Analytic Evaluation of the NNLO virtual corrections to Muon-Electron scattering

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INFN & University of Padova

In collaboration with

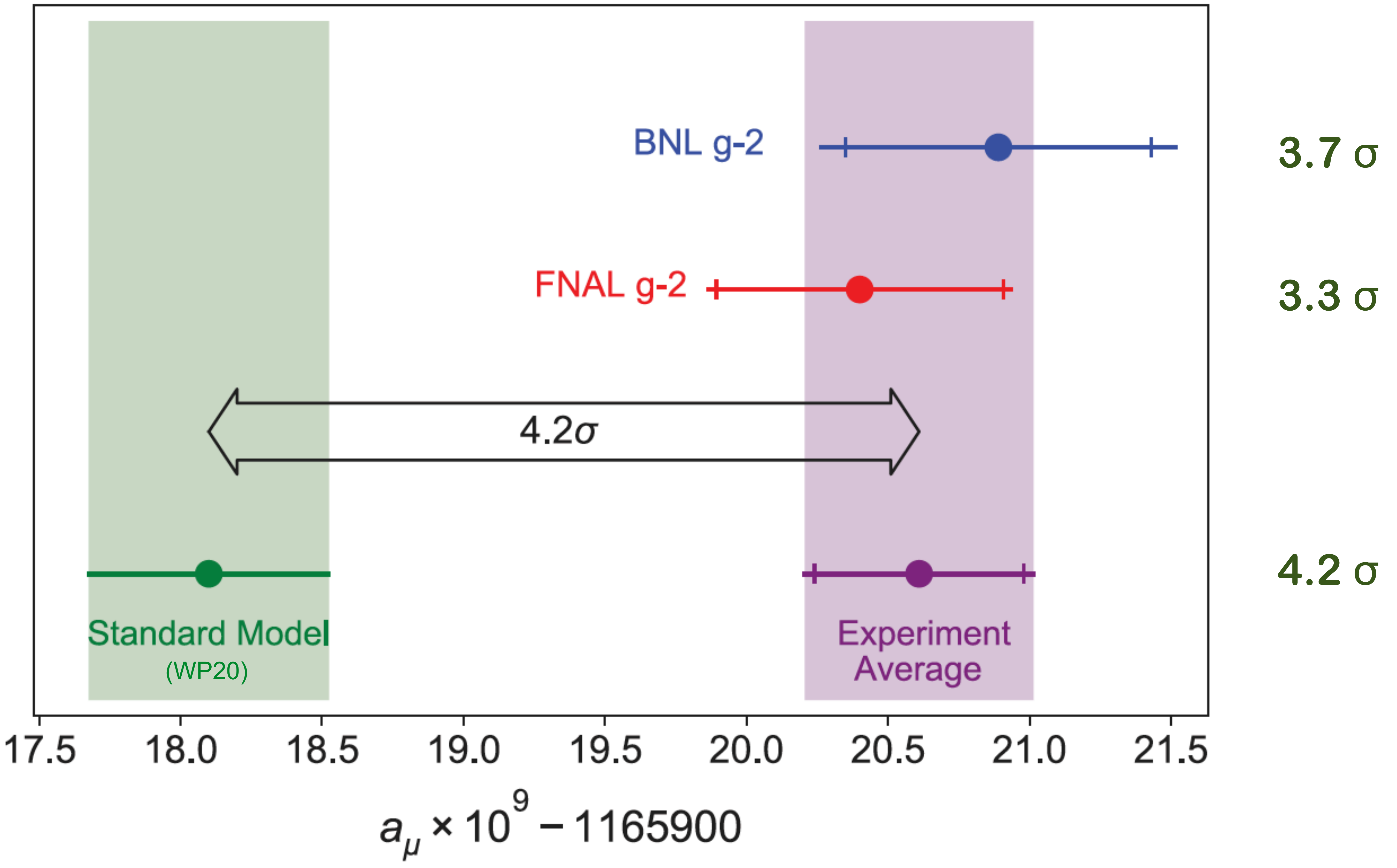
R. ***Bonciani****, A.* ***Broggio****, S.* ***Di Vita****, A.* ***Ferroglia****, S.* ***Laporta****, P.* **Mastrolia***, L.* ***Mattiazzi****, M.* ***Passera****,*
A. ***Primo****, J.* ***Ronca****, U.* ***Schubert****, W. J.* ***Torres Bobadilla****, and F.* ***Tramontano***

Strong 2020, Remote Seminar
26th November, 2021

Fellini
Fellowship for Innovation at INFN
H2020 MSCA COFUND
G.A. 754496

INFN
Istituto Nazionale di Fisica Nucleare

Motivation



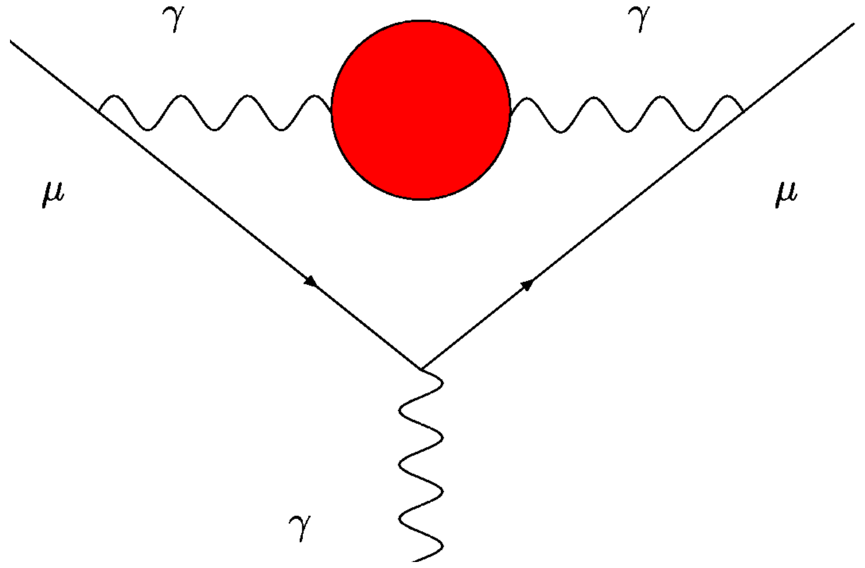
$a_\mu^{EXP} = (116592089 \pm 63) \times 10^{-11}$ [0.54ppm] BNL E821

$a_\mu^{EXP} = (116592040 \pm 54) \times 10^{-11}$ [0.46ppm] FNAL E989 Run 1

$a_\mu^{EXP} = (116592061 \pm 41) \times 10^{-11}$ [0.35ppm] WA

Motivation:

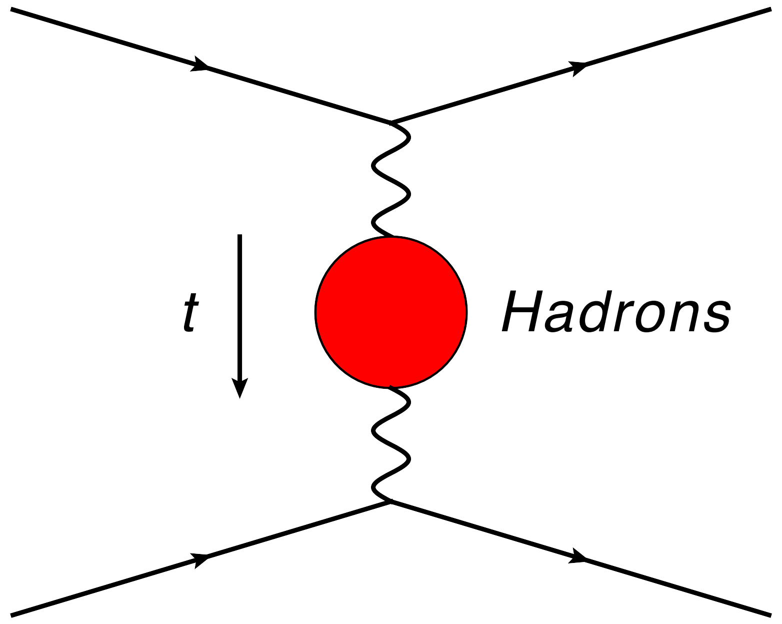
Leading hadronic contribution computed via the usual dispersive (timelike) formula:



