\hbar}{\sqrt{s}}$

Schwarzschild radius
 Gm or $G\sqrt{s}$

Impact parameter
 b

λ

$\ll$

R

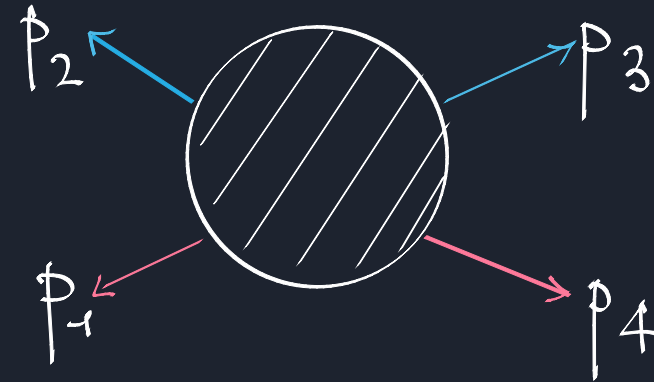
$\ll$

b

CLASSICAL LIMIT

PM REGIME

Amplitude:



$$= \mathcal{A}(s, q)$$

PERTURBATIVE
MOMENTUM
TRANSFER

$$q = p_4 + p_1$$

$$s = E^2 = -(p_1 + p_2)^2 = m_1^2 + 2m_1 m_2 \sigma + m_2^2$$

Large-b / Small-q Expansion

- Impact-parameter space representation:

$$\tilde{A}(s, b) = \frac{1}{4u_1 u_2 \sqrt{\sigma^2 - 1}} \int e^{ib \cdot q} A(s, q) \frac{d^2 q}{(2\pi)^{2-2\epsilon}}$$

$$D = 4 - 2\epsilon$$

$$\left\{ \begin{array}{l} \sigma = \frac{1}{\sqrt{1-v^2}} \\ E|\vec{p}| = u_1 u_2 \sqrt{\sigma^2 - 1} \end{array} \right.$$

* Large b

$\Leftrightarrow$ Small q^2

* Long-range in b $\Leftrightarrow$ Non-Analytic in q^2

- Loop-expansion: $A = A_0 + A_1 + A_2 + \dots$

$$\tilde{A}_0 = c_0 \left(\frac{R}{\lambda} \right)$$

$$\tilde{A}_1 = c_{1,0} \left(\frac{R}{\lambda} \right)^2 + c_{1,1} \left(\frac{R}{\lambda} \right) \left(\frac{R}{b} \right) + c_{1,2} \left(\frac{R}{b} \right)^2 \left[1 + \mathcal{O}\left(\frac{\lambda}{b} \right) \right]$$

$$\tilde{A}_2 = c_{2,0} \left(\frac{R}{\lambda} \right)^3 + c_{2,1} \left(\frac{R}{\lambda} \right)^2 \left(\frac{R}{b} \right) + c_{2,2} \left(\frac{R}{\lambda} \right) \left(\frac{R}{b} \right)^2 + c_{2,3} \left(\frac{R}{b} \right)^3 \left[1 + \mathcal{O}\left(\frac{\lambda}{b} \right) \right]$$

$$R = Gm(b^{2\epsilon})$$

$$\lambda = \frac{\hbar}{m}$$

$$\frac{R}{\lambda} \gg 1, \frac{R}{b} \ll 1$$

CLASSICAL
LIMIT

PM
REGIME

Eikonal Exponentiation

cf. Bern et al. - 2101.07254
 Damgaard, Planté, Vanhove - 2107.12891,
 Brandhuber, Chen, Travaglini, Wen - 2108.04216

Quantum remainder

$$\Delta = \Delta_1 + \Delta_2 + \dots$$

$$\Delta_{n-1} \sim \left(\frac{R}{b}\right)^n \left[1 + \mathcal{O}\left(\frac{\lambda}{b}\right)\right]$$

(Classical) Eikonal

$$\left[\sim e^{\frac{i}{\hbar} S_{\text{classical}}} \right]$$

$$\delta = \delta_0 + \delta_1 + \delta_2 + \dots$$

$$\delta_{n-1} \sim \left(\frac{R}{\lambda}\right) \left(\frac{R}{b}\right)^{n-1}$$

$$R = Gm(b^{2\epsilon})$$

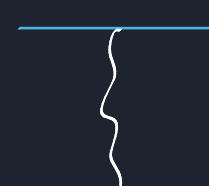
$$\lambda = \frac{\hbar}{m}$$

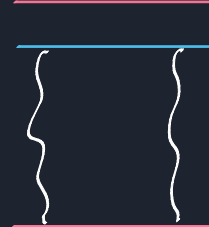
$$\frac{R}{\lambda} \gg 1, \frac{R}{b} \ll 1$$

CLASSICAL
LIMIT

PM
REGIME

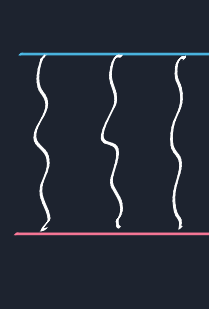
For "small G ":

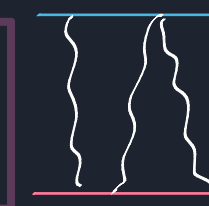
$$i\tilde{A}_0(s,b) = 2i\delta_0$$


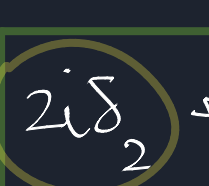
$$i\tilde{A}_1(s,b) = \frac{(2i\delta_0)^2}{2!}$$


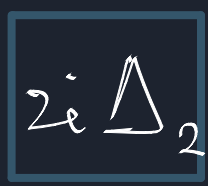
$$+ 2i\delta_1$$

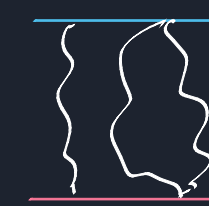
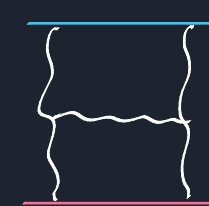

$$+ 2i\Delta_1$$


$$i\tilde{A}_2(s,b) = \frac{(2i\delta_0)^3}{3!}$$


$$+ 2i\delta_0 2i\delta_1$$


$$+ 2i\delta_2 + 2i\delta_0 2i\Delta_1$$


$$+ 2i\Delta_2$$




Eikonal & Deflection Angle

- STATIONARY PHASE APPROXIMATION: after exponentiation

$$A(s, \vec{Q}) \sim \int e^{-i\vec{b} \cdot \vec{Q} + 2i\delta} d^{2-2\epsilon} \vec{b}$$

