J. Phys. G42, 085004 (2015)
Eur. Phys. J. C77, 844 (2017)
ISBN: 9780128034392 (2017)
J. Phys. G46, 115006 (2019)

Hadronic contribution to the muon anomalous magnetic moment within DPT

A.V. Nesterenko

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna, Russian Federation
nesterav ⟨at⟩ theor.jinr.ru , nesterav ⟨at⟩ gmail.com

“STRONG 2020” Workshop on spacelike and timelike determination of the hadronic leading order contribution to the muon $g-2$ (24–26 November 2021)

INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- **electron–positron annihilation into hadrons**
- **inclusive τ lepton hadronic decay**
- **muon anomalous magnetic moment**
- **running of the electromagnetic coupling**

The relevant energy scales span from IR to UV domain.

QCD PERTURBATIVE PREDICTIONS

The scalar part of the hadronic vacuum polarization tensor

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\langle 0|\, T\{J_\mu(x)\, J_\nu(0)\}\,|0\rangle = i(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{\Pi(q^2)}{12\pi^2}$$

satisfies the inhomogeneous RG equation

$$\left[\frac{\partial}{\partial \ln \mu^2} + \frac{\partial a_{\mathrm{s}}(\mu^2)}{\partial \ln \mu^2}\frac{\partial}{\partial a_{\mathrm{s}}}\right]\Pi\left(q^2, \mu^2, a_{\mathrm{s}}\right) = \gamma\left(a_{\mathrm{s}}\right), \quad \gamma^{(\ell)}_{\mathrm{pert}}\left(a_{\mathrm{s}}\right) = \sum_{j=0}^{\ell}\gamma_j\left[a^{(\ell)}_{\mathrm{s}}(\mu^2)\right]^j,$$

where $q^2 < 0$**,** $\mu^2 > 0$**,** $a_{\mathrm{s}}(\mu^2) = \alpha_{\mathrm{s}}(\mu^2)\beta_0/(4\pi)$**,** $\beta_0 = 11 - 2n_{\mathrm{f}}/3$**.**

In practice it is convenient to deal with the Adler function

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad \left[\frac{\partial}{\partial \ln \mu^2} + \frac{\partial a_{\mathrm{s}}(\mu^2)}{\partial \ln \mu^2}\frac{\partial}{\partial a_{\mathrm{s}}}\right] D\left(Q^2, \mu^2, a_{\mathrm{s}}\right) = 0,$$

with $Q^2 = -q^2 > 0$ **being the spacelike kinematic variable**

- **Adler, Phys. Rev. D10, 3714 (1974).**

Strong running coupling

$$\frac{\partial a_s(\mu^2)}{\partial \ln \mu^2} = \beta(a_s), \quad \beta^{(\ell)}_{\rm pert}(a_s) = -\sum_{i=0}^{\ell-1} B_i \left[a_s^{(\ell)}(\mu^2)\right]^{i+2}, \quad B_i = \frac{\beta_i}{\beta_0^{i+1}}.$$

The coefficients β_i are known up to the 5–loop level $(i=0...4)$:

- **Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 118, 082002 (2017); Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP02, 090 (2017); Luthe, Maier, Marquard, Schroder, JHEP10 166 (2017).**

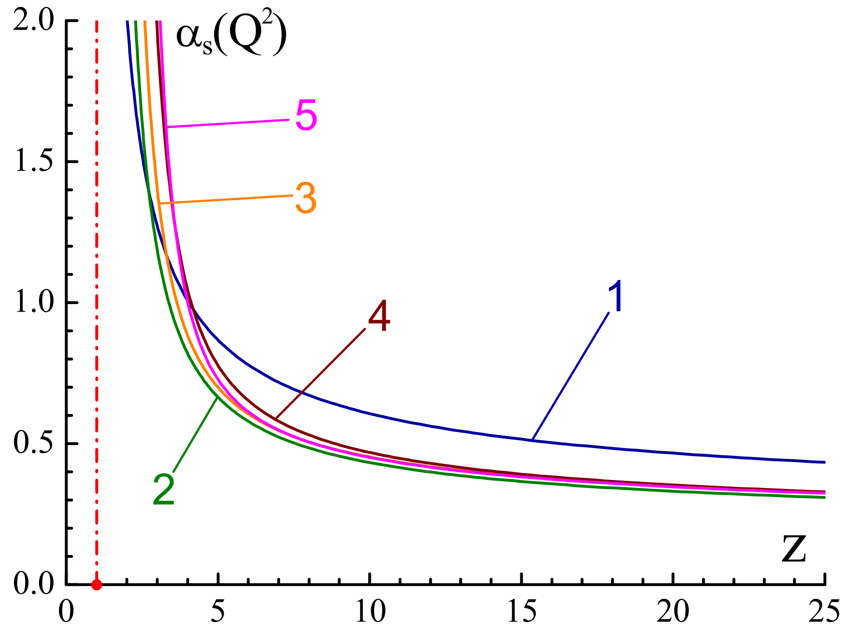
Perturbative QCD running coupling at the ℓ–loop level:

$$a_s^{(\ell)}(Q^2) = \sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m \frac{\ln^m(\ln z)}{\ln^n z}, \quad z = \frac{Q^2}{\Lambda^2},$$

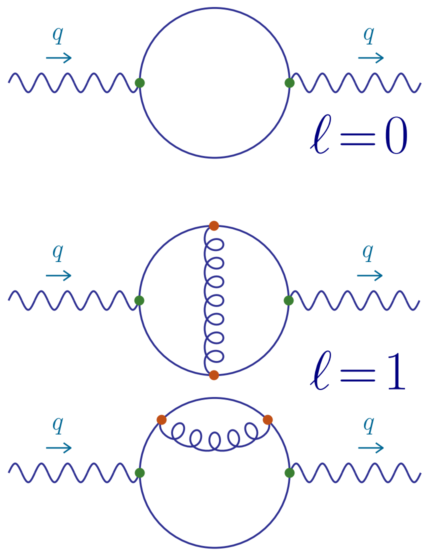
where $b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -B_1$, etc.

