Wilson loops as defects in $\mathcal{N} = 2$ conformal field theories

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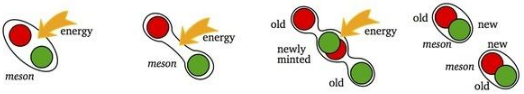
Context and motivations

- Quantum CromoDynamics (QCD): strong interaction between quarks and gluons.
- Theoretical approach to QCD: **non-abelian Gauge Theories**
  - High-energy → Asymptotic freedom
  - Low-energy → Confinement mechanism.



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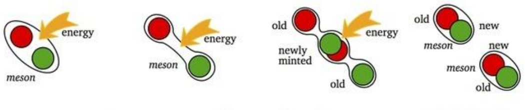
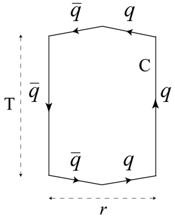


- Confined phase of quarks described by the **Wilson loop**
  - Closed path C traced out by a $q\bar{q}$ pair.
  - Non-local operator inside the gauge theory

$$W(C) \sim \mathrm{e}^{\mathrm{i} \int_C A}$$



Superconformal Field Theories (SCFTs) in a nutshell

How to approach the confined phase of Gauge Theories? We study QFTs highly constrained by the action of **extra spacetime symmetries**.

Translational and rotational invariance get enhanced with:

- Conformal symmetry: action is invariant under scale transformations

$$S = \int d^4x \, (\partial\phi)^2 + \frac{\lambda}{4}\phi^4 \,,$$

$$x \to \mu x \,, \quad \phi(x) \to \mu^n \phi(\mu x) \,, \quad n = \text{conformal dimension} \,.$$

- Supersymmetry: additional anticommuting charges $\mathcal{Q}^{A=1,\dots\mathcal{N}}$ such that

$$\mathcal{Q}\, |\text{boson}\rangle = |\text{fermion}\rangle \quad \text{and viceversa} \,.$$

No individual fields but multiplets $(\phi, \psi, A_\mu, \dots)$ with same quantum numbers.

Why Superconformal Field theories?

- Treatable behavior at the quantum level.
- Possibility to obtain **exact results**.

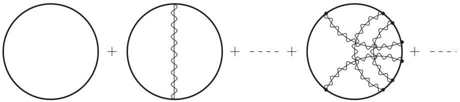
- The most famous example of 4-dim SCFT is $\mathcal{N} = 4$ Super Yang-Mills theory. **Maximally supersymmetric** gauge theory, unique multiplet $\left(A_\mu, \psi^{A=1,\dots,4}, \varphi^{i=1,\dots,6}\right)$

- Supersymmetric Wilson loop

$$W(C) = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left\{ g \oint_C d\tau \Big[\mathrm{i}\, A_\mu\, \dot{x}^\mu + |\dot{x}|\, \theta_i\, \varphi^i \Big] \right\}$$

- $\langle W(C) \rangle$ computed **exactly** [1] using field theory techniques:

$\langle W(C) \rangle =$



$+ \cdots = \frac{1}{N} L^1_{N-1}\left(-\frac{g^2}{4} \right) \exp\left[\frac{g^2}{8} \right]$

$L^\beta_\alpha \rightarrow$ Laguerre polynomials

- What about the less constrained $\mathcal{N} = 2$ case? We need special tools to compute observables in presence of a Wilson loop.

[1] [Erikson, Semenoff, Zarembo, 2000], [Drukker, Gross, 2001]

Tools for $\mathcal{N} = 2$ SCFTs (1): SUSY Localization

- The path integral is mapped to a sphere S_4 and reduced to a finite dimensional integral[2]

$$Z_{\mathbb{R}^4} = \int [\mathcal{D}\Phi] \xrightarrow{Q} \mathcal{Z}_{S^4} = \int da$$

  a is a $N \times N$ matrix.

- In $\mathcal{N} = 4$ purely Gaussian matrix model $\Rightarrow$ exact results.