$$a_\mu^{\rm HLO} = \frac{1}{4\pi^3}\int_{m_\pi^2}^{\infty} ds\, K(s)\, \sigma^{(0)}_{\rm had}(s)$$

$$K(s) = \int_0^1 dx\, \frac{x^2\,(1-x)}{x^2 + (1-x)\,(s/m_\mu^2)}$$

Alternatively, simply exchanging the x and s integrations:



$$a_\mu^{\rm HLO} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\,\Delta\alpha_{\rm had}[t(x)]$$

$$t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$

Lautrup, Peterman, de Rafael, 1972

Eur. Phys. J. C (2017) 77:139
DOI 10.1140/epjc/s10052-017-4633-z

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Experimental Physics

Measuring the leading hadronic contribution to the muon g-2 via μe scattering

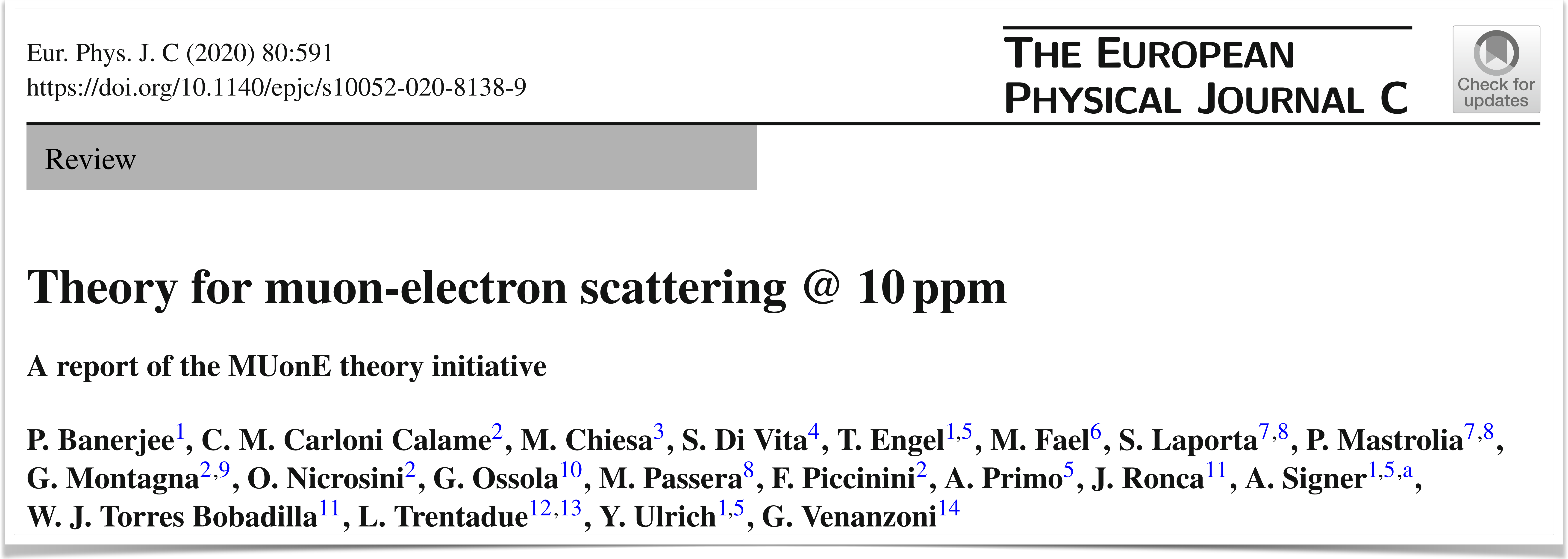
G. Abbiendi[1,a], C. M. Carloni Calame[2,b], U. Marconi[3,c], C. Matteuzzi[4,d], G. Montagna[2,5,e], O. Nicrosini[2, M. Passera[6,g], F. Piccinini[2,h], R. Tenchini[7,i], L. Trentadue[8,4,j], G. Venanzoni[9,k]

See talk by Stefano Laporta, Umberto Marconi, Giovanni Abbiendi

$\Delta\alpha_{\rm had}(t)$ is the hadronic contribution to the space-like running of α: proposal to measure $a_\mu^{\rm HLO}$ via scattering data!

Carloni Calame, MP, Trentadue, Venanzoni, 2015

Muon-Electron Scattering @ NNLO


Eur. Phys. J. C (2020) 80:591
https://doi.org/10.1140/epjc/s10052-020-8138-9
THE EUROPEAN PHYSICAL JOURNAL C

Review

Theory for muon-electron scattering @ 10 ppm

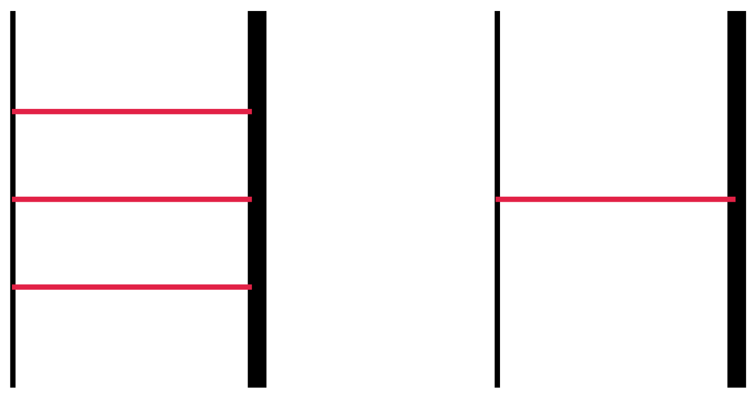
A report of the MUonE theory initiative

P. Banerjee[1], C. M. Carloni Calame[2], M. Chiesa[3], S. Di Vita[4], T. Engel[1,5], M. Fael[6], S. Laporta[7,8], P. Mastrolia[7,8], G. Montagna[2,9], O. Nicrosini[2], G. Ossola[10], M. Passera[8], F. Piccinini[2], A. Primo[5], J. Ronca[11], A. Signer[1,5,a], W. J. Torres Bobadilla[11], L. Trentadue[12,13], Y. Ulrich[1,5], G. Venanzoni[14]

[Banerjee, Engel, Signer, Ulrich (2020)]
[Banerjee, Engel, Schalch, Signer, Ulrich (2021)]
[Budassi, Carloni Calame, Chiesa, Del Pio, Hasan, Montagna, Nicrosini, Piccinini (2021)]
[Carloni Calame, Chiesa, Hasan, Montagna, Nicrosini, Piccinini (2020)]

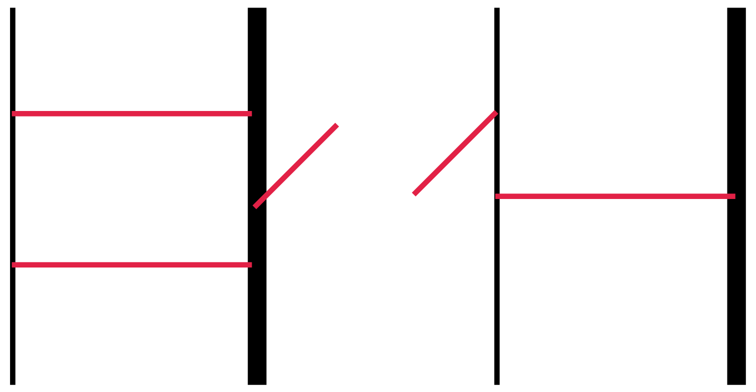
See talk by Ettore Budassi, Tim Engel

$$\int \left[\frac{VV_4}{\epsilon^4} + \frac{VV_3}{\epsilon^3} + \frac{VV_2}{\epsilon^2} + \frac{VV_1}{\epsilon^1} + VV_0 \right] d\phi_2$$