CLASSICAL
LIMIT

$$\vec{Q} = \frac{\partial}{\partial \vec{b}} \text{Re}(2\delta)$$

- $\delta = \delta_0 + \delta_1 + \delta_2 + \dots$, schematically

$$2\delta_0 \propto G m_1 m_2 \frac{b^{2\epsilon}}{2\epsilon}$$

$$\Rightarrow \chi_{1PM} \propto \frac{G m_1 m_2}{|\vec{p}| b} \sim \frac{R}{b}$$

$$2\delta_1 \propto G m_1 m_2 \frac{G(m_1 + m_2)}{b^{1-4\epsilon}}$$

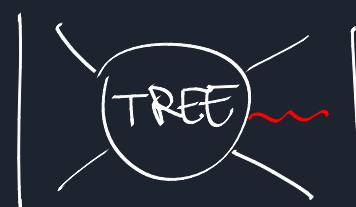
$$\Rightarrow \chi_{2PM} \propto \frac{G^2 m_1 m_2 (m_1 + m_2)}{|\vec{p}| b^2} \sim \left(\frac{R}{b}\right)^2$$

$$2\delta_2 \propto G m_1 m_2 \frac{G^2 m_1 m_2}{b^{2-6\epsilon}} \left[(\dots) + \frac{i}{\epsilon} (\dots) \right]$$

IR finite
REAL PART

IR divergent
IMAGINARY PART

$$\Rightarrow \chi_{3PM} \propto \frac{G^3 (m_1 m_2)^2}{|\vec{p}| b^3} \sim \left(\frac{R}{b}\right)^3$$

due to  ²

(Weinberg '65)

Nontrivial part:

precise dependence on $|\vec{Q}|$

$\sigma \rightarrow 1$ STATIC LIMIT

$\sigma \rightarrow \infty$ ULTRAREL. LIMIT

Outline

- Classical Gravity from Scattering Amplitudes

EIKONAL EXPONENTIATION

- $N=8$ as a Theoretical Laboratory

POTENTIAL + RADIATION REACTION @ 3PM

- Radiative Effects in General Relativity

3-PARTICLE CUT & RADIATION

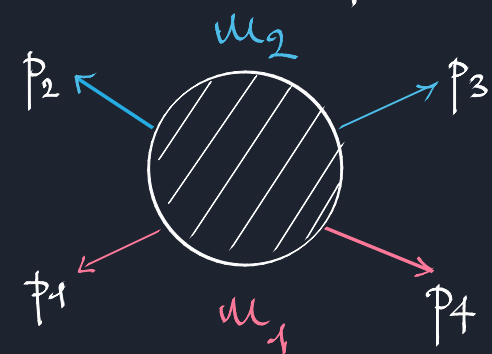
$\mathcal{N}=8$ SUGRA

- Useful toy model:
 - * simpler amplitudes compared to GR
 - * same technical and conceptual challenges

- s-u symmetric amplitude for two massive scalars

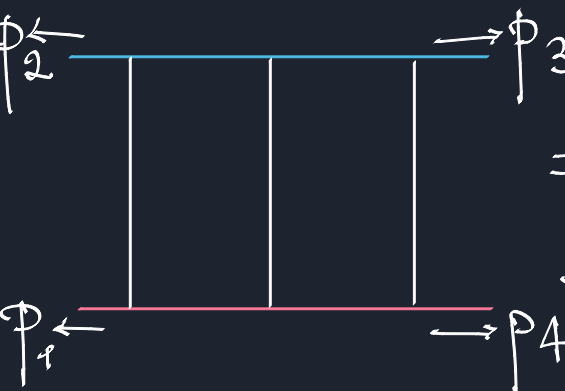
$$A_2(s, q) = \frac{(8\pi G)^3}{2} \left[(s - m_1^2 - m_2^2)^4 + (u - m_1^2 - m_2^2)^4 - (q^2)^4 \right]$$

Caron-Huot, Zahraee - 1810.04694
Parra-Martinez, Ruf, Zeng - 2005.04236



$$\begin{aligned} & \times \left[(s - m_1^2 - m_2^2)^2 \left(\begin{array}{c} \text{Three diagrams: 1. Three vertical lines. 2. Two vertical lines with a diagonal cross. 3. Two vertical lines with a diagonal cross in the opposite orientation.} \end{array} \right) \right. \\ & + (u - m_1^2 - m_2^2)^2 \left(\begin{array}{c} \text{Three diagrams: 1. Two vertical lines with a diagonal cross. 2. Two vertical lines with a diagonal cross in the opposite orientation. 3. Two vertical lines with a diagonal cross in the opposite orientation.} \end{array} \right) \\ & \left. + (q^2)^2 \left(\begin{array}{c} \text{Two diagrams: 1. Two vertical lines with a horizontal line connecting them. 2. Two vertical lines with a diagonal cross.} \end{array} + \dots \right) \right] \end{aligned}$$

Loop Integrals for Small q^2

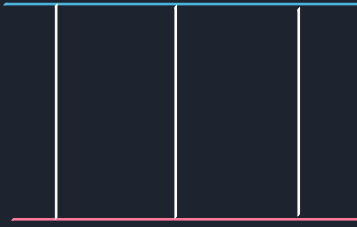


$$= \int \frac{d^D l_1}{d^D l_2} \frac{[-2\bar{p}_1 \cdot l_1 - i0 + (l_1^2 - l_1 \cdot q)] [-2\bar{p}_2 \cdot l_1 - i0 + (l_1^2 - l_1 \cdot q)] [2\bar{p}_1 \cdot l_2 - i0 + (l_2^2 - l_2 \cdot q)] [2\bar{p}_2 \cdot l_2 - i0 + (l_2^2 - l_2 \cdot q)]}{(l_1^2 - i0)(l_2^2 - i0)[(l_1 + l_2 - q)^2 - i0]}$$

As $q^2 \rightarrow 0$, METHOD OF REGIONS

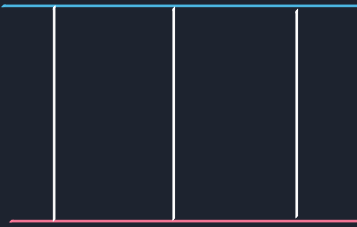
Beneke, Smirnov
- hep-ph/9781391

● **HARD REGION:** $l_{1,2} \sim \mathcal{O}(m)$



$$\stackrel{(H)}{=} c_0 (q^2)^0 + c_1 (q^2)^1 + c_2 (q^2)^2 + \dots$$

● **SOFT REGION:** $l_{1,2} \sim \mathcal{O}(q)$



$$\stackrel{(S)}{=} \frac{c_{sscl}}{(q^2)^{1+2\epsilon}} + \frac{c_{scl}}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{c_{cl}}{(q^2)^{2\epsilon}} + \dots$$

$$\begin{cases} p_1 = -\bar{p}_1 + \frac{1}{2}q \\ p_2 = -\bar{p}_2 - \frac{1}{2}q \\ p_3 = \bar{p}_2 - \frac{1}{2}q \\ p_4 = \bar{p}_1 + \frac{1}{2}q \end{cases}$$

analytic in q^2
 $\Rightarrow$ irrelevant for the long-range behavior

relevant non-analytic terms

IBP + ODE Toolkit

- **Integration-by-Parts (IBP)** identities reduce the calculation to a few **SOFT** master integrals [LiteRed, FIRE6]

$$\text{Diagram: A rectangle with three vertical lines and two horizontal lines (top and bottom).} \quad \stackrel{(s)}{=} \sum_{j=1}^{14} c_j(\epsilon; x) f_j(\epsilon; x)$$

$$\begin{cases} y \simeq \tau + \mathcal{O}(q^2) \\ x = y - \sqrt{y^2 - 1} \end{cases}$$