- In $\mathcal{N} = 2$ one gets a **matrix model** with a Gaussian term, a 1-loop determinant and the instanton contributions

$$\mathcal{Z}_{S^4} \sim \int da \; e^{-\frac{8\pi^2}{g^2}\operatorname{tr} a^2} |\mathcal{Z}_{\text{1-loop}}\, \mathcal{Z}_{\text{inst}}|^2 \,.$$

- Observables from $\mathbb{R}^4$ field theory can be localized and computed using such matrix model

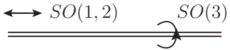
$$\begin{aligned}\langle O_1(a) \dots O_n(a) \rangle_{S^4} &= \frac{1}{\mathcal{Z}_{S^4}} \int da \; O_1(a) \dots O_n(a) \; e^{-\frac{8\pi^2}{g^2}\operatorname{tr} a^2} |\mathcal{Z}_{\text{1-loop}}\, \mathcal{Z}_{\text{inst}}|^2 \\ &= f(N, g)\end{aligned}$$

[2] [Pestun, 2007]

Tools for $\mathcal{N}=2$ SCFTs (2): Wilson loop as a conformal defect

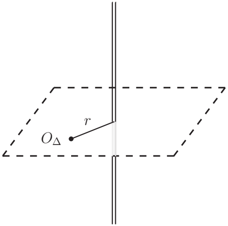
4-d conformal symmetry is broken by the line:

$$\mathrm{SO}(1,5) \to \mathrm{SO}(1,2) \times \mathrm{SO}(3)$$



Residual conformal symmetry is sufficient to constrain the one-point function of a bulk operator

$$\langle O_\Delta(x)\rangle_W = \frac{\langle O_\Delta(x) W(C)\rangle}{\langle W(C)\rangle} = \frac{A_O}{(2\pi r)^\Delta}$$



or the two-point function of a defect operator

$$\left\langle D_{\Delta_1}(\tau) D_{\Delta_2}(0)\right\rangle_W = C_D \frac{\delta_{\Delta_1 \Delta_2}}{\tau^{2\Delta}}$$

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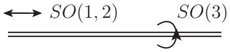

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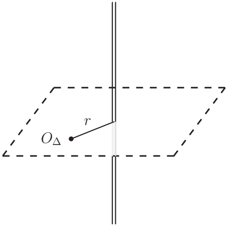

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Key point:
interplay between localization and Defect CFT $\Rightarrow \langle W\rangle$, A_O or C_D as functions of the coupling.

1. Wilson loop vev in $\mathcal{N} = 2$ SCFTs

2. Wilson loop observables: emitted radiation

$\mathcal{N}=2$ conformal matrix model

- 4-d $\mathcal{N}=2$ gauge theory with SU(N) gauge group and conformal matter content:

$$V=(A_\mu,\lambda^a,\varphi)_{\text{adj}}\ ,\qquad H=\left(q,\tilde{q},\psi,\tilde{\psi}\right)_{\mathcal{R}}$$

- Example: Superconformal QCD: $\qquad V_{\text{adj}}+2N\,H_{\text{fund}}$

- Matrix model with perturbative approach [3]:

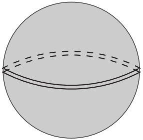
$$\left|Z_{\text{inst}}\right|^2\to 1\ ,\qquad \left|Z_{1-\text{loop}}\right|^2=\mathrm{e}^{-S_{\text{int}}(a)}\ ,\qquad \mathcal{Z}_{S^4}=\int da\ \ \mathrm{e}^{-\mathrm{tr}\,a^2-S_{\text{int}}(a)}\ .$$

$$S_{\text{int}}(a)=-\left(\frac{g^2}{8\pi^2}\right)^2\frac{\zeta(3)}{2}\,S_4(a)\ +\left(\frac{g^2}{8\pi^2}\right)^3\frac{\zeta(5)}{3}\,S_6(a)+\dots$$

- $\mathcal{O}(g^2)\to 0$ no 1-loop term.
- $S_{2n}(a)$ stands for the combination $\left(\mathrm{Tr}_{\mathcal{R}}-\mathrm{Tr}_{\text{adj}}\right)a^{2n}$

[3] [Billò, Fucito, Lerda, Morales, Stanev, Wen, 2017], [Billò, FG, Lerda, 2019]