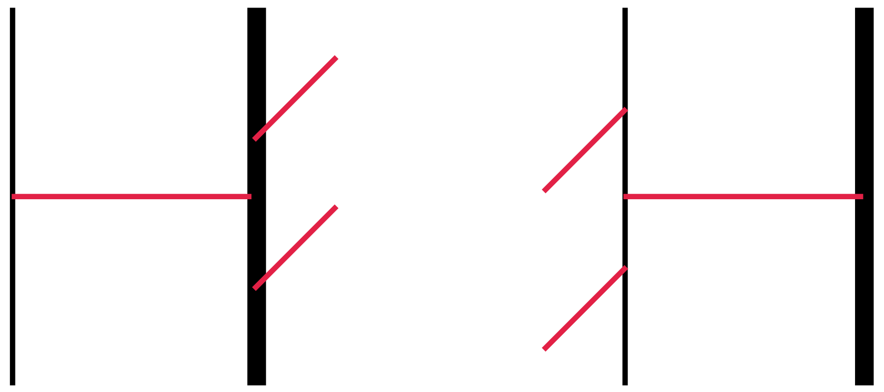
Double Virtual

$$\int \left[\frac{RV_2}{\epsilon^2} + \frac{RV_1}{\epsilon^1} + RV_0 \right] d\phi_3$$



Real Virtual

$$\int [RR_0]\, d\phi_4$$



Double Real

Muon-Electron Scattering @ NNLO

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Check for updates

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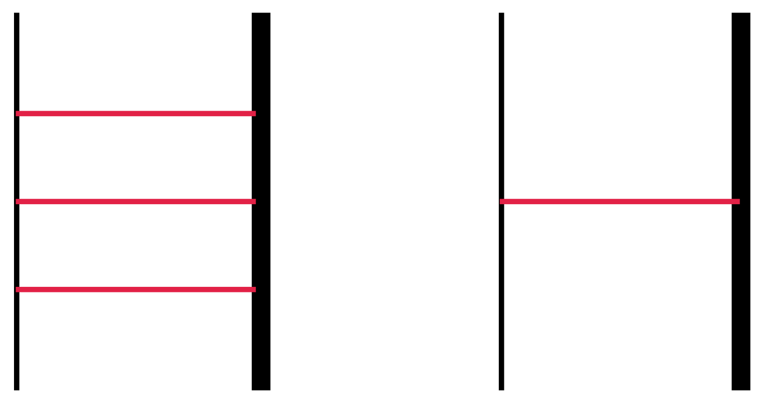
$$\int \left[\frac{VV_4}{\epsilon^4} + \frac{VV_3}{\epsilon^3} + \frac{VV_2}{\epsilon^2} + \frac{VV_1}{\epsilon^1} + VV_0\right] d\phi_2$$



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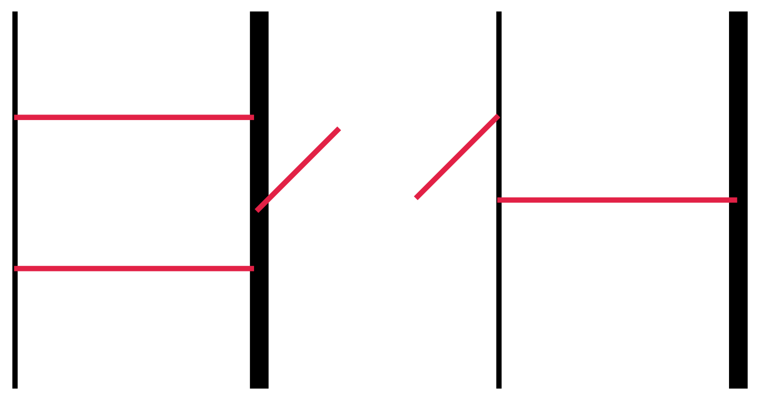

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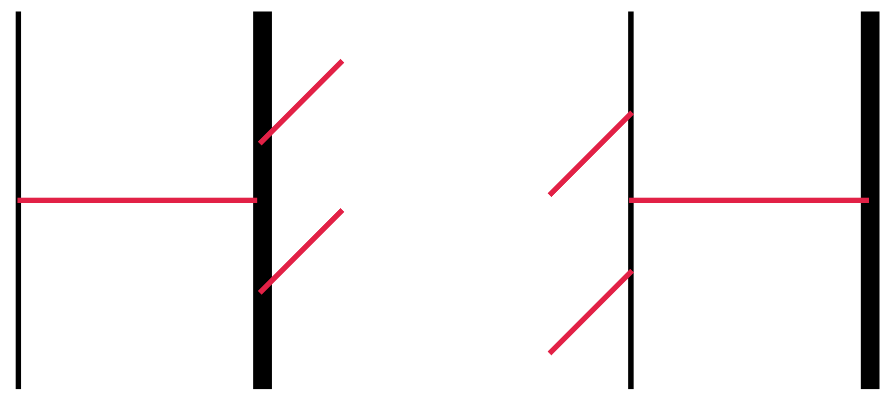

Double Real

This talk is towards the computation of the complete 2-Loop Virtual Amplitude

Muon-Electron Scattering @ NNLO

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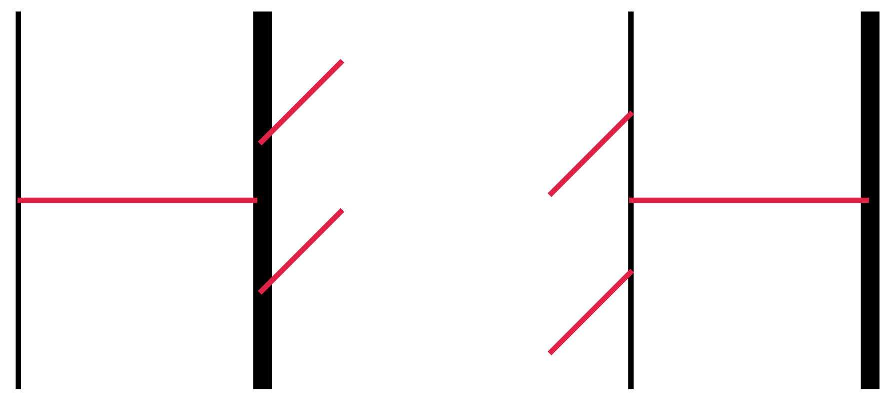
$$\int \left[\frac{RV_2}{\epsilon^2} + \frac{RV_1}{\epsilon^1} + RV_0 \right] d\phi_3$$

$$\int [RR_0]\, d\phi_4$$

The two-loop four-fermion scattering amplitude in QED

R. Bonciani,[1,*] A. Broggio,[2,†] S. Di Vita,[3,4] A. Ferroglia,[5,6,‡] M. K. Mandal,[7,8,§] P. Mastrolia,[8,7,¶] L. Mattiazzi,[7,8,‖] A. Primo,[9,**] J. Ronca,[10,††] U. Schubert,[11,‡‡] W. J. Torres Bobadilla,[12,§§] and F. Tramontano[10,¶¶]

Real Virtual



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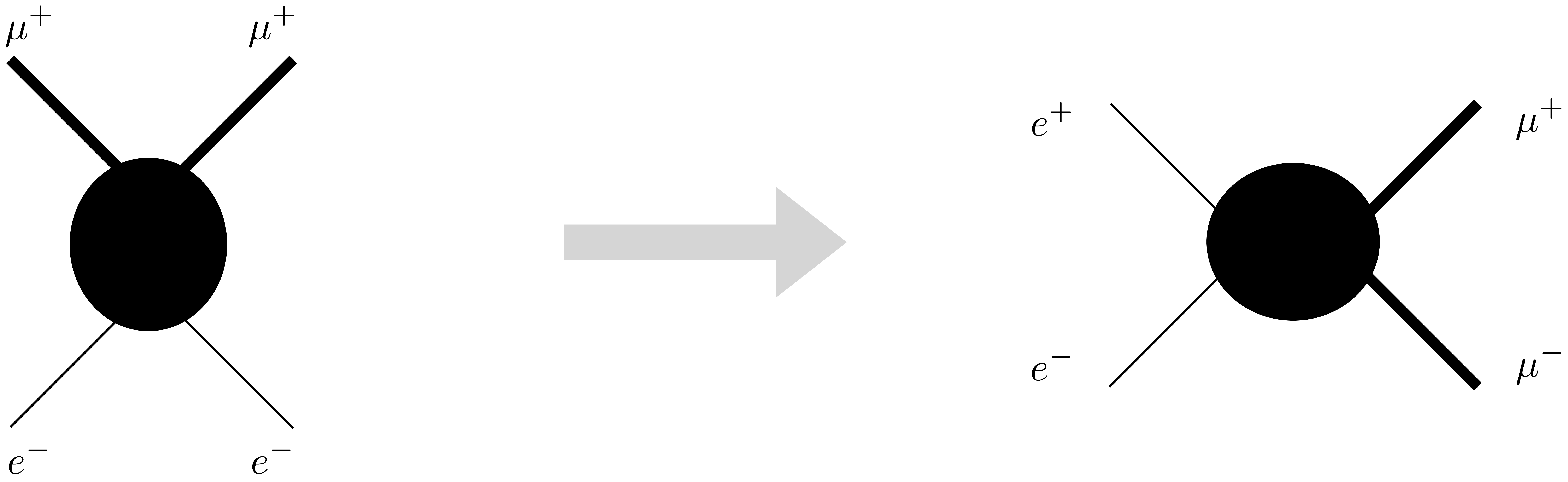
Di-Muon Production