The one–loop expression reads

$$\alpha_s^{(1)}(Q^2) = \frac{4\pi}{\beta_0} a_s^{(1)}(Q^2), \quad a_s^{(1)}(Q^2) = \frac{1}{\ln z}.$$



Perturbative expressions for $\Pi(q^2)$ and $D(Q^2)$



$$\Pi^{(\ell)}_{\text{pert}}(q^2,\mu^2,a_s) = \sum_{j=0}^{\ell}\left[a_s^{(\ell)}(\mu^2)\right]^j \sum_{k=0}^{j+1} \Pi_{j,k} \ln^k\left(\frac{\mu^2}{-q^2}\right),$$

$$D^{(\ell)}_{\text{pert}}(Q^2,\mu^2,a_s) = \sum_{j=0}^{\ell}\left[a_s^{(\ell)}(\mu^2)\right]^j \sum_{k=0}^{j+1} k\Pi_{j,k} \ln^{k-1}\left(\frac{\mu^2}{Q^2}\right).$$

Explicit form of the RG relations for the higher–order coefficients $\Pi_{j,k}$ at an arbitrary loop level:

$$\Pi_{0,1} = \gamma_0, \quad \Pi_{1,1} = \gamma_1, \quad \Pi_{j,j+1} = 0 \;\; (j \geq 1), \quad \Pi_{j,1} = \gamma_j + \sum_{k=1}^{j-1} k\,\Pi_{k,0}\,B_{j-k-1} \;\; (j \geq 2),$$

$$\Pi_{j,2} = \frac{1}{2}\sum_{i=1}^{j-1} iB_{j-i-1}\Pi_{i,1} \quad (j \geq 2), \qquad B_i = \frac{\beta_i}{\beta_0^{i+1}}, \qquad \mathfrak{B}_n = \frac{1}{4}\sum_{i=0}^{n} B_i\,B_{n-i},$$

$$\Pi_{j,k} = \frac{1}{T_{k-1}}\sum_{i=k-2}^{j-2} i\,(i+j)\mathfrak{B}_{j-i-2}\Pi_{i,k-2} \quad (j \geq k,\;\; k \geq 3), \qquad T_n = \frac{1}{2}n(n+1)$$

■ **Nesterenko, J. Phys. G46, 115006 (2019); 47, 105001 (2020)** [**higher-order π^2-terms in $R(s)$**].

For a general choice of μ^2 all the coefficients $\Pi_{j,k}$ contribute to the resulting expression for the Adler function, whereas the native choice $\mu^2 = Q^2$ casts $D^{(\ell)}_{\text{pert}}(Q^2, \mu^2, a_\text{s})$ to ($\Pi_{0,1} = 1$)

$$D^{(\ell)}_{\text{pert}}(Q^2) = \sum_{j=0}^{\ell} \Pi_{j,1} \left[a_\text{s}^{(\ell)}(Q^2)\right]^j = 1 + \sum_{j=1}^{\ell} d_j \left[a_\text{s}^{(\ell)}(Q^2)\right]^j, \quad d_j = \Pi_{j,1}$$

(prefactor $N_\text{c} \sum_{f=1}^{n_\text{f}} Q_f^2$ is omitted unless otherwise specified).

The coefficients d_j are known up to the 4–loop level ($j = 1 \ldots 4$):

- **Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 101, 012002 (2008);
Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 104, 132004 (2010);
Baikov, Chetyrkin, Kuhn, Rittinger, Phys. Lett. B714, 62 (2012).**

The numerical estimations of d_5 are also available, see, e.g.,

- **Kataev, Starshenko, Mod. Phys. Lett. A10, 235 (1995).**

The one–loop expression for the Adler function reads

$$D^{(1)}_{\text{pert}}(Q^2) = 1 + d_1 a_\text{s}^{(1)}(Q^2), \quad d_1 = \frac{4}{\beta_0}, \quad a_\text{s}^{(1)}(Q^2) = \frac{1}{\ln(Q^2/\Lambda^2)}.$$

Expression for $\Delta\Pi^{(\ell)}_{\text{pert}}(-Q^2,-Q_0^2)$ at the one–loop level $(\ell=1)$:

$$\Delta\Pi^{(1)}_{\text{pert}}(-Q^2,-Q_0^2) = -\ln\left(\frac{Q^2}{Q_0^2}\right) - d_1 \ln\left[\frac{a_{\text{s}}^{(1)}(Q_0^2)}{a_{\text{s}}^{(1)}(Q^2)}\right], \quad a_{\text{s}}^{(1)}(Q^2)=\frac{1}{\ln z}, \quad z=\frac{Q^2}{\Lambda^2}$$

- **Moorhouse, Pennington, Ross, Nucl. Phys. B124, 285 (1977); Pennington, Ross, Phys. Lett. B102, 167 (1981); Pennington, Roberts, Ross, Nucl. Phys. B242, 69 (1984); Pivovarov, Nuovo Cim. A105, 813 (1992).**

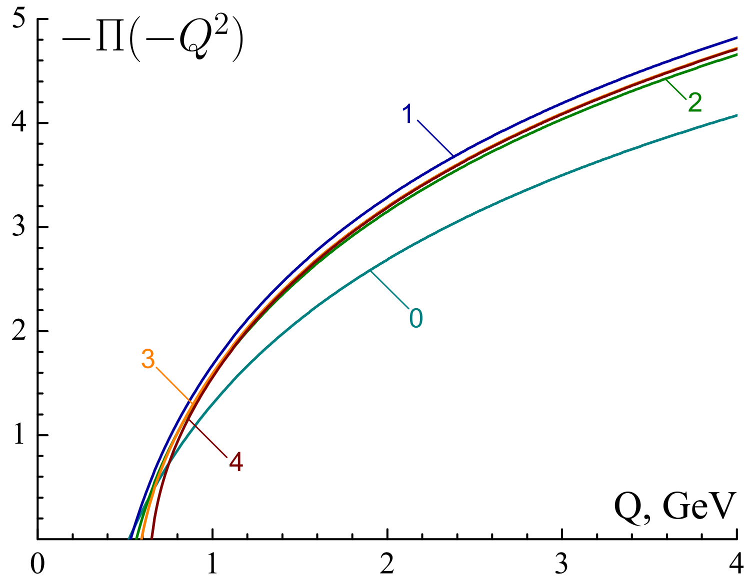
Expression for $\Delta\Pi^{(\ell)}_{\text{pert}}(-Q^2,-Q_0^2)$ at an arbitrary loop level:

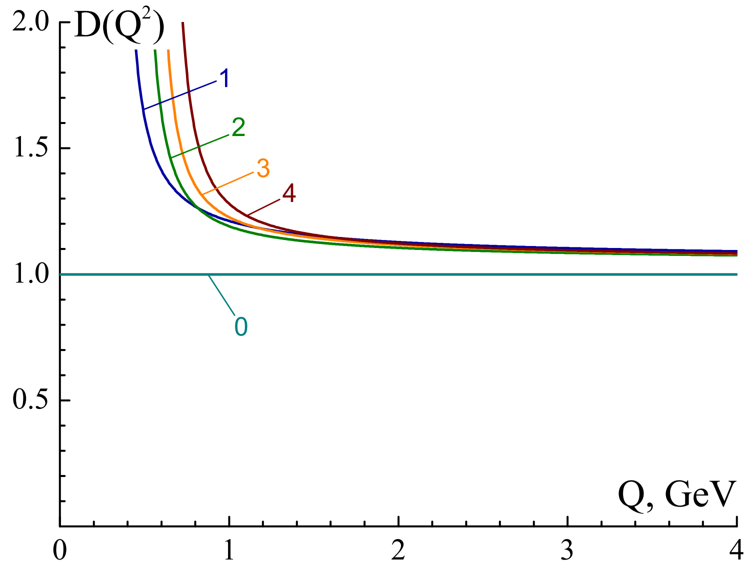
$$\Delta\Pi^{(\ell)}(-Q^2,-Q_0^2) = -\ln\left(\frac{Q^2}{Q_0^2}\right) + \sum_{j=1}^{\ell} d_j \left[p_j^{(\ell)}(Q^2) - p_j^{(\ell)}(Q_0^2)\right],$$

$$p_j^{(\ell)}(Q^2) = \sum_{n_1=1}^{\ell} \dots \sum_{n_j=1}^{\ell} \sum_{m_1=0}^{n_1-1} \dots \sum_{m_j=0}^{n_j-1} \left(\prod_{i=1}^{j} b_{n_i}^{m_i}\right) J\left(Q^2, \sum_{i=1}^{j} n_i, \sum_{i=1}^{j} m_i\right),$$

$$J(Q^2,n,m) = \begin{cases} -\dfrac{\ln^{m+1}(\ln z)}{m+1}, & \text{if } n=1, \\ \displaystyle\sum_{k=0}^{m} \frac{m!}{k!}(n-1)^{k-m-1} \frac{\ln^k(\ln z)}{\ln^{n-1} z}, & \text{if } n \geq 2 \end{cases}$$

- **Nesterenko, J. Phys. G46, 115006 (2019).**





The incorporation of the strong corrections makes the theoretical description of the functions on hand more precise at high energies, but at the same time it entails the appearance of the infrared unphysical singularities.

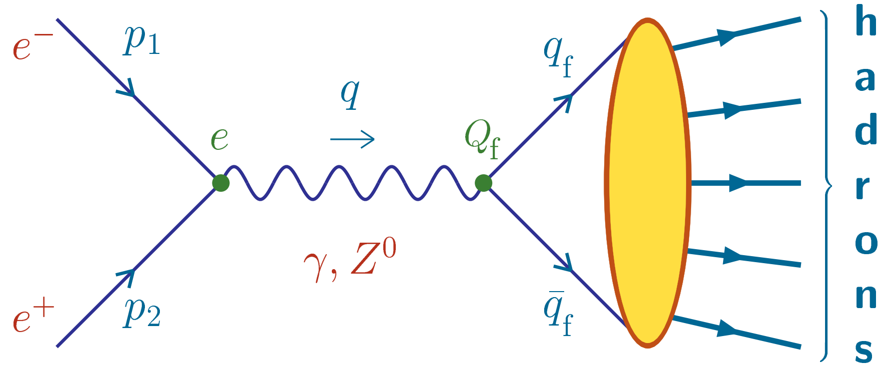
Nonetheless, it appears that the dispersion relations provide certain nonperturbative constraints on the low energy behavior of the functions on hand, that can be used to obviate some inherent difficulties of the QCD perturbation theory.

GENERAL DISPERSION RELATIONS

Cross–section of $e^+e^- \to$ hadrons:

$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$ **[TL],**



$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma)\langle 0|J_\mu(-q)|\Gamma\rangle\langle\Gamma|J_\nu(q)|0\rangle,$$

and $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ **is the electromagnetic quark current.**

Kinematic restriction**: the hadronic tensor** $\Delta_{\mu\nu}(q^2)$ **assumes non–zero values only for** $q^2 \geq 4m_\pi^2 = m^2$**, since otherwise no hadron state** Γ **could be excited**

■ **Feynman (1972); Adler (1974).**

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

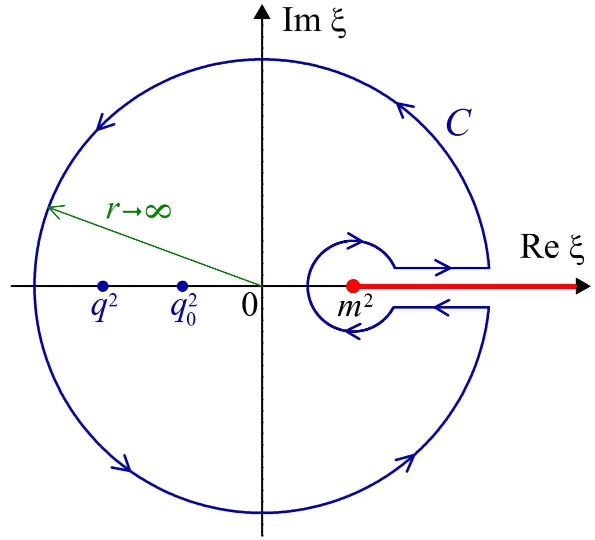
$$\Pi_{\mu\nu}(q^2) = i \int e^{iqx} \langle 0 |\, T\{J_\mu(x)\, J_\nu(0)\} \,| 0 \rangle \, d^4x = i\big(q_\mu q_\nu - g_{\mu\nu} q^2\big) \frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ **has the only cut** $s = q^2 \geq m^2$**.**

Dispersion relation for $\Pi(q^2)$**:**

$$\Delta\Pi(q^2,\, q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2) \oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\, d\xi$$

$$= (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)}\, ds,$$



where $\Delta\Pi(q^2,\, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ **and** $R(s)$ **denotes the measurable ratio of two cross–sections**