The matrix model expression for $W(C)$ is



$$\mathcal{W}(a) = \frac{1}{N}\,\mathrm{tr}\exp\left(\frac{g}{\sqrt{2}}\,a\right)$$

We compute $\langle \mathcal{W}(a) \rangle$ perturbatively for each transcendentality, using Gaussian integral recursion relations [4]:

$$\begin{aligned}\langle \mathcal{W}(a) \rangle &= \left\langle \mathrm{e}^{-S_{\mathrm{int}}(a,g)}\,\mathcal{W}(a) \right\rangle_0 \\ &= \langle \mathcal{W}(a) \rangle_0 - \left(\frac{g^2}{8\pi^2}\right)^2 \frac{\zeta(3)}{2}\,\langle S_4\,\mathcal{W}(a) \rangle_0 + \left(\frac{g^2}{8\pi^2}\right)^3 \frac{\zeta(5)}{3}\,\langle S_6\,\mathcal{W}(a) \rangle_0 + \ldots\end{aligned}$$

[4] [Billò, FG, Gregori, Lerda, 2018]

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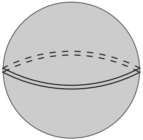

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$\downarrow$

$W(g, N)$
$\mathcal{N} = 4$ result
exact in g, N

[4] [Billò, FG, Gregori, Lerda, 2018]

Wilson loop vev in $\mathcal{N} = 2$ matrix model

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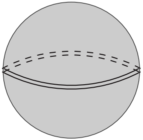

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$\mathcal{N} = 4$ result
exact in g, N

$$\langle S_{2m}\mathcal{W}(a)\rangle_0 \leftrightarrow \left\{g\,\partial_g^{(m)}\,W(g,N)\right\}$$

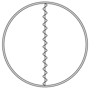
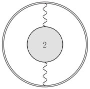
using recursion relations
exact in g, N for each $\zeta(2n-1)$

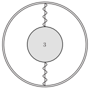
[4] [Billò, FG, Gregori, Lerda, 2018]

1/2 BPS Wilson loop with Feynman diagrams

$$W(C) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left\{g\oint_C d\tau\Big[\mathrm{i}\,A_\mu\,\dot{x}^\mu + \frac{1}{\sqrt{2}}\big(\varphi+\bar{\varphi}\big)\Big]\right\}$$

The perturbative results follow the matrix model organization:

$\langle W(C)\rangle = \Big(1 +$  $+\cdots\Big) + \zeta(3)\Big($ 

$+\cdots\Big) + \zeta(5)\Big($ 

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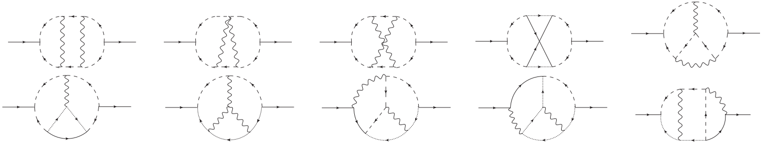
We were able to compute $\langle W(C)\rangle$ at four-loops order using $\mathcal{N}=1$ superspace formalism:



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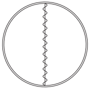
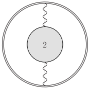
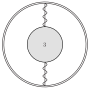
$\langle W(C)\rangle = \Big(1 +$  $+\cdots\Big) + \zeta(3)\Big($ 

$+\cdots\Big) + \zeta(5)\Big($ 

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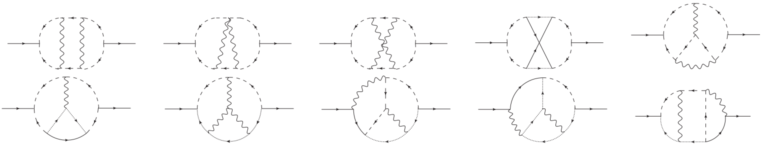

Take-home messages:

- The matrix model is a guideline for field theory computations.
- Perturbative expansions can be pushed very far by using matrix model techniques [5]

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1. Wilson loop vev in $\mathcal{N} = 2$ SCFTs

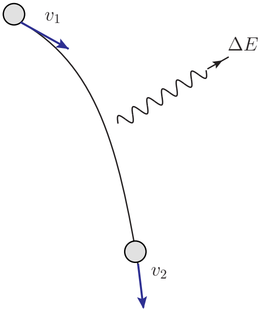
2. Wilson loop observables: emitted radiation

Emitted energy by a charged particle

- A charged particle in an accelerated motion emits energy following relativistic Larmor formula

$$\Delta E = 2\pi\, B \int dt (\dot{v})^2$$

- *B* (Bremsstrahlung function) contains the coupling dependence
- For simple theories this dependence is **exact** (e.g. electrodynamics: $B = e^2/12\pi^2$).