[Bonciani, Broggio, Di Vita, Ferroglia, MKM, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla, Tramontano (2021)]



$e^- + \mu^+ \to e^- + \mu^+$

$e^- + e^+ \to \mu^- + \mu^+$

Crossing

Amplitude for Di-muon Production

[Bonciani, Broggio, Di Vita, Ferroglia, MKM, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla, Tramontano (2021)]

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_3) + \mu^+(p_4)$$

$$m_e = 0$$
$$m_\mu = M$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_2 - p_3)^2$$
$$s + t + u = 2M^2$$

$$\mathcal{A}_\mathrm{b}(\alpha_\mathrm{b}) = 4\pi\alpha_\mathrm{b} S_\epsilon \mu^{-2\epsilon} \left[\mathcal{A}_\mathrm{b}^{(0)} + \left(\frac{\alpha_\mathrm{b}}{\pi}\right)\mathcal{A}_\mathrm{b}^{(1)} + \left(\frac{\alpha_\mathrm{b}}{\pi}\right)^2 \mathcal{A}_\mathrm{b}^{(2)}\right]$$

Bare Amplitude

LO NLO NNLO

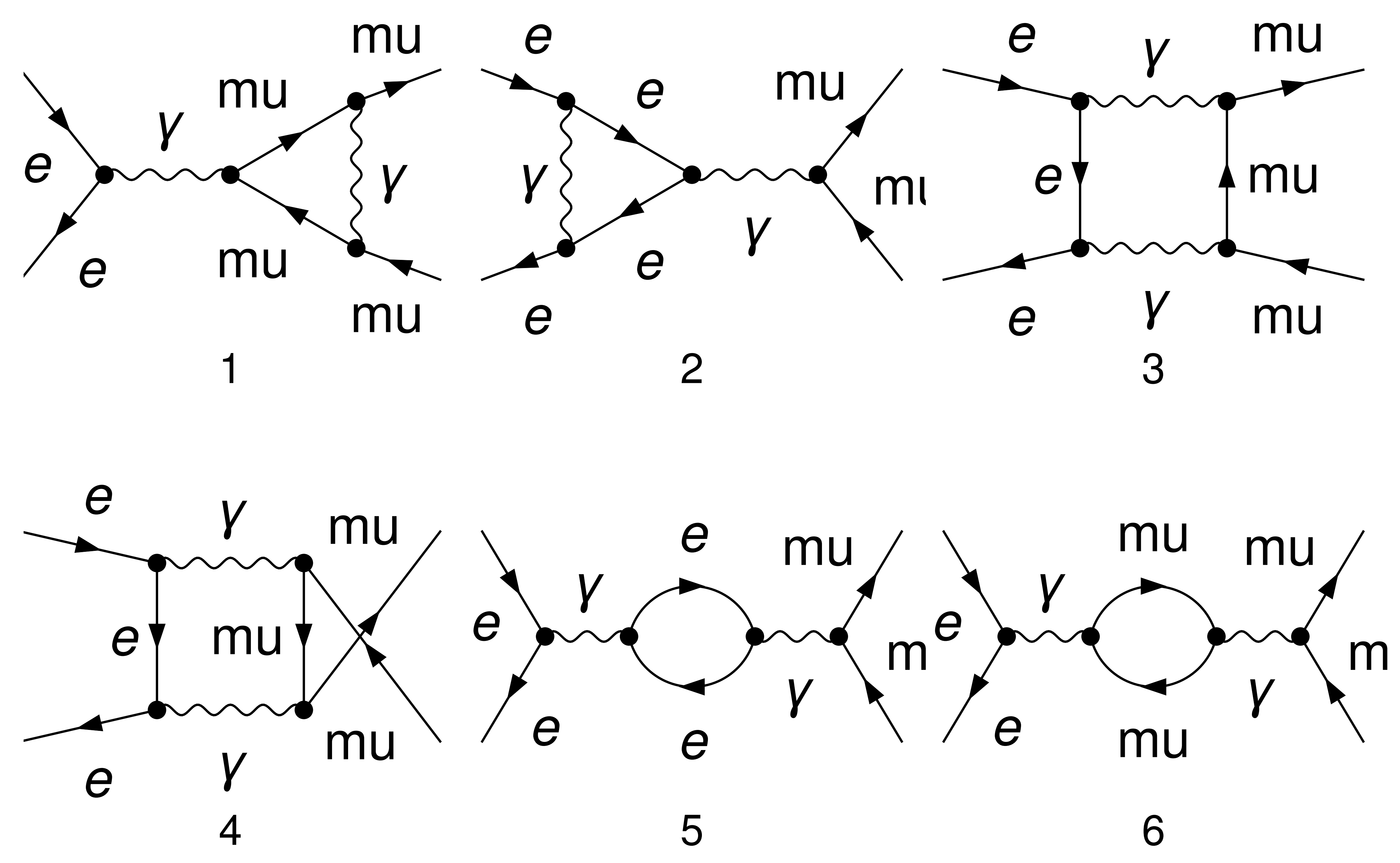
$$\mathcal{M}_\mathrm{b}^{(0)} = \frac{1}{4}\sum_\mathrm{spins} |\mathcal{A}_\mathrm{b}^{(0)}|^2 = \frac{1}{s^2}\left[2(1-\epsilon)s^2 + 4\left(t - M^2\right)^2 + 4st\right]$$

$$\mathcal{M}_\mathrm{b}^{(n)} = \frac{1}{4}\sum_\mathrm{spins} 2\,\mathrm{Re}(\mathcal{A}_\mathrm{b}^{(0)*}\mathcal{A}_\mathrm{b}^{(n)})$$