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{\sigma(e^+e^- \to \text{hadrons};\, s)}{\sigma(e^+e^- \to \mu^+\mu^-;\, s)}.$$

Kinematic restriction: $R(s) = 0$ **for** $s = q^2 < m^2$**. [TL]**

It is convenient to employ the so–called Adler function

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds, \qquad Q^2 = -q^2 > 0.$$

[SL]

■ **Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).**

It plays a valuable role for congruous analysis of SL/TL data: the dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

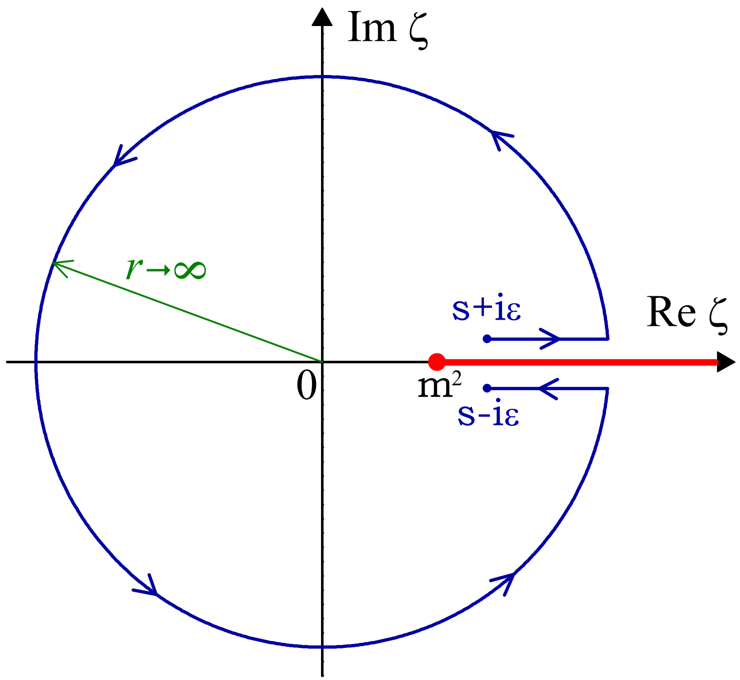
The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$

■ **Radyushkin (1982); Krasnikov, Pivovarov (1982).**

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\sigma)\, \frac{d\sigma}{\sigma}$$

■ **Pennington, Ross (1981); Pivovarov (1992).**



The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma = - \int_{-q_0^2}^{-q^2} D(\sigma)\, \frac{d\sigma}{\sigma},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\,\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma.$$

Derivation of these relations requires only the location of cut of $\Pi(q^2)$ and its UV asymptotic. Neither additional approximations nor phenomenological assumptions are involved.

Kinematic restrictions ⟶ Dispersion relations ⟶ Nonperturbative constraints:

- $\Pi(q^2)$: **has the only cut** $q^2 \geq m^2$
- $R(s)$: **vanishes for** $s < m^2$, **embodies "SL ⟶ TL" effects**
- $D(Q^2)$: **vanishes at** $Q^2 \to 0$, **has the only cut** $Q^2 \leq -m^2$

DISPERSIVE APPROACH TO QCD

Underlying concept: **merge these nonperturbative constrains with pertinent perturbative input in a self–consistent way.**

Functions on hand in terms of the common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\frac{m^2 - q_0^2}{m^2 - q^2}\right)\frac{d\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2)\int_{s}^{\infty} \rho(\sigma)\,\frac{d\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2}\int_{m^2}^{\infty} \rho(\sigma)\,\frac{\sigma - m^2}{\sigma + Q^2}\,\frac{d\sigma}{\sigma},$$

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p(\sigma - i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d(-\sigma - i\varepsilon),$$

where $\Delta\Pi^{(0)}(q^2,\, q_0^2)$, $R^{(0)}(s)$, $D^{(0)}(Q^2)$ **denote the leading–order terms and** $p(q^2)$, $r(s)$, $d(Q^2)$ **stand for the strong corrections**

- **Nesterenko, Phys. Rev. D88, 056009 (2013); J. Phys. G42, 085004 (2015); ISBN: 9780128034392 (2017).**

Derivation of obtained representations involves neither additional approximations nor model–dependent assumptions, with all the nonperturbative constraints being embodied.

The leading–order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2,\, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\!\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\!\varphi_0}, \qquad \sin^2\!\varphi = \frac{q^2}{m^2},$$

$$R^{(0)}(s) = \theta(s - m^2)\left(1 - \frac{m^2}{s}\right)^{3/2}, \qquad \sin^2\!\varphi_0 = \frac{q_0^2}{m^2},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\!\left(\xi^{1/2}\right)\right], \qquad \xi = \frac{Q^2}{m^2}.$$

■ **Feynman (1972); Akhiezer, Berestetsky (1965).**

The spectral density $\rho(\sigma)$ brings in the perturbative input:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\,\frac{d\,\mathrm{Im}\,p_{\text{pert}}(\sigma - i0_+)}{d\,\ln\sigma} = -\frac{d\,r_{\text{pert}}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\,d_{\text{pert}}(-\sigma - i0_+).$$

One–loop: $\rho^{(1)}_{\text{pert}}(\sigma) = 4/\left[\beta_0\left(\ln^2(\sigma/\Lambda^2) + \pi^2\right)\right]$; early attempts for the higher–loops: **Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetic (2015–2018).**

Note on the massless limit

In the limit $m = 0$ the obtained integral representations read

$$\Delta\Pi(q^2,\, q_0^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) + \int_0^\infty \rho(\sigma) \ln\left[\frac{1-(\sigma/q^2)}{1-(\sigma/q_0^2)}\right]\frac{d\sigma}{\sigma},$$

$$R(s) = 1 + \int_s^\infty \rho(\sigma)\,\frac{d\sigma}{\sigma}, \qquad D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\,d\sigma.$$

For $\rho(\sigma) = \rho_{\mathrm{pert}}(\sigma)$ two highlighted massless equations become identical to those of the APT ■ Shirkov, Solovtsov, Milton (1997–2007).