- The master integrals can be determined by solving **ODE's** in canonical form [epsilon]

$$d\vec{f} = \epsilon \left[A_0 d\log x + A_{+1} d\log(x+1) + A_{-1} d\log(x-1) \right] \vec{f}$$

Factorized
 ϵ -dependence
Constant Matrices

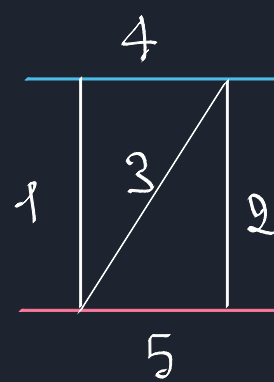
Parra-Martinez, Ruf, Zeng - 2005.04236

- One still needs the **Boundary Conditions**...

Soft Boundary Conditions

- Boundary conditions in the static limit $\sigma \rightarrow 1^+$

- EXAMPLE



$$(s) \frac{1}{u_1 u_2} \int_{\mathbb{R}_+^3} \frac{dt_1 dt_2 dt_3}{T + e} e^{-\frac{t_1 t_2 t_3}{T} q^2} \left[\int_{\mathbb{R}_+^2} dx_4 dx_5 e^{-\left(t_{23} x_5^2 + 2t_3(-\sigma - i0)x_4 x_5 + t_{13} x_4^2 \right)} \right]$$

$T = t_1 t_2 + t_2 t_3 + t_3 t_1$
 $t_{23} = t_2 + t_3$
 $t_{13} = t_1 + t_3$

As $\sigma \rightarrow 1^+$, METHOD OF REGIONS ($\tau \equiv \sqrt{\sigma^2 - 1}$)

* "ORDINARY" region: $t_{1,2,3} \sim \mathcal{O}(\tau^0)$

$$[\dots]^{(s)} = \int_{\mathbb{R}_+^2} dx_4 dx_5 e^{-\left(t_{23} x_5^2 - 2t_3 x_4 x_5 + t_{13} x_4^2 \right)} = \text{Real} \text{ (\& Finite)}$$

* "SINGULAR" region: $t_{1,2} \sim \mathcal{O}(\tau^0)$ $t_3 \sim \mathcal{O}(\tau^{-2})$

$$[\dots]^{(s)} = \frac{1}{2} \int_{-\infty}^{+\infty} d\beta e^{-t_3 \beta^2} \int_0^\infty d\alpha e^{-\frac{\alpha^2}{4} (t_{12} + t_3(-\tau^2 - i0))} = \frac{i\pi}{2\sqrt{t_3} \sqrt{t_3 \tau^2 + (-t_{12} + i0)}}$$

($\alpha = x_5 + x_4$, $\beta = x_5 - x_4$)

$2\delta_2$ from the Soft-Region

Potential Contribution

Parra-Martinez, Ruf, Zeng - 2005.04236

Radiation-Reaction

DVHRV - 2008.12743

$$\bullet \operatorname{Re}(2\delta_2) = \frac{16 G^3 m_1^2 m_2^2 \sigma^4}{b^2 (\sigma^2 - 1)} \left[-\operatorname{arccosh} \sigma + \frac{1}{\sigma^2 - 1} \left(\sigma^2 + \frac{\sigma(\sigma^2 - 2)}{\sqrt{\sigma^2 - 1}} \operatorname{arccosh} \sigma \right) \right]$$

✓ IR finite

✓ UR limit $\operatorname{Re}(2\delta_2) \sim \frac{16 G^3 (m_1 m_2 \sigma)^2}{b^2}$ as $\sigma \rightarrow \infty$

(UNIVERSAL)

Amati, Ciafaloni, Veneziano '90

Linked by ANALYTICITY
and CROSSING SYMMETRY

DVHRV - 2104.03256

IR divergence due to
soft particles in the
3-particle cut

$$\bullet \operatorname{Im}(2\delta_2) = -\frac{16 G^3 m_1^2 m_2^2}{\pi b^2} \frac{\sigma^4}{(\sigma^2 - 1)^2} \left\{ \left[\frac{1}{\epsilon} - \log(4(\sigma^2 - 1)) + 3 \log(\pi b^2 e^{\gamma_E}) \right] \left(\sigma^2 + \frac{\sigma(\sigma^2 - 2)}{\sqrt{\sigma^2 - 1}} \operatorname{arccosh} \sigma \right) \right. \\ \left. + (\sigma^2 - 1) \left[1 + \frac{\sigma(\sigma^2 - 2)}{\sqrt{\sigma^2 - 1}} \right] \left(\operatorname{arccosh} \sigma \right)^2 + \frac{\sigma(\sigma^2 - 2)}{\sqrt{\sigma^2 - 1}} \operatorname{Li}_2 \left(1 - z^2 \right) + 2\sigma^2 \right\} + \mathcal{O}(\epsilon)$$

$z \equiv \sigma - \sqrt{\sigma^2 - 1}$

Towards GR

- IDEA: add RR effects $\text{Re}(2\delta_2^{\text{RR}})$ to the (known) POTENTIAL PART
- By ANALYTICITY

DVHRV -
2101.05772,
2104.03256

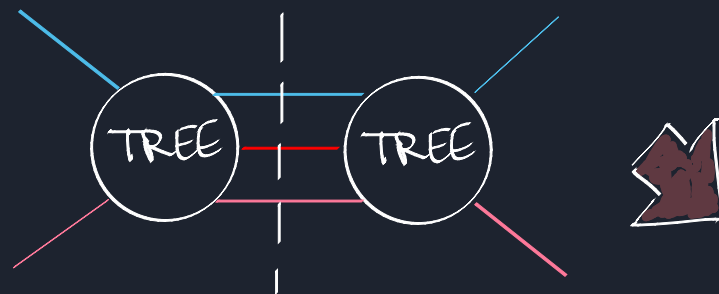
$$\boxed{\log(\sigma^2 - 1)} = \Im \left[i \log(-(\sigma^2 - 1) - i0) \right] \Rightarrow \text{Re} \left[i \log(-(\sigma^2 - 1) - i0) \right] = \boxed{+\pi}$$

- $\log(\sigma^2 - 1)$ term of $\Im(2\delta_2) \Rightarrow \text{Re}(2\delta_2^{\text{RR}})$

- By UNITARITY,

$$\left[\Im(2A_2) \right]_{\text{3PC}}$$

3-particle
cut



$$\boxed{\left[\Im \tilde{A}_2 \right]_{\text{3PC}} = \Im 2\delta_2}$$

in b-space.