1-Loop Diagrams

6 Diagrams

$$\mathcal{M}^{(1)} = A^{(1)} + n_l\, B_l^{(1)} + n_h\, C_h^{(1)}$$

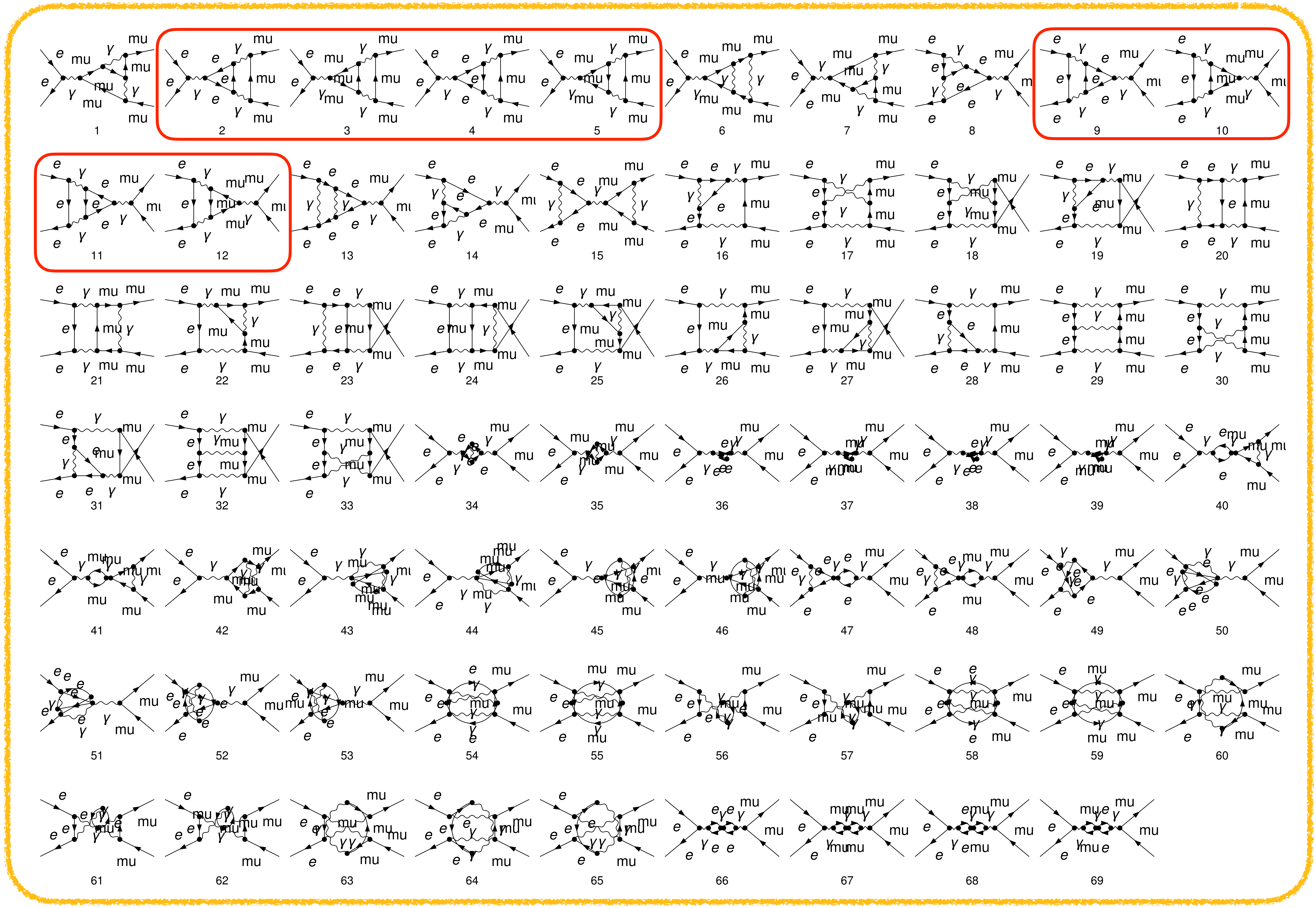


2-Loop Diagrams

69 Diagrams

0 Due to Furry's Theorem

$$\mathcal{M}^{(2)} = A^{(2)} + n_l\, B_l^{(2)} + n_h\, C_h^{(2)} + n_l^2 D_l^{(2)}$$
$$+ n_h\, n_l\, E_{hl}^{(2)} + n_h^2\, F_h^{(2)}$$



Computation of the Loop Amplitude

Mathematica Based Package AIDA [Mastrolia, Peraro, Primo, Ronca, Torres Bobadilla (To be Published)]

Generation of Diagram by FeynArts

↓

Spin sums, Dirac Algebra, Trace by FeynCalc

↓

Adaptive Integrand Decomposition

↓

IBP Reduction via Reduze and KIRA

↓

Master Integral evaluation

$$\mathcal{M}_{\mathrm{b}}^{(n)} = (S_\epsilon)^n \int \prod_{i=1}^{n} \frac{d^d k_i}{(2\pi)^d} \sum_G \frac{N_G}{\prod_{\sigma \in G} D_\sigma}$$

$$\mathcal{M}_{\mathrm{b}}^{(n)} = \mathbb{C}^{(n)} \cdot \mathbf{I}^{(n)}$$

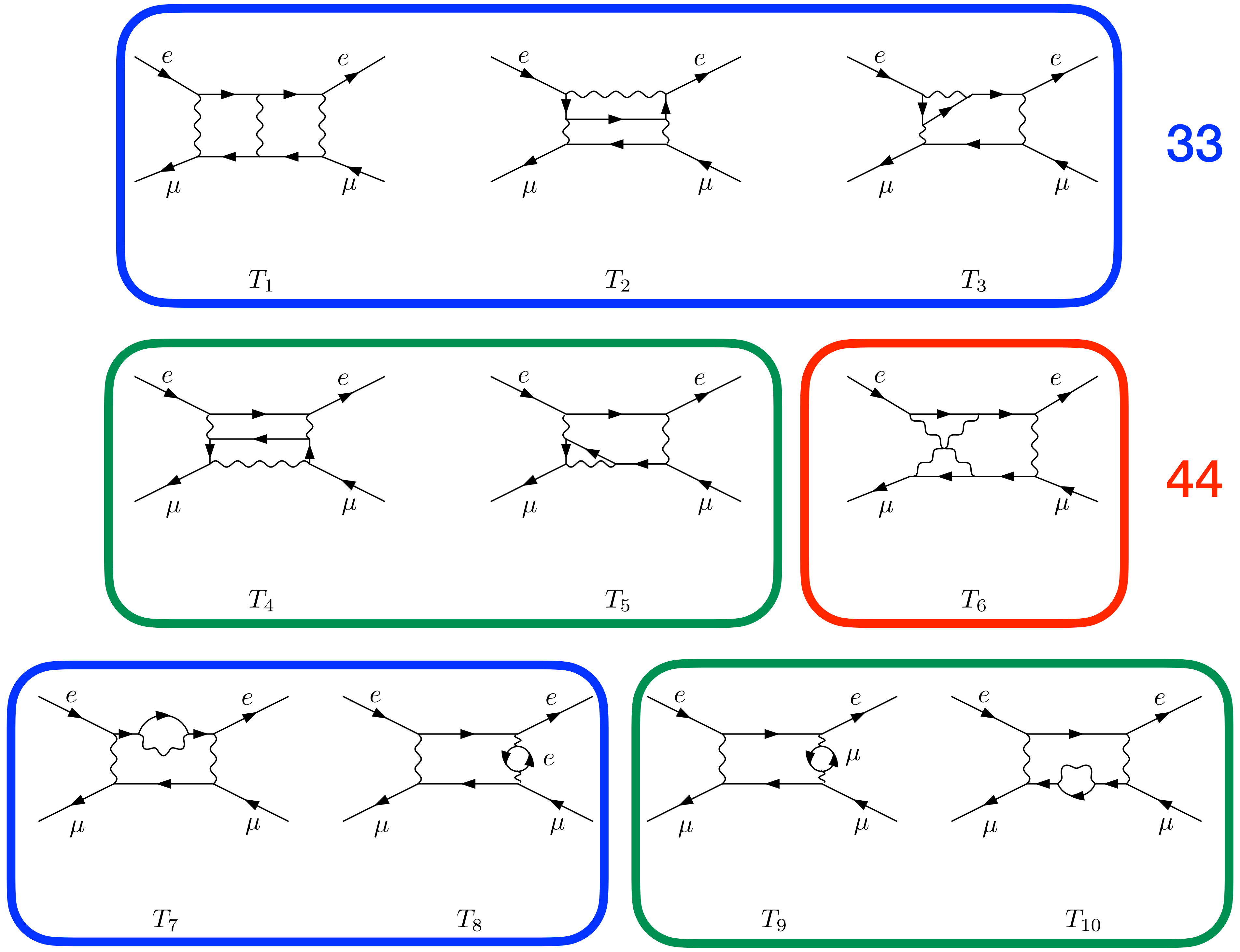
Master Integrals

Master Integrals

◆ The differential equation method has been the most successful in the computation of the MIs

[Kotikov (1990)] [Gehrmann, Remiddi (1999)]

[Henn (2013)] [Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi (2014)]



33 MIs

44 MIs

42 MIs

[Bonciani, Ferroglia, Gehrmann, von Manteuffel (2008-13)]

[Mastrolia, Passera, Primo, Schubert (2017)]

[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]