[$\Pi(q^2)$ was not studied in the framework of the APT]

However, it is essential to keep the threshold m nonvanishing:

- **massless limit loses some of nonperturbative constraints**
- **effects due to $m \neq 0$ become substantial at low energies**

The perturbative spectral function at the ℓ–loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j \, \bar{\rho}^{(\ell)}_j(\sigma), \qquad \bar{\rho}^{(\ell)}_j(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left\{ \left[a^{(\ell)}_{\text{s}}(-\sigma - i\varepsilon) \right]^j - \left[a^{(\ell)}_{\text{s}}(-\sigma + i\varepsilon) \right]^j \right\}.$$

Explicit expression for $\rho^{(\ell)}_{\text{pert}}(\sigma)$ valid at an arbitrary loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j \sum_{k=0}^{K(j)} \binom{j}{2k+1} (-1)^k \, \pi^{2k} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n \, u^m_n(\sigma) \right]^{j-2k-1} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n \, v^m_n(\sigma) \right]^{2k+1}$$

- **Nesterenko, Eur. Phys. J. C77, 844 (2017).**

This makes the higher–loop calculations within approach on hand easily accessible. Here ℓ denotes the loop level,

$$u^m_n(\sigma) = \begin{cases} u^0_n(\sigma), & \text{if } m = 0, \\ u^0_n(\sigma) u^m_0(\sigma) - \pi^2 v^0_n(\sigma) v^m_0(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v^m_n(\sigma) = \begin{cases} v^0_n(\sigma), & \text{if } m = 0, \\ v^0_n(\sigma) u^m_0(\sigma) + u^0_n(\sigma) v^m_0(\sigma), & \text{if } m \geq 1, \end{cases}$$

where

$$v_0^m(\sigma) = \sum_{k=0}^{K(m)} \binom{m}{2k+1} (-1)^{k+1} \pi^{2k} \Big[L_1(y)\Big]^{m-2k-1} \Big[L_2(y)\Big]^{2k+1},$$

$$u_0^m(\sigma) = \sum_{k=0}^{K(m+1)} \binom{m}{2k} (-1)^{k} \pi^{2k} \Big[L_1(y)\Big]^{m-2k} \Big[L_2(y)\Big]^{2k},$$

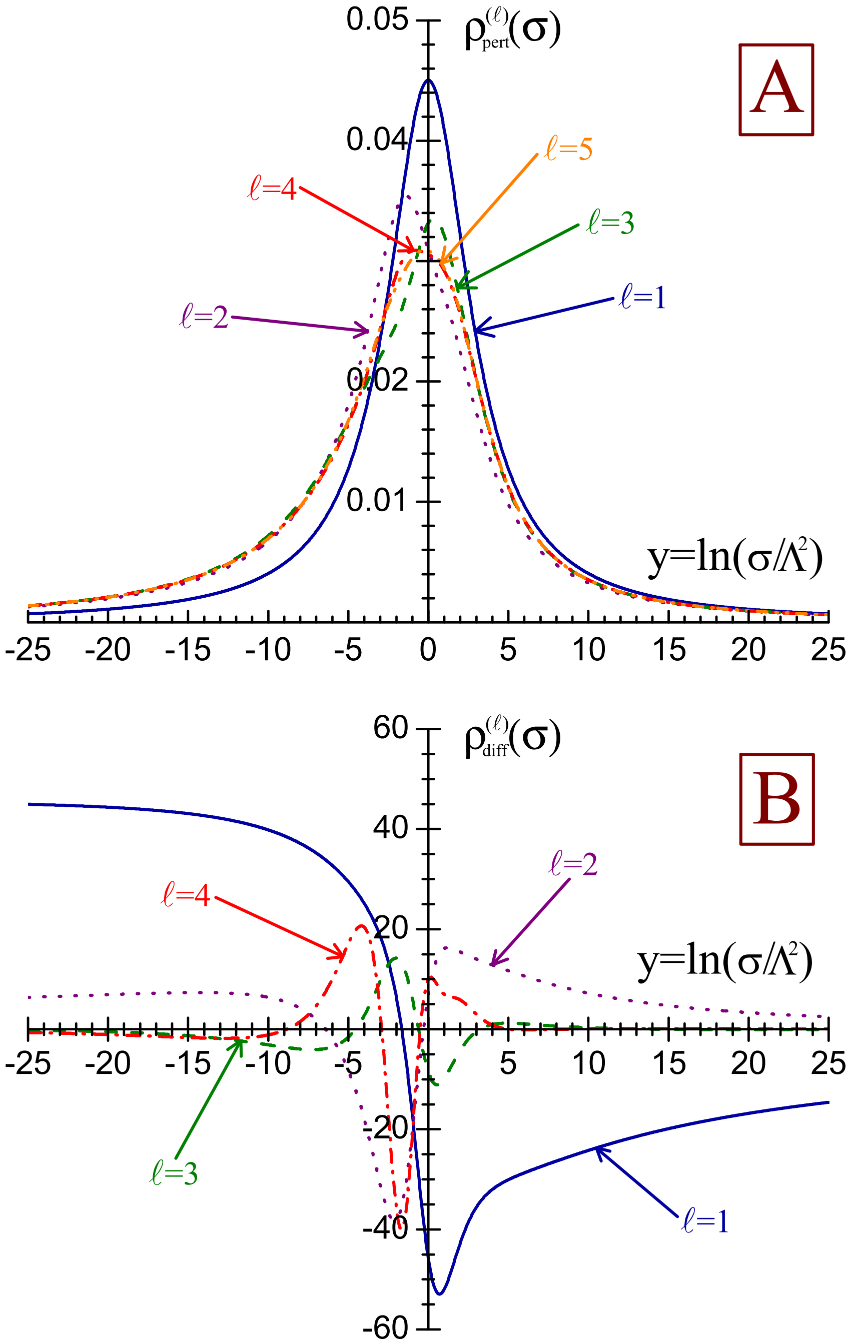
$$v_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n)} \binom{n}{2k+1} (-1)^{k} \pi^{2k} y^{n-2k-1}, \qquad L_1(y) = \ln\sqrt{y^2+\pi^2},$$

$$u_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n+1)} \binom{n}{2k} (-1)^{k} \pi^{2k} y^{n-2k}, \qquad L_2(y) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{y}{\pi}\right),$$

$$K(n) = \frac{n-2}{2} + \frac{n \bmod 2}{2}, \qquad \binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \qquad y = \ln\left(\frac{\sigma}{\Lambda^2}\right)$$

and b_n^m stands for a combination of the β function perturbative expansion coefficients ($b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -\beta_1/\beta_0^2$, etc.)