Outline

- Classical Gravity from Scattering Amplitudes

EIKONAL EXPONENTIATION

- $N=8$ as a Theoretical Laboratory

POTENTIAL + RADIATION REACTION @ 3PM

- Radiative Effects in General Relativity

3-PARTICLE CUT & RADIATION

Exponentiation & Unitarity

$$i \tilde{A}_0(s, b) = 2i\delta_0$$

$$i \tilde{A}_1(s, b) = \frac{(2i\delta_0)^2}{2!} + 2i\delta_1 + 2i\Delta_1$$

$$i \tilde{A}_2(s, b) = \frac{(2i\delta_0)^3}{3!} + 2i\delta_0 2i\delta_1 + 2i\delta_2 + 2i\delta_0 2i\Delta_1 + 2i\Delta_2$$

$S^\dagger S = 1$

$2 \operatorname{Im} \left(\text{1-loop self-energy} \right) = \sum_X \left(\text{tree} \right) \left(\text{1-loop} \right)$

$$2 \operatorname{Im} \left(\text{1-loop} \right) = \left(\text{tree} \right) \left(\text{tree} \right) \longleftrightarrow \frac{(2\delta_0)^2}{2!}$$

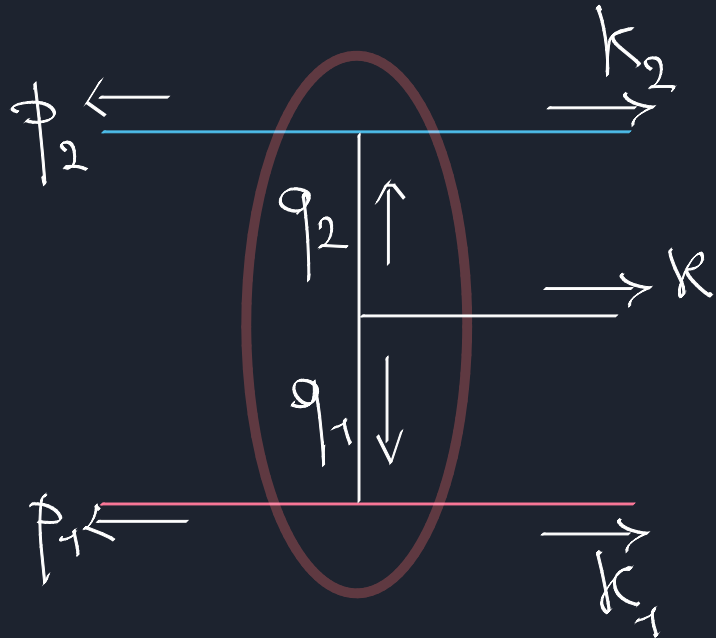
$$2 \operatorname{Im} \left(\text{2-loops} \right) = \left(\text{tree} \right) \left(\text{1-loop} \right) + \left(\text{1-loop} \right) \left(\text{tree} \right) + \left(\text{tree} \right) \left(\text{tree} \right)$$

$\uparrow \quad \quad \uparrow$
 $2\delta_0 \quad 2\operatorname{Re} \Delta_1$

$\downarrow$
 $\operatorname{Im} 2\delta_2$

5-pt Amplitudes in the Classical Limit $q_i \rightarrow 0$

Goldberger, Ridgway; Luna, Nicholson, O'Connell, White; Mogull, Plefka, Steinhoff



$$A_{5pt}(P_1, P_2, K_1, K_2, K) = (8\pi G)^{\frac{3}{2}} \left\{ \frac{8(P_1 \cdot K P_2^M - P_2 \cdot K P_1^M)(P_1 \cdot K P_2^N - P_2 \cdot K P_1^N)}{q_1^2 q_2^2} \right.$$

$$+ 8P_1 P_2 \left[\left(\frac{P_1^M P_2^N K \cdot P_1}{K \cdot P_1} - P_1^{(M} P_2^{N)} \right) \frac{1}{q_2^2} + 1 \leftrightarrow 2 \right] - 2 \frac{P_1 \cdot K P_2^{(M} q_1^{N)} - P_2 \cdot K P_1^{(M} q_1^{N)}}{q_1^2 q_2^2} \left. \right]$$

Leading Weinberg term $(K_i \ll q)$ $\left[+ \beta \left[- \left(\frac{P_1^M P_1^N K \cdot q_1}{(P_1 \cdot K)^2 q_2^2} + 1 \leftrightarrow 2 \right) + 2 \left(\frac{P_1^{(M} q_1^{N)}}{P_1 \cdot K q_2^2} - \frac{P_2^{(M} q_1^{N)}}{P_2 \cdot K q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2} \right) \right] \right]$

$$P_1^2 = 0 = P_2^2,$$

$$|A_{5pt}|^2 \equiv A_{5pt}^{MN}(P_1, P_2, K_1, K_2, K) \prod_{MN, RS} A_{5pt}^{RS}(P_4, P_3, -K_1, -K_2, -K)$$

$$\boxed{\mathcal{N} = 8}$$

$$P_1 = (p_1, 0, 0, 0, 0, 0, m_1)$$

$$P_2 = (p_2, 0, 0, 0, 0, 0, m_2, 0)$$

$$\beta = 4m_1^2 m_2^2 \sigma^2, \quad \prod_{MN, RS} \eta_{MR} \eta_{NS}$$

$$\boxed{GR}$$

$$P_1 = (p_1, 0, 0, 0, 0, 0, 0)$$

$$P_2 = (p_2, 0, 0, 0, 0, 0, 0)$$

$$\beta = 4m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{D-2} \right), \quad \prod_{MN, RS} \eta_{MR} \eta_{NS} \frac{\eta_{MN} \eta_{RS}}{D-2}$$

Evaluation of the Cut Integrals

DVHRV - 2104.03256

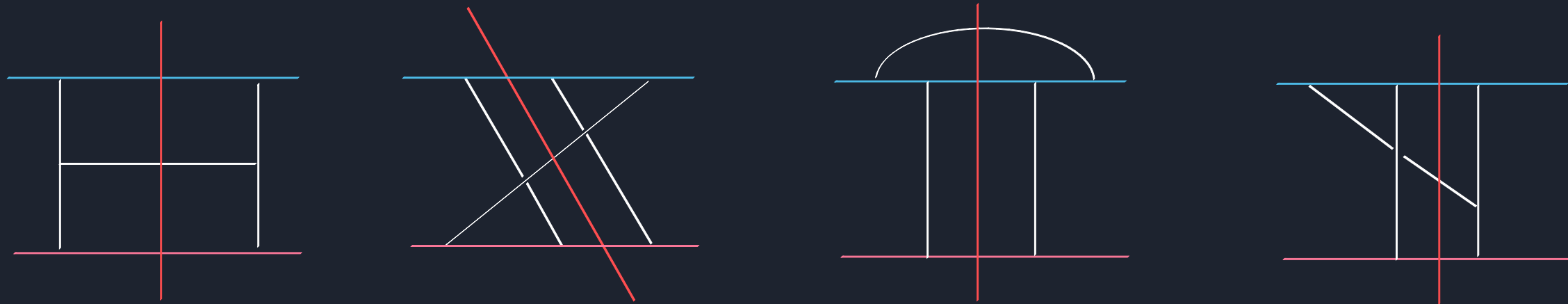
- $$|A_{5pt}|^2 = \frac{N(u_1 l_2, u_2 l_2, l_1 q, l_2 q, u_1 l_1, u_2 l_1, l_1^2, l_1 l_2, l_2^2)}{D_1^{d_1} D_2^{d_2} D_3 D_4 D_5 D_6 \cancel{D_7} \cancel{D_8} \cancel{D_9}}$$