Generalized Polylogarithms

$$G(w_n,\ldots,w_1;\tau) \equiv \int_0^\tau \frac{dt}{t-w_n} G(w_{n-1},\ldots,w_1;t)$$

$$G(w_1;t) \equiv \log(1-t/w_1)$$

UV Renormalization

Wave functions of External particles - Onshell

Mass of the muon - Onshell

$$\psi_{\rm b} = \sqrt{Z_2}\,\psi, \quad A^\mu_{\rm b} = \sqrt{Z_3}\,A^\mu, \quad M_{\rm b} = Z_M M$$

Renormalized QED Vertex

$$\mathcal{L}_{\rm int} = e_{\rm b}\,\bar\psi_{\rm b}\,\not{A}_{\rm b}\,\psi_{\rm b} = e\,Z_1\,\bar\psi\,\not{A}\,\psi$$

$$e\,Z_1 = e_{\rm b}\,Z_2\sqrt{Z_3}$$

$$Z_1 = Z_2$$ QED Ward Identity

Coupling constant - $\overline{\rm MS}$

$$\alpha_{\rm b} \equiv e_{\rm b}^2/4\pi \qquad \alpha_{\rm b}\,S_\epsilon = \alpha(\mu^2)\,\mu^{2\epsilon}\,Z_\alpha^{\overline{\rm MS}}$$

$$\mathcal{A} = Z_{2,f}\,Z_{2,F}\,\mathcal{A}_{\rm b}\,(\alpha_{\rm b} = \alpha_{\rm b}(\alpha), M_{\rm b} = M_{\rm b}(M))$$

$$Z_j = 1 + \left(\frac{\alpha}{\pi}\right)\delta Z_j^{(1)} + \left(\frac{\alpha}{\pi}\right)^2\delta Z_j^{(2)} + \mathcal{O}(\alpha^3)$$

UV Renormalization

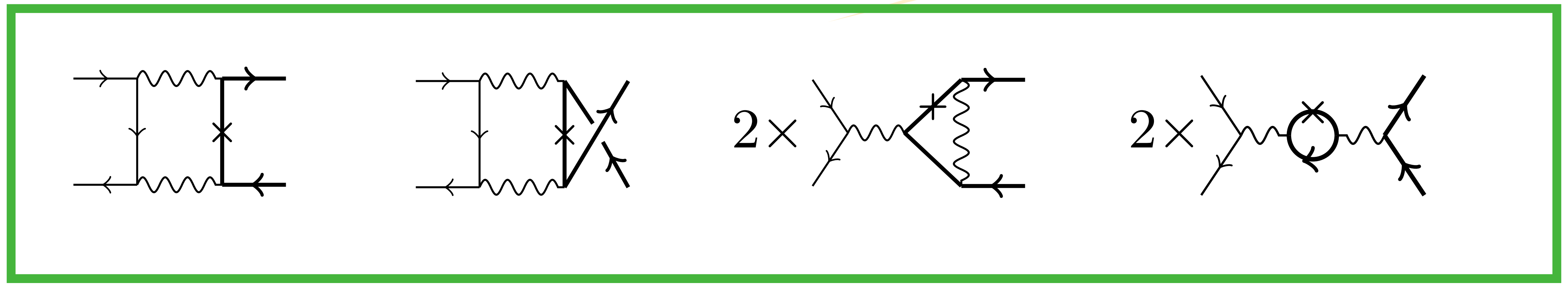
Renormalized Amplitude

$$\mathcal{A}(\alpha) = 4\pi\alpha\left[\mathcal{A}^{(0)} + \left(\frac{\alpha}{\pi}\right)\mathcal{A}^{(1)} + \left(\frac{\alpha}{\pi}\right)^2\mathcal{A}^{(2)}\right]$$

$$\mathcal{A}^{(0)} = \mathcal{A}_\text{b}^{(0)}$$

$$\mathcal{A}^{(1)} = \mathcal{A}_\text{b}^{(1)} + \left(\delta Z_\alpha^{(1)} + \delta Z_F^{(1)}\right)\mathcal{A}_\text{b}^{(0)}$$

$$\mathcal{A}^{(2)} = \mathcal{A}_\text{b}^{(2)} + \left(2\delta Z_\alpha^{(1)} + \delta Z_F^{(1)}\right)\mathcal{A}_\text{b}^{(1)} + \left(\delta Z_\alpha^{(2)} + \delta Z_F^{(2)} + \delta Z_f^{(2)} + \delta Z_F^{(1)}\delta Z_\alpha^{(1)}\right)\mathcal{A}_\text{b}^{(0)} + \delta Z_M^{(1)}\mathcal{A}_\text{b}^{(1,\text{mass CT})}$$



$$\delta Z_f^{(2)} = n_h\left(\frac{L_\mu}{8} + \frac{1}{16\epsilon} - \frac{5}{96}\right)$$

[Czakon, Mitov, Moch (2007)]

IR Factorization

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}}\, \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8}\left[\left(Z_2^{\text{IR}} - \left(Z_1^{\text{IR}}\right)^2\right)\mathcal{M}^{(0)} + 2\, Z_1^{\text{IR}}\, \mathcal{M}^{(1)}\right]\Big|_{\text{poles}}$$

[Becher, Neubert (2009)]

[Hill (2017)]

IR Renormalization Factor

$$\ln Z_{\text{IR}} = \frac{\alpha}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right) + \left(\frac{\alpha}{4\pi}\right)^2\left(-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right) + \mathcal{O}\left(\alpha^3\right)$$

$$\Gamma'(\alpha) \equiv \frac{\partial}{\partial \ln\mu}\Gamma(\alpha) = \sum_{n=0}^{\infty}\Gamma_i'\left(\frac{\alpha}{4\pi}\right)^{n+1}$$

$$\Gamma = \gamma_{\text{cusp}}(\alpha)\ln\left(-\frac{s}{\mu^2}\right) + 2\gamma_{\text{cusp}}(\alpha)\ln\left(\frac{t-M^2}{u-M^2}\right) + \gamma_{\text{cusp,M}}(\alpha, s) + 2\gamma_h(\alpha) + 2\gamma_\psi(\alpha)$$

Cusp Anomalous dimension

Anomalous dimension

Numerical Result

$$\mathcal{M}^{(0)} = \frac{1}{s^2}\left[2(1-\epsilon)s^2 + 4\left(t-M^2\right)^2 + 4st\right]$$

$$\mathcal{M}^{(1)} = A^{(1)} + n_l\, B_l^{(1)} + n_h\, C_h^{(1)}$$

$$\mathcal{M}^{(2)} = A^{(2)} + n_l\, B_l^{(2)} + n_h\, C_h^{(2)} + n_l^2\, D_l^{(2)} + n_h\, n_l\, E_{hl}^{(2)} + n_h^2\, F_h^{(2)}$$

- At 2 Loop there are 4063 GPLs up to weight 4 $\quad G(w_n,\dots,w_1;\tau) \equiv \int_0^\tau \frac{dt}{t-w_n} G(w_{n-1},\dots,w_1;t)$

- 18 Letters $\quad w_i = w_i(x,y,z)$

$$-t/M^2 = x$$
$$-s/M^2 = (1-y)^2/y$$
$$-(u-M^2)/(t-M^2) = z^2/y$$

- The GPLs are evaluated by Ginac [PolyLogTools interface] and HandyG

[Vollinga, Weinzierl (2004)] [Duhr, Dulat (2019)]

[Naterop, Signer, Y. Ulrich (2019)]

☑ **We have obtained complete agreement between the predicted IR poles and the 2-Loop UV renormalized amplitude**

Further checks

Kinematical point $s/M^2 = 5,\ t/M^2 = -5/4,\ \mu = M$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ
$\mathcal{M}^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{100}$	1.99877525	22.0079572	-11.7311017
$B_l^{(1)}$	-	-	-	-	-0.069056030	4.94328573
$C_h^{(1)}$	-	-	-	-	-2.24934027	2.54943566
$A^{(2)}$	$\frac{181}{400}$	-0.499387626	-35.4922919	19.4997261	48.8842283	-
$B_l^{(2)}$	-	$-\frac{181}{400}$	0.785712779	-16.1576674	-3.75247701	-
$C_h^{(2)}$	-	-	1.12467013	-9.50785825	-25.8771503	-
$D_l^{(2)}$	-	-	-	-	-3.96845688	-
$E_{hl}^{(2)}$	-	-	-	-	-4.88512563	-
$F_h^{(2)}$	-	-	-	-	-0.158490810	-