- **Nesterenko, Eur. Phys. J. C77, 844 (2017).**



Function $\rho^{(4)}_{\rm diff}(\sigma)$ is scaled by the factor of 10

The perturbative spectral function $\rho^{(\ell)}_{\rm pert}(\sigma)$ is remarkably stable with respect to the higher–loop corrections. In particular, the range of y, where the difference between $\rho^{(\ell)}_{\rm pert}(\sigma)$ and $\rho^{(\ell+1)}_{\rm pert}(\sigma)$ is sizable, is located in the vicinity of $y = 0$ and becomes smaller at larger ℓ.

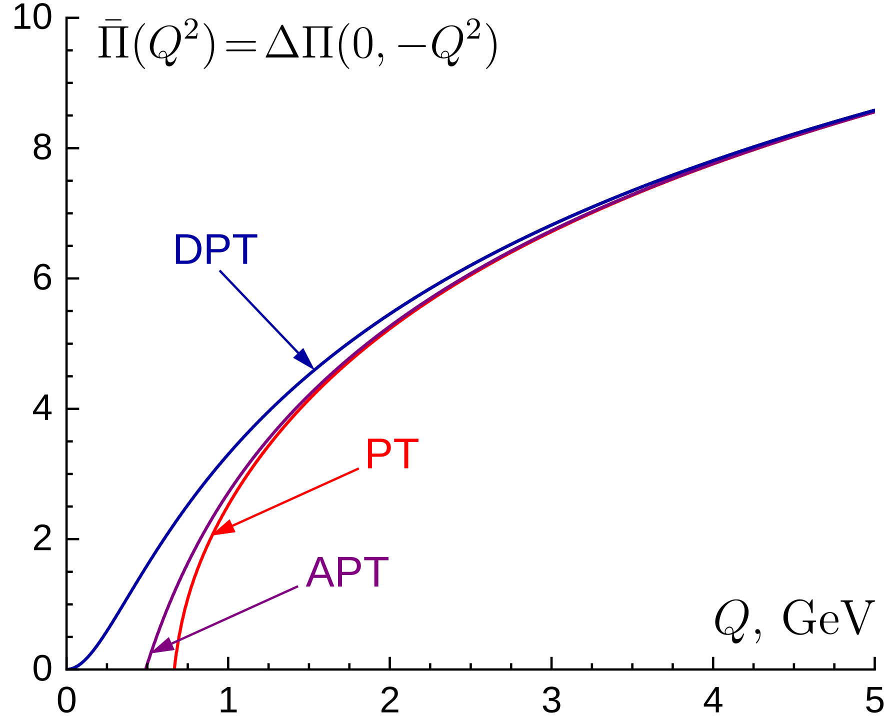
Plot A: $\rho^{(\ell)}_{\rm pert}(\sigma)$ **for** $\ell = 1 \ldots 5$

Plot B: $\rho^{(\ell)}_{\rm diff}(\sigma)$ **for** $\ell = 1 \ldots 4$

$$\rho^{(\ell)}_{\rm diff}(\sigma) = \left[1 - \frac{\rho^{(\ell)}_{\rm pert}(\sigma)}{\rho^{(\ell+1)}_{\rm pert}(\sigma)}\right] \times 100\%$$

■ **Nesterenko, Eur. Phys. J. C77, 844 (2017).**

HADRONIC VACUUM POLARIZATION FUNCTION



- **PT:** $\Pi(q^2)$ **possesses unphysical singularities at low energies**
- **APT:** $\Pi(q^2)$ **diverges in the infrared limit**
- **DPT:** $\Pi(q^2)$ **contains no unphysical singularities**

Both PT and APT fail to describe $\Pi(q^2)$ **at low energies, whereas DPT is applicable in the entire energy range.**

MUON ANOMALOUS MAGNETIC MOMENT

The theoretical description of $a_\mu = (g_\mu - 2)/2$ is a long–standing challenging issue of the elementary particle physics

Experiment: $a_\mu^{\text{exp}} = 11659206.1 \pm 4.1 \ \ (0.35\,\text{ppm})$

- **BNL E821 (2002–2006); FNAL E989 Run–1 (2021).**

Theory: $a_\mu^{\text{theor}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HVP,LO}} + a_\mu^{\text{HVP,NLO}} + a_\mu^{\text{HVP,NNLO}} + a_\mu^{\text{HLbL,LO}} + a_\mu^{\text{HLbL,NLO}}$

$a_\mu^{\text{QED}} = 11658471.8931 \pm 0.0104, \quad a_\mu^{\text{EW}} = 15.36 \pm 0.10,$

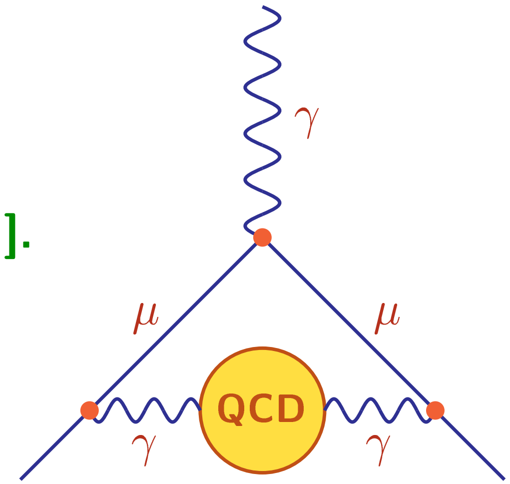
$a_\mu^{\text{HVP,LO}} = 693.1 \pm 4.0, \quad a_\mu^{\text{HVP,NLO}} = -9.83 \pm 0.07, \quad a_\mu^{\text{HVP,NNLO}} = 1.24 \pm 0.01,$

$a_\mu^{\text{HLbL,LO}} = 9.2 \pm 1.9, \quad a_\mu^{\text{HLbL,NLO}} = 0.2 \pm 0.1$

(the values of a_μ are given in units of 10^{-10})

- **Aoyama et al., Phys. Rept. 887, 1 (2020) [and references therein].**

The uncertainty of evaluation of a_μ^{theor} is largely dominated by the $a_\mu^{\text{HVP,LO}}$ term →