$$\begin{cases} q_1 = -l_1 \\ q_2 = l_1 + l_2 \\ k = -l_2 \end{cases}$$

$$D_1 = u_1 l_2, D_2 = u_2 l_2, D_3 = l_1^2, D_4 = (l_1 + l_2)^2, D_5 = (l_1 + q)^2, D_6 = (l_1 + l_2 + q)^2$$

$$D_7 = l_2^2, D_8 = u_1 l_1, D_9 = u_2 (l_1 + l_2)$$

- This can be reduced to



- These integrals can be decomposed into the same MASTER INTEGRALS discussed previously up to the choice of (cut) BOUNDARY CONDITIONS

[REVERSE UNITARITY]

Anastasiou, Melnikov - 0207004

Herrmann, Parra-Martinez, Ruf, Zeng - 2101.07255, 2104.03957

3PM Eikonal in GR

POTENTIAL

Bern, Cheung, Shen, Solon, Ruf, Zeng -1901.04424

$$\bullet \operatorname{Re} 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2} \left\{ \underbrace{\frac{s(12\sigma^4 - 10\sigma^2 + 1)}{2m_1 m_2 (\sigma^2 - 1)^{\frac{3}{2}}}}_{\text{PROBE LIMIT}} - \underbrace{\frac{\sigma(14\sigma^2 + 25)}{3\sqrt{\sigma^2 - 1}}}_{\text{RADIATION REACTION}} + \underbrace{\frac{-4\sigma^4 + 12\sigma^2 + 3}{\sigma^2 - 1} \operatorname{arccosh} \sigma}_{\text{ANALYTICITY \& CROSSING}} \right.$$

DVHRV - 2101.05772, 2104.03256
(Agrees with Damour - 2010.01641)

$$+ \frac{(2\sigma^2 - 1)^2}{2(\sigma^2 - 1)^2} \left[\frac{8 - 5\sigma^2}{3} + \frac{\sigma(2\sigma^2 - 3)}{\sqrt{\sigma^2 - 1}} \operatorname{arccosh} \sigma \right]$$

$$\bullet \operatorname{Im} 2\delta_2 = \frac{2G^3 m_1^2 m_2^2}{\pi b^2} \frac{(2\sigma^2 - 1)^2}{(\sigma^2 - 1)^2} \left\{ \left[\underbrace{-\frac{1}{\epsilon}}_{\text{DIVERGENT PART}} + \log(4(\sigma^2 - 1)) - 3 \log(\pi b^2 e^{\gamma_E}) \right] \left[\frac{8 - 5\sigma^2}{3} + \frac{\sigma(2\sigma^2 - 3)}{\sqrt{\sigma^2 - 1}} \operatorname{arccosh} \sigma \right] \right.$$

$$+ (\operatorname{arccosh} \sigma)^2 \left[\frac{\sigma(3 - 2\sigma^2)}{\sqrt{\sigma^2 - 1}} - 2 \frac{4\sigma^6 - 16\sigma^4 + 9\sigma^2 + 3}{(2\sigma^2 - 1)^2} \right]$$

$$+ \sigma \operatorname{arccosh} \sigma \frac{88\sigma^6 - 240\sigma^4 + 240\sigma^2 - 97}{3(2\sigma^2 - 1)^2 \sqrt{\sigma^2 - 1}}$$

$$+ \frac{\sigma(3 - 2\sigma^2)}{\sqrt{\sigma^2 - 1}} \operatorname{Li}_2(1 - z^2) + \frac{-140\sigma^6 + 220\sigma^4 - 127\sigma^2 + 56}{9(2\sigma^2 - 1)^2} \left. \right\}$$

The $-\frac{1}{\epsilon}$ always accompanies $\log(\sigma^2 - 1) \Rightarrow$ the DIVERGENT PART governs RR effects

Explicitly checked in
Bjerrum-Bohr, Damgaard, Planté, Vanhove
- 2105.05218

The $\frac{1}{\epsilon}$ Part: Unitarity + Soft Theorems

DVHRV - 2101.05772

- $\left[\mathcal{I}m(2\mathcal{A}_2) \right]_{3PC} = \int d\vec{k} \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \delta^{(D)}(p_1 + p_2 + k_1 + k_2 + k_3) \delta_+(k^2) \delta_+(k_1^2 + m_1^2) \delta_+(k_2^2 + m_2^2) \left| \mathcal{A}_{5pt} \right|^2$

- Weinberg limit $k \ll q \ll m$

- $\mathcal{I}m 2\mathcal{S}_2 = \left[\mathcal{I}m(2\tilde{\mathcal{A}}_2) \right]_{3PC}$

$$\sim \int \frac{d^{D-1} \vec{k}}{|\vec{k}|} \left(\frac{1}{|\vec{k}|} \hat{A}(\hat{k}, b) \right) \left(\frac{1}{|\vec{k}|} \hat{A}(\hat{k}, b) \right)^*$$