☑ We recovered the Abelian part of the QCD result of $q\bar{q} \to t\bar{t}$ at 1- and 2-Loop [Czakon(2008)] [Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)] [Bärnreuther, Czakon, Fiedler (2014)]

☑ n_h contributions were checked independently [Fael, Passera (2019)] [Fael (2018)]

Results: One Loop

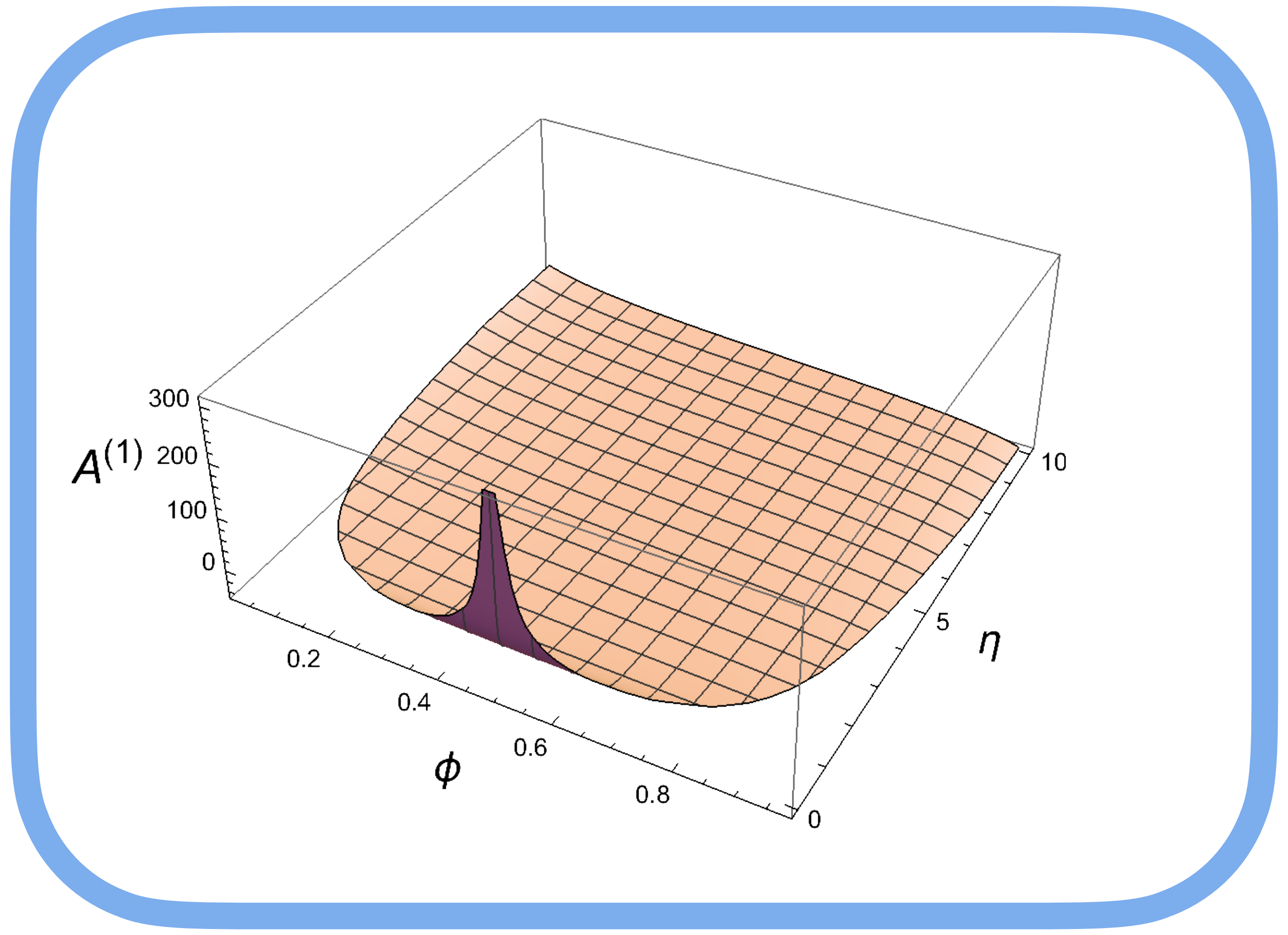
$$\mathcal{M}^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)}$$

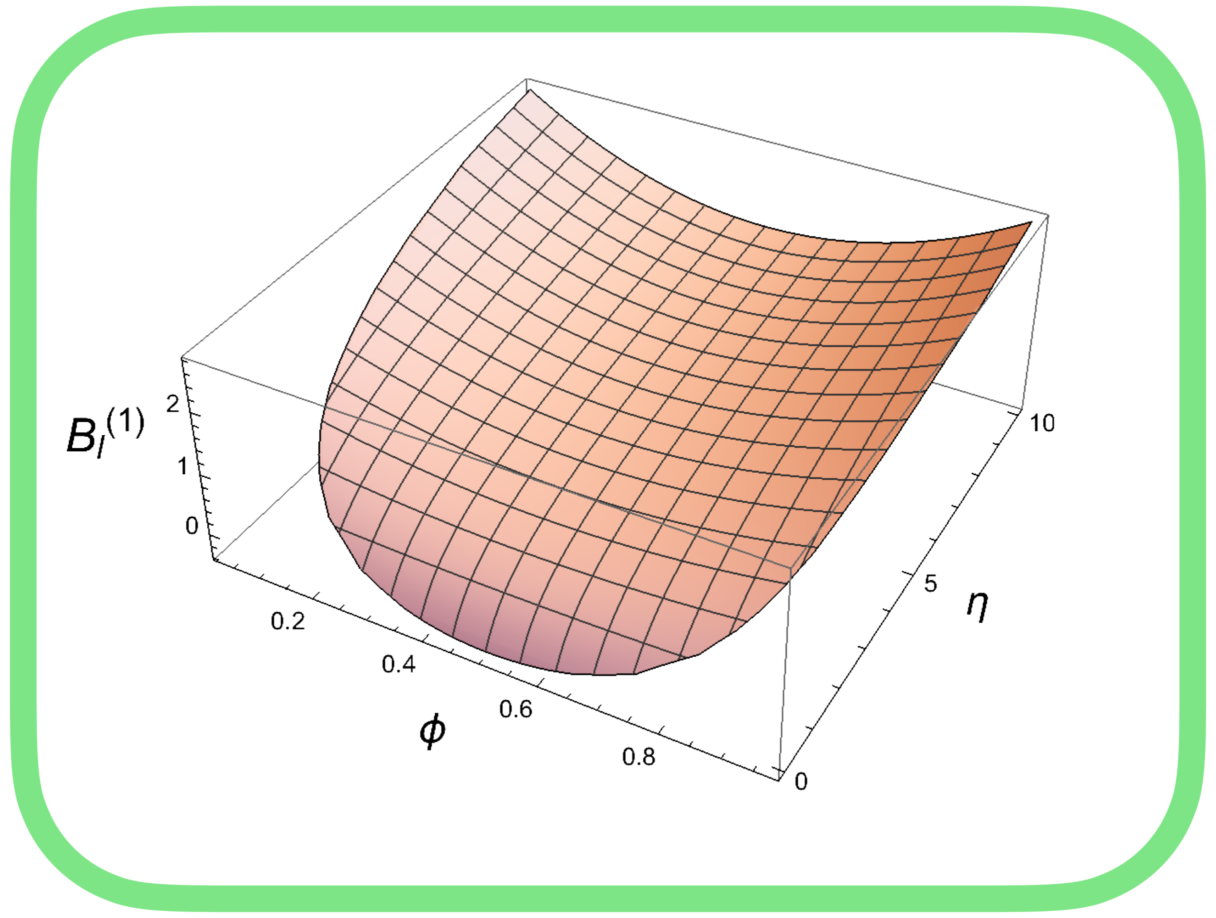
$$\mathcal{M}^{(1)} = \frac{\mathcal{M}_{-2}^{(1)}}{\epsilon^2} + \frac{\mathcal{M}_{-1}^{(1)}}{\epsilon} + \mathcal{M}_0^{(1)}$$