The latter involves the integration of $\bar{\Pi}(Q^2)$ over the low energies inaccessible within PT ■ Lautrup, Peterman, de Rafael, Phys. Rep. 3, 193 (1972).

$$a_\mu^{\rm HVP,LO} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 (1-x)\bar{\Pi}\left(\frac{x^2 m_\mu^2}{1-x}\right) dx$$

TL: $a_\mu^{\rm HVP,LO}$ is expressed in terms of $R(s)$ and the data are used

SL: direct integration of $\bar{\Pi}(Q^2) = \Delta\Pi(0, -Q^2)$, no data are used

DPT assessment of $a_\mu^{\rm HVP,LO}$ (**SL**):

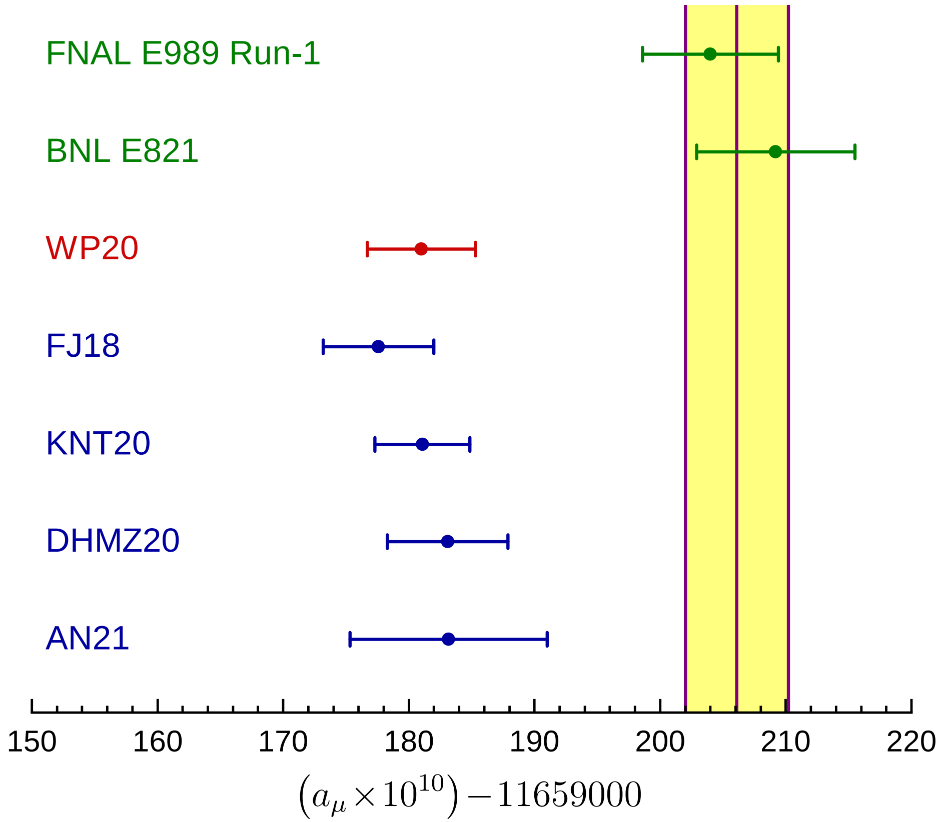
- no data on $R(s)$ are used
- PDG20 $\alpha_s(M_Z^2)$ as input
- 4–loop [5–loop] level

$a_\mu^{\rm HVP,LO} = 695.1 \pm 7.6 \; [694.9 \pm 7.7]$.

The complete SM prediction

$$a_\mu = 11659183.2 \pm 7.8$$

differs from $a_\mu^{\rm exp}$ by 2.6 standard deviations ■ Nesterenko, J. Phys. G42, 085004 (2015); in preparation (2021).



APPLICATION TO THE MUonE PROJECT

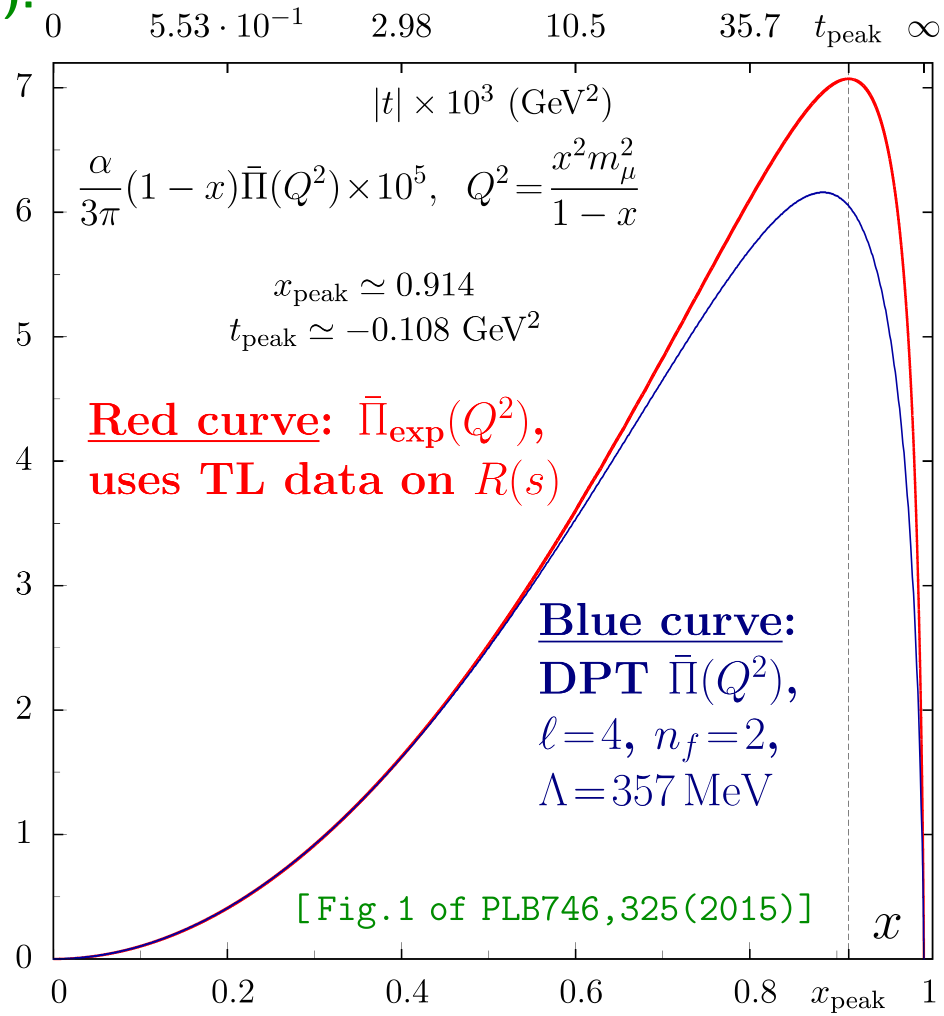
One can evaluate $a_\mu^{\mathrm{HVP,LO}}$ by making use of the spacelike data on $\bar{\Pi}(Q^2)$ extracted from the $\mu\, e \to \mu\, e$ elastic scattering