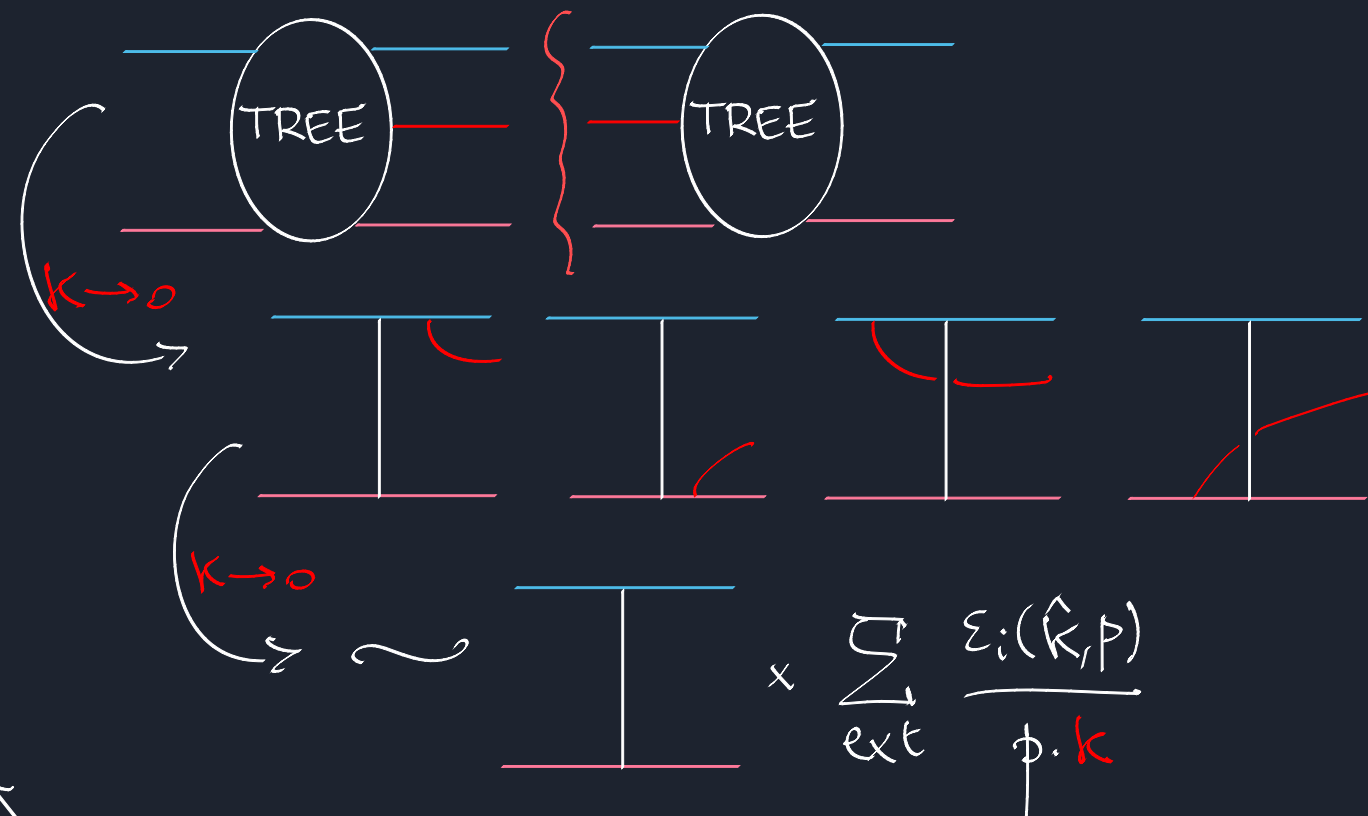
$$\sim \int_0^{\omega_{MAX}} \frac{d\omega}{\omega^{1+2\epsilon}} \int d\Omega_{2-2\epsilon}(\hat{k}) \left| \hat{A}(\hat{k}, b) \right|^2$$

$$\sim -\frac{1}{2\epsilon} \frac{1}{\omega_{MAX}^{2\epsilon}} = -\frac{1}{2\epsilon} + \log \omega_{MAX} \sim \text{small velocity}$$

$$\boxed{-\frac{1}{2\epsilon} + \log v}$$

generic velocity

$$\boxed{\frac{1}{2} \left[-\frac{1}{\epsilon} + \log(v^2 - 1) \right]}$$



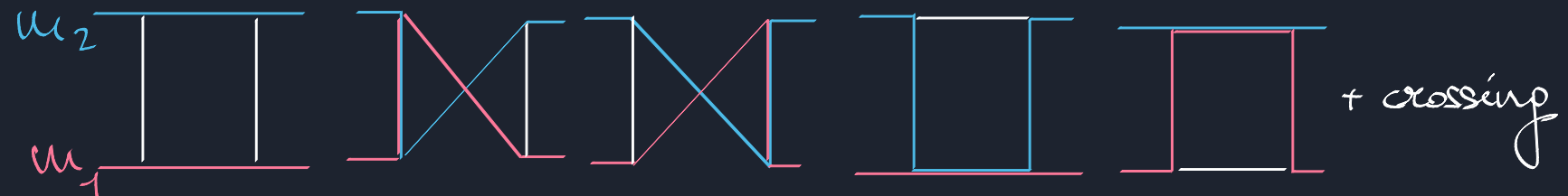
going to b-space BEFORE integrating over $\vec{k}$

The $\frac{1}{\epsilon}$ Part: Exponentiation of IR Divergences

H - 2105.04594

• $\mathcal{N} = 8$ $A_1 = \frac{1}{2} \frac{(8\pi G)^2}{(4\pi)^{2-\epsilon}} \left[(s-m_1^2-m_2^2)^4 + (u-m_1^2-m_2^2)^4 - t^4 \right] (\text{box} + \text{cross} + \text{cross})$

$A = A_0 e^{\hat{A}_1} (1 + \hat{A}_2 + \dots)$



$\hat{A}_1 \equiv A_1/A_0$

IR FINITE

IR-DIVERGENT PART $\Rightarrow A \sim \frac{A_1^2}{2A_0} \Rightarrow 2\delta_2 \sim -\frac{i}{\pi\epsilon} \frac{16G^3 m_1^2 m_2^2 \sigma^5}{b^2(\sigma^2-1)^2} \left[\sigma + (\sigma^2-2) \frac{\text{arccosh} \sigma}{\sqrt{\sigma^2-1}} \right]$

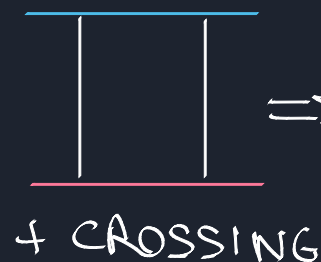
• GR (Weinberg '65)

$A = e^W (A_0 + A_1^0 + A_2^0 + \dots)$

IR FINITE

$W = \frac{G}{2\pi\epsilon} \sum_{n,m} m_n m_m \left(\sigma_{nm}^2 - \frac{1}{2} \right) \frac{\pm \text{arccosh} \sigma_{nm} - i\pi}{\sqrt{\sigma_{nm}^2 - 1}}$
 $\sigma_{nm} = -\frac{p_n \cdot p_m}{m_n m_m}$
 $\left. \begin{matrix} \text{out/out} \\ \text{in/in} \\ \text{in/out} \end{matrix} \right\} \pm$
 only if $n \neq m$, both in (out)

$A \sim W A_1 + \text{ANALYTIC in } q^2$



$\Rightarrow 2\delta \sim -\frac{i}{\pi\epsilon} \frac{(2G)^3 m_1^2 m_2^2 (\sigma^2 - \frac{1}{2})^2}{b^2(\sigma^2-1)^2} \left[\frac{8-5\sigma^2}{3} + \sigma(2\sigma^2-3) \frac{\text{arccosh} \sigma}{\sqrt{\sigma^2-1}} \right]$

Summary of RR from Analyticity + Crossing

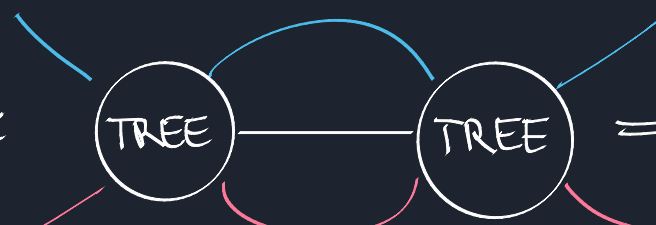
- Unitarity + Soft Th
DVHRV - 2101.05772



$$\Rightarrow \lim_{\epsilon \rightarrow 0} 2\delta_2 \sim -\frac{1}{\pi\epsilon} \left[\frac{1}{2} - \epsilon \log v \right] (\dots)_{RR}$$

small v

- Full Unitarity:
DVHRV - 2104.03256



$$\Rightarrow \text{Im } 2\delta_2 = -\frac{1}{\pi\epsilon} \left[\frac{1}{2} - \epsilon \log(\sigma^2 - 1) \right] (\dots)_{RR}$$

$$+ \epsilon^0 (\text{analytic as } \sigma \rightarrow 1) + \mathcal{O}(\epsilon)$$