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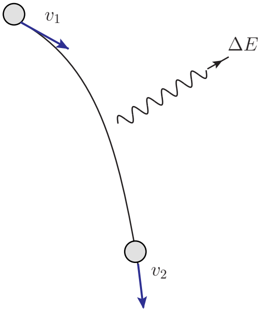
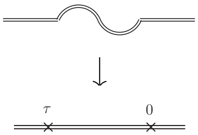


Can we face this physical set up in *Superconformal theories*?

- The Wilson loop represents the worldline of the charged particle
- Bremsstrahlung ⇔ Wilson loop observables
- Techniques from Defect Conformal Field Theories

Deformations of the Wilson loop are associated to operators (Displacement) inserted on the defect ⇒ non trivial 2-pt function

$$\left\langle \mathbb{D}_i(\tau)\mathbb{D}_j(0)\right\rangle_W = 12B\,\frac{\delta_{ij}}{\tau^4}$$



6[Correa, Henn, Maldacena, Sever, 2012], [Lewkowycz, Maldacena, 2013]

Stress tensor and displacement operator in DCFTs

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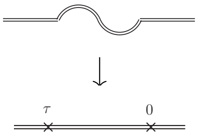
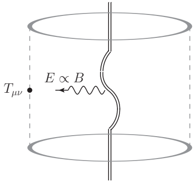


Measuring the emitted energy:[6]

The energy emitted by the Wilson loop is parametrized by B and measured by a stress energy tensor insertion.



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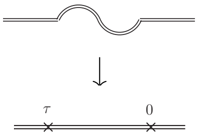

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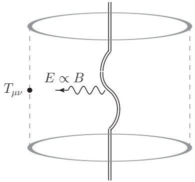
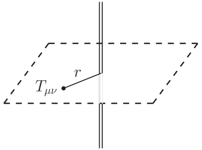


The related observable is the stress tensor one point function:

$$\langle T_{00}\rangle_W = \frac{h_W}{r^4}\,, \qquad \left\langle T_{ij}\right\rangle_W = -\frac{h_W}{r^4}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)$$



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Computing the Emitted energy in $\mathcal{N} = 2$ SCFTs[9]

- **Relation between B and h_W**

  Using superconformal Ward identities it has been proved that [7]

$$B = 3h_W$$

[7] [Bianchi, Lemos, Meineri, 2018]

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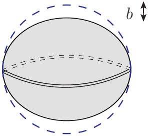
- **Compute h_W?**

  • We define $\langle W \rangle$ on a 4-d sphere in a superconformal-preserving way.

  • Insertion of a stress tensor ↔ small variation of the action w.r.t. the geometry

$$\frac{\sqrt{g}}{2} T_{\mu\nu} = -\frac{\partial S}{\partial g^{\mu\nu}}$$



  • We find an **exact result**[8]

$$h_W = \frac{1}{12\pi^2} \partial_b \log \langle W \rangle_b \Big|_{b=1}$$

  where $\langle W \rangle_b = f(g, N)$ can be computed using localization.

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Resume

- We discussed gauge theories in presence of additional space-time symmetries (**Supersymmetry** and **conformal symmetry**), in presence of an extended operator (**Wilson loop**).
- Main techniques: supersymmetric localization - Defect conformal field theory
- Computation of observables in $\mathcal{N} = 2$ SCFTs:
  - **Wilson loop vev** $\langle W(C) \rangle$ using both the matrix model and Feynman diagrams up to four loops.
  - **Emitted radiation** related to a **deformation of the geometry** on a 4-dim squashed sphere.

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THANK YOU!