- Carloni Calame, Passera, Trentadue, Venanzoni, Phys. Lett. B746, 325 (2015); Abbiendi et al., Eur. Phys. J. C77, 139 (2017).

At low energies ($x \lesssim 0.5$) the function $\bar{\Pi}(Q^2)$ obtained within DPT [blue curve, PDG20 $\alpha_s(M_Z^2)$ is used] can also be employed as a supplementing infrared input for

- **MUonE project**
- **lattice studies**

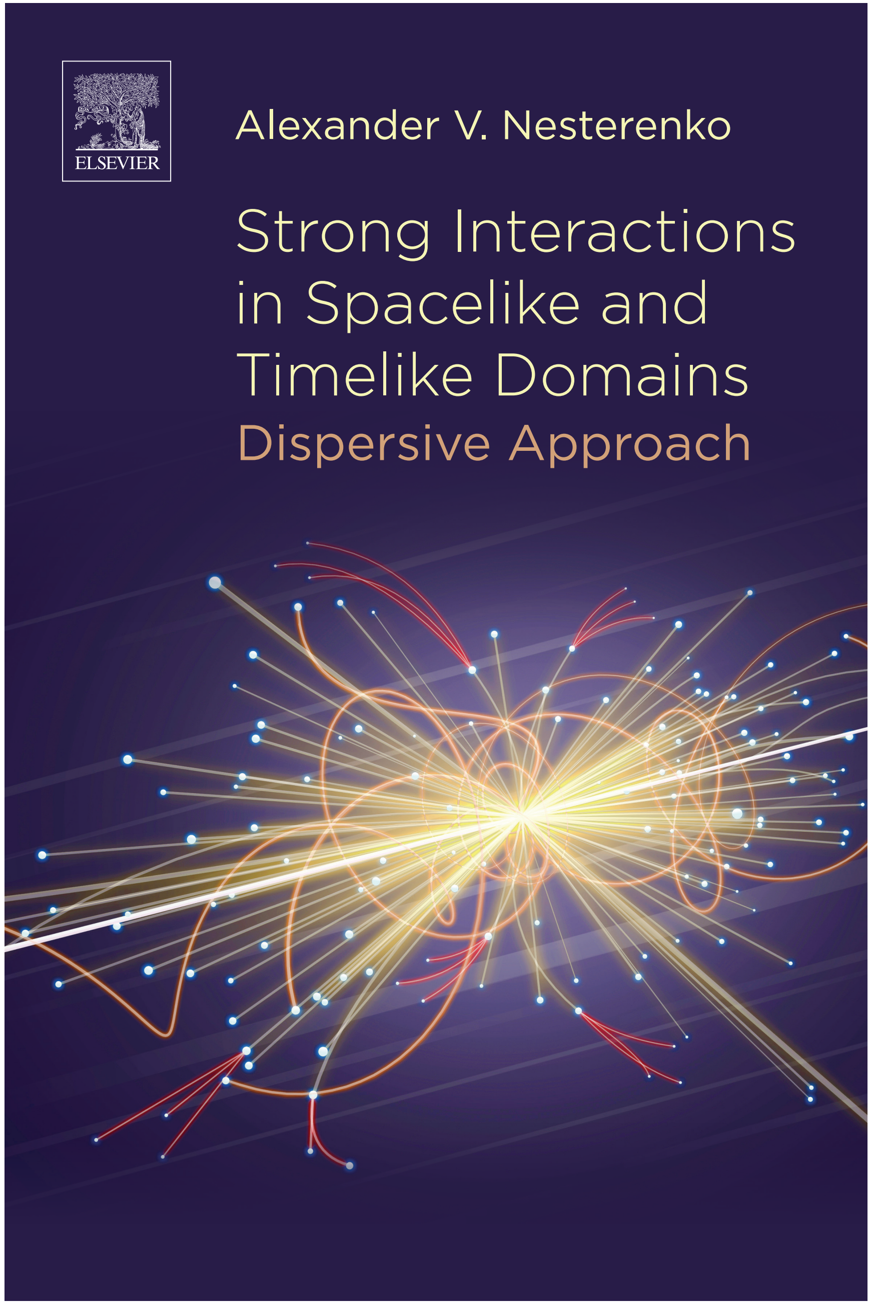
- Nesterenko, in preparation (2021).



SUMMARY

- **The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are obtained in the framework of dispersive approach to QCD.**
- **These representations merge, in a self–consistent way, the corresponding perturbative input with intrinsically non-perturbative constraints, which originate in the respective dispersion relations and play a substantial role in the studies of the strong interaction processes at low energies.**
- **The DPT enables one to properly account for all SL $\longrightarrow$ TL effects, which play a valuable role in the studies of electron–positron annihilation into hadrons.**

- **The explicit expression for the perturbative spectral function valid at an arbitrary loop level is obtained, that substantially facilitates the practical calculations within DPT.**
- **The hadronic contribution to the muon anomalous magnetic moment evaluated within DPT agrees with its recent assessments.**
- **The DPT hadronic vacuum polarization function can also be employed as a supplementing infrared input for the MUonE project and lattice studies.**



The detailed discussion of the presented results and other related topics can also be found in:

A.V. Nesterenko

Strong interactions in spacelike and timelike domains: Dispersive approach

Elsevier, Amsterdam, 2017

https://www.elsevier.com/books/isbn/9780128034392