- IR exponentiation: $A = e^{\text{1-loop div}} A_{\text{FINITE}} \Rightarrow 2\delta_2 \sim -\frac{i}{2\pi\epsilon} (\dots)_{RR}$
H - 2105.04594

Then by Analyticity + Crossing:

$$(\text{Re } 2\delta_2^{RR})_{D=4} = \lim_{\epsilon \rightarrow 0} [-\pi\epsilon \text{Im } 2\delta_2]$$

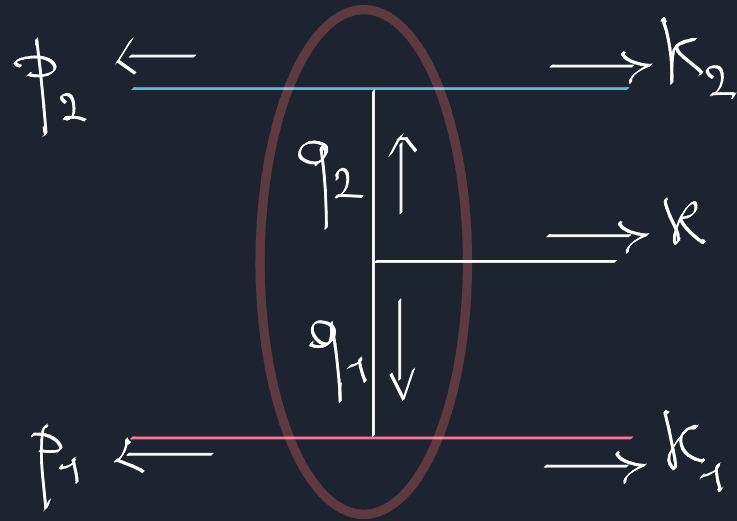
Bonus: Zero-frequency limit of the emission spectrum

$$\left. \frac{dE}{d\omega} \right|_{\omega=0, D=4}^{\text{GW}} = \lim_{\epsilon \rightarrow 0} -4\epsilon \text{Im } 2\delta_2$$

Waveforms

- Going to b-space BEFORE integrating over $\vec{k}$:

$$\begin{cases} \vec{q}_1 + \vec{q}_2 = 0 \\ \vec{q}_1 - \vec{q}_2 = \frac{1}{2} \vec{\Delta} \end{cases}$$



$$\tilde{A}_{5pt, i} = \int \frac{d^{2-2\epsilon} \underline{\Delta}}{(2\pi)^{2-2\epsilon}} \frac{e^{i \underline{b} \cdot \underline{\Delta}}}{4m_1 m_2 \sqrt{s-1}} \epsilon_{i, \mu\nu} A_{5pt}^{\mu\nu}$$

POLARIZATIONS $\times, +$

- Waveform in the far zone: $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa H_{\mu\nu}$ ($\kappa = \sqrt{8\pi G}$)
 $\square H^{\mu\nu} = \tau^{\mu\nu}$, with $\hat{\tau}^{\mu\nu} = \frac{1}{\kappa} A_{5pt}^{\mu\nu}$

$$\Rightarrow_{D=4} H_{ret}^{\mu\nu} = \frac{\kappa}{4\pi R} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} A_{5pt}^{\mu\nu}(\vec{k} = \omega \hat{x})$$

detector's distance

retarded time

observation direction

Waveforms: X

cf. Kovacs, Thorne (1978)

Jakobsen, Mogull, Plefka, Steinhoff - 2101.10256,

Mougiakakos, Riva, Vernizzi - 2102.08339

We work in the CM frame:

$$\begin{aligned} \tilde{\mathcal{A}}_{5pt, X} = & \frac{\kappa^3}{4pE} \frac{i \hat{\mathbf{b}} \cdot \vec{e}_\phi}{2\pi} \left\{ \beta \left[-\frac{m_2 s(\theta)}{\sqrt{s}} e^{i \frac{\mathbf{b} \cdot \mathbf{k}}{2}} K_2(bc_2 |\mathbf{k}|) - \frac{m_1 s(\theta)}{\sqrt{s}} e^{-i \frac{\mathbf{b} \cdot \mathbf{k}}{2}} K_1(bc_1 |\mathbf{k}|) \right. \right. \\ & + i \int_0^1 dx e^{i \frac{\mathbf{b} \cdot \mathbf{k}}{2} (1-2x)} \left(\hat{\mathbf{b}} \cdot \vec{e}_\theta b \sqrt{f} K_1(b |\mathbf{k}| f) + i \left(x - \frac{1}{2}\right) \mathbf{k} \cdot \vec{e}_\theta b K_0(b |\mathbf{k}| \sqrt{f}) \right) \Big] \\ & \left. + \left(4 p_1 p_2 pE \omega s(\theta) + \beta s(\theta) \frac{q_1^L - q_2^L}{2} \right) \int_0^1 dx e^{i \frac{\mathbf{b} \cdot \mathbf{k}}{2} (1-2x)} b K_0(b |\mathbf{k}| \sqrt{f}) \right\} \end{aligned}$$

$$\begin{pmatrix} s(\theta) \equiv \sin \theta \\ c(\theta) \equiv \cos \theta \end{pmatrix}$$

$$f = x(1-x) + c_1^2 x + c_2^2 (1-x)$$

$$|\mathbf{k}| c_1 = \frac{\mathbf{p}_2 \cdot \mathbf{k}}{m_2 \sqrt{\sigma^2 - 1}}$$

$$|\mathbf{k}| c_2 = \frac{\mathbf{p}_1 \cdot \mathbf{k}}{m_1 \sqrt{\sigma^2 - 1}}$$

$$\vec{e}_\phi = (-s(\phi), c(\phi), 0)$$

$$\vec{e}_\theta = (c(\theta) c(\phi), c(\theta) s(\phi), -s(\theta))$$

$$\begin{aligned} k^\mu &= \omega (1, s(\theta) c(\phi), s(\theta) s(\phi), c(\theta)) \\ \mathbf{k} &= \omega s(\theta) (c(\phi), s(\phi)) \end{aligned}$$

**WARNING,
PRELIMINARY**

Waveforms: +

u-space expressions of X, t agree with soft theorems dictating the large- u behavior

(Sahoo, Sen - 2105.08739 and refs. therein)

$$\begin{aligned} \tilde{A}_{5pt, +} = & \frac{k^3}{4E_p} \frac{1}{4\pi} \left\{ \beta \left(p_1 k p_2 \vec{e}_\theta - p_2 k p_1 \vec{e}_\theta \right)^2 \int_0^1 dx e^{i \frac{b \cdot k}{2} (1-2x)} \frac{b}{2} \frac{K_1(b|k|\sqrt{f})}{|k|\sqrt{f}} \right. \\ & + \beta \left[\frac{(p_1 \cdot \vec{e}_\theta)^2}{(p_1 \cdot k)^2} e^{i \frac{k \cdot b}{2}} k^2 \left(d_2 K_0(bc_2|k|) + \frac{i k \cdot b}{b|k|} c_2 K_1(bc_2|k|) \right) + \frac{(p_2 \cdot \vec{e}_\theta)^2}{(p_2 \cdot k)^2} e^{-i \frac{k \cdot b}{2}} k^2 \left(d_1 K_0(bc_1|k|) - \frac{i k \cdot b}{b|k|} c_1 K_1(bc_1|k|) \right) \right. \\ & - \frac{p_i \cdot \vec{e}_\theta}{p_i \cdot k} e^{i \frac{b \cdot k}{2}} \left(\vec{e}_\theta \cdot k + s(\theta) (q_1^L - q_2^L) K_0(bc_2|k|) + 2i \vec{e}_\theta \cdot \hat{b} c_2 |k| K_1(bc_2|k|) \right) + \frac{s(\theta)^2 (q_1^L - q_2^L)^2}{2} \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} \frac{b}{2} \frac{K_1(b|k|\sqrt{f})}{|k|\sqrt{f}} \\ & - \frac{p_i \cdot \vec{e}_\theta}{p_i \cdot k} e^{-i \frac{b \cdot k}{2}} \left(\vec{e}_\theta \cdot k - s(\theta) (q_1^L - q_2^L) K_0(bc_1|k|) - 2i \vec{e}_\theta \cdot \hat{b} c_1 |k| K_1(bc_1|k|) \right) + \frac{1}{2} \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} (\vec{e}_\theta \cdot k)^2 (1-2x)^2 \frac{b}{2} \frac{K_1(b|k|\sqrt{f})}{|k|\sqrt{f}} \\ & \left. \left. - s(\theta) (q_1^L - q_2^L) \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} \left(-\vec{e}_\theta \cdot k (1-2x) \frac{b}{2} \frac{K_1(b|k|\sqrt{f})}{|k|\sqrt{f}} - i \vec{e}_\theta \cdot \hat{b} b K_0(b|k|\sqrt{f}) \right) + \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} \left(-s(\theta)^2 K_0(b|k|\sqrt{f}) + [\vec{e}_\theta \cdot \hat{b}]^2 (\vec{e}_\theta \cdot \hat{b})^2 \right) b |k| \sqrt{f} K_1(b|k|\sqrt{f}) \right] \right. \\ & + \beta p_1 p_2 \left[\left(\frac{(p_1 \cdot \vec{e}_\theta)^2 k \cdot p_2}{k \cdot p_1} - p_1 \cdot \vec{e}_\theta p_2 \cdot \vec{e}_\theta \right) e^{i \frac{k \cdot b}{2}} K_0(bc_2|k|) + \left(\frac{(p_2 \cdot \vec{e}_\theta)^2 k \cdot p_1}{k \cdot p_2} - p_2 \cdot \vec{e}_\theta p_1 \cdot \vec{e}_\theta \right) e^{-i \frac{k \cdot b}{2}} K_0(bc_1|k|) \right. \\ & \left. + \left(p_1 k p_2 \vec{e}_\theta - p_2 k p_1 \vec{e}_\theta \right) \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} \left(i \vec{e}_\theta \cdot \hat{b} b K_0(b|k|\sqrt{f}) + s(\theta) (q_1^L - q_2^L) \frac{b}{2} \frac{K_1(b|k|\sqrt{f})}{|k|\sqrt{f}} \right) \right. \\ & \left. + i \int_0^1 dx e^{i \frac{k \cdot b}{2} (1-2x)} \vec{e}_\theta \cdot k \vec{e}_\theta \cdot \hat{b} (1-2x) b K_0(b|k|\sqrt{f}) \right\} \end{aligned}$$

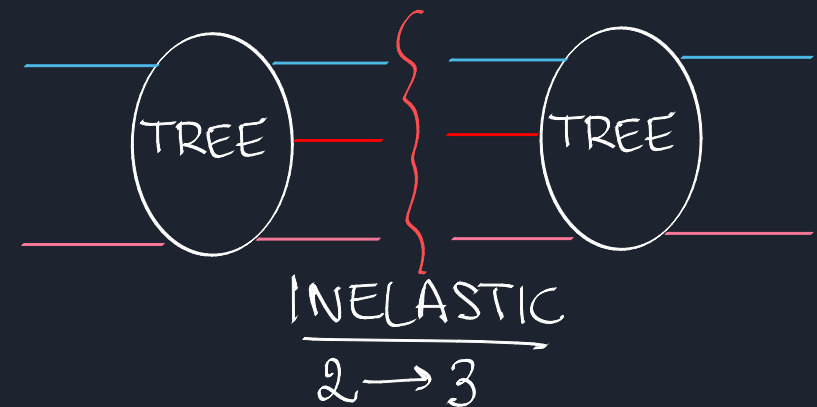
cf. Kovacs, Thorne (1978)
Jakobsen, Mogull, Plefka, Steinhoff - 2101.10256,
Mougiakakos, Riva, Vernizzi - 2102.08339

WARNING,
PRELIMINARY

Towards an Eikonal with Real Radiation

- $1 + i\tilde{A}(s, b) \sim e^{2i\delta(s, b)}$ fails to be "unitary" because
 $\uparrow$
 ELASTIC $2 \rightarrow 2$ CLASSICAL
 $\Im 2\delta_2 \neq 0$ (in fact $\sim \frac{1}{\epsilon}$)

- $\Im 2\delta_2$ cancels against $[\Im \tilde{A}_2]_{3PC} = \int d\Gamma_{int}$



Ciafaloni, Colferai, Veneziano - 1812.08137

- The appropriate way to restore unitarity is to combine $2 \rightarrow 2$ and $2 \rightarrow 3$ via a Hermitian eikonal OPERATOR

- The 1st step exponentiation of the $2 \rightarrow 3$ amplitude

↑ The $\frac{1}{\epsilon}$, superclassical term in $A_{5pt, N=8}^{1-loop, massless}$ is compatible with $2i\delta_0 \cdot A_{5pt, N=8}^{tree, massless}$

WARNING,
PRELIMINARY

Conclusions & Outlook

- 3PM eikonal: $\text{Re} \delta_2$ (Potential & Radiation Reaction) ☒
($N=8$ & GR) $\text{Im} \delta_2$ ☒
- Radiation-Reaction from:
 - ANALYTICITY + CROSSING + UNITARITY ☒
 - ANALYTICITY + CROSSING + IR EXPONENTIATION ☒
- Leading-Order Waveforms $\mathcal{O}(G^2)$ ☒
- Inelastic Eikonal ☐
- Eikonal Operator @ 4PM ☐