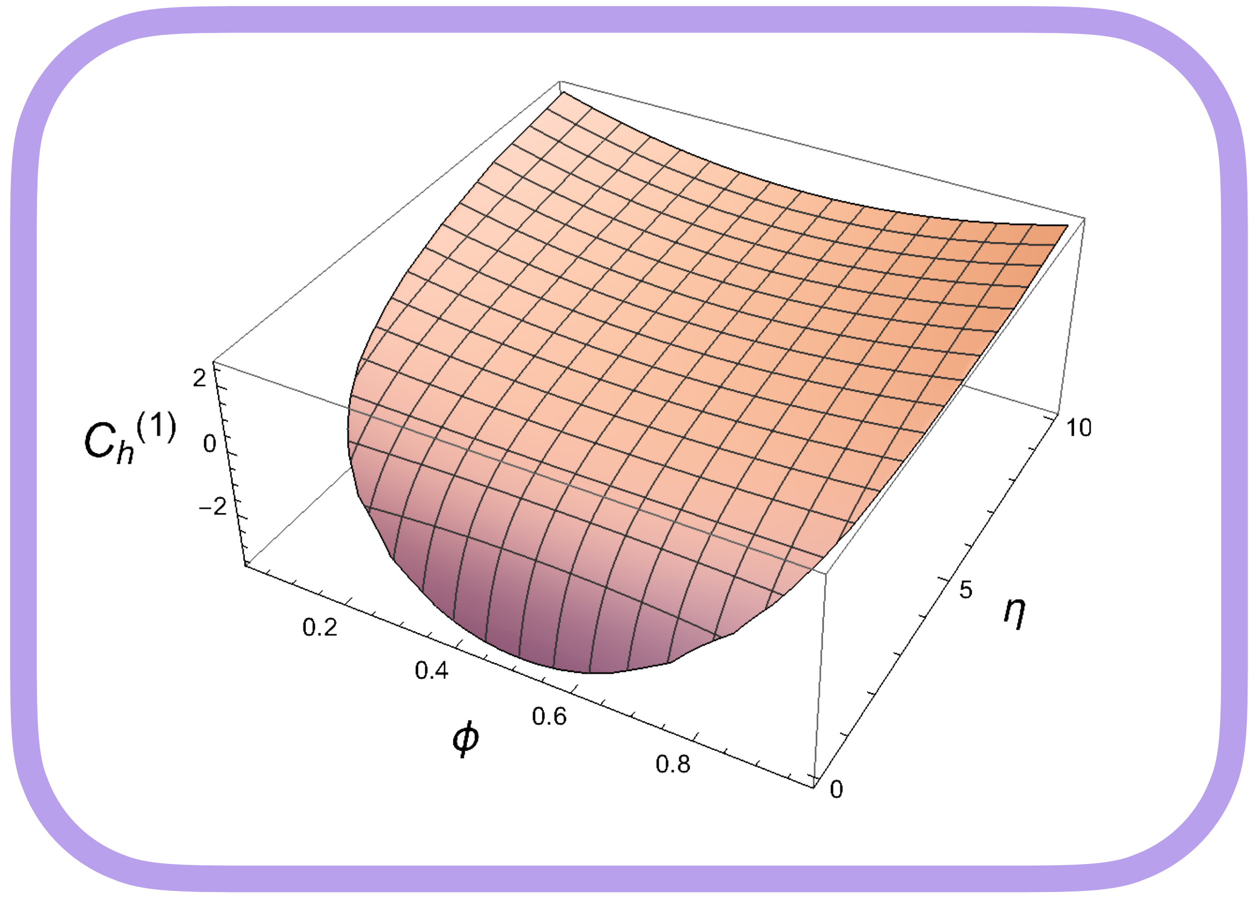
$$\eta = s/(4M^2) - 1$$

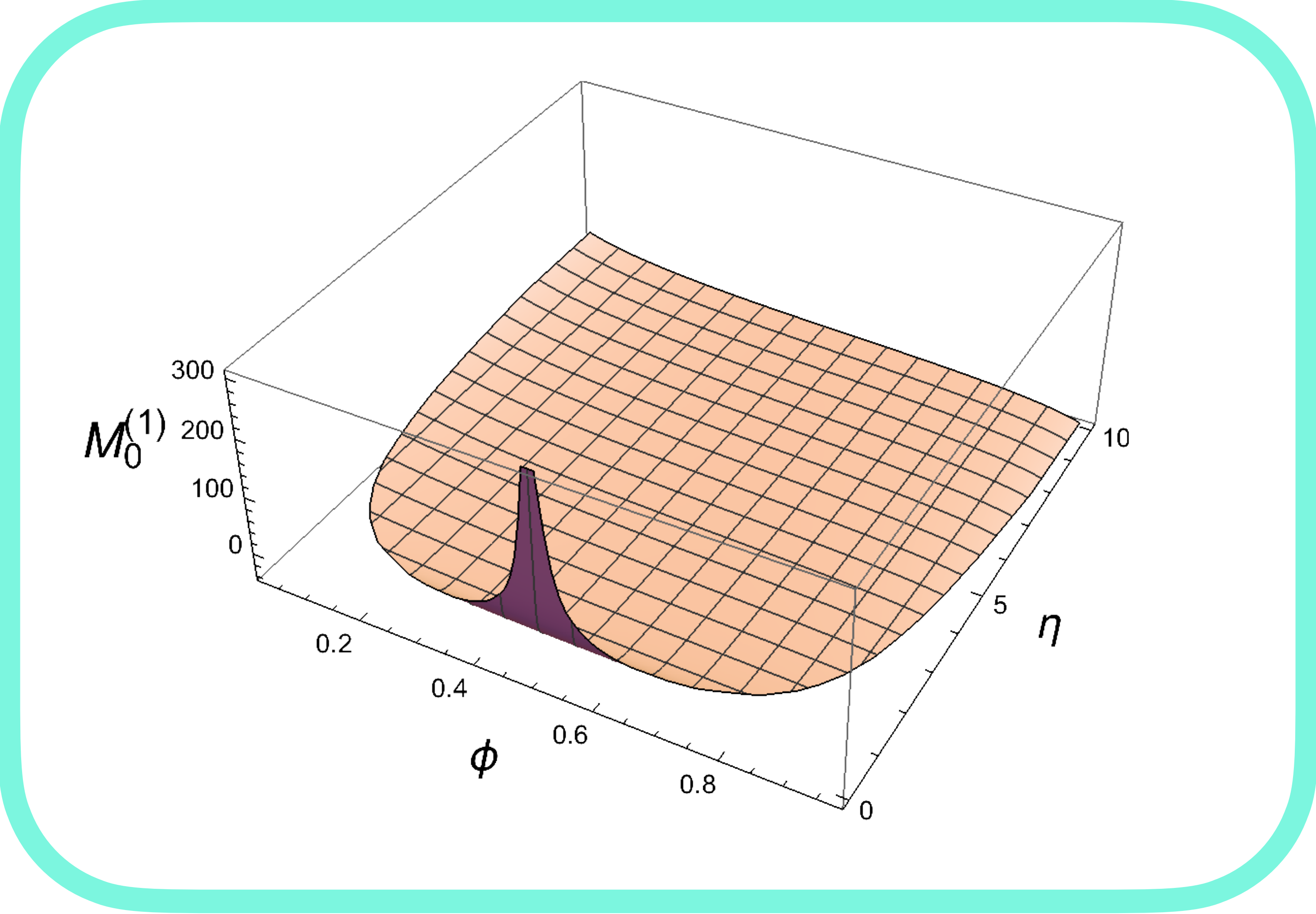
$$\phi = -(t - M^2)/s$$

$$\frac{1}{2}\left(1 - \sqrt{\frac{\eta}{1+\eta}}\right) \leq \phi \leq \frac{1}{2}\left(1 + \sqrt{\frac{\eta}{1+\eta}}\right)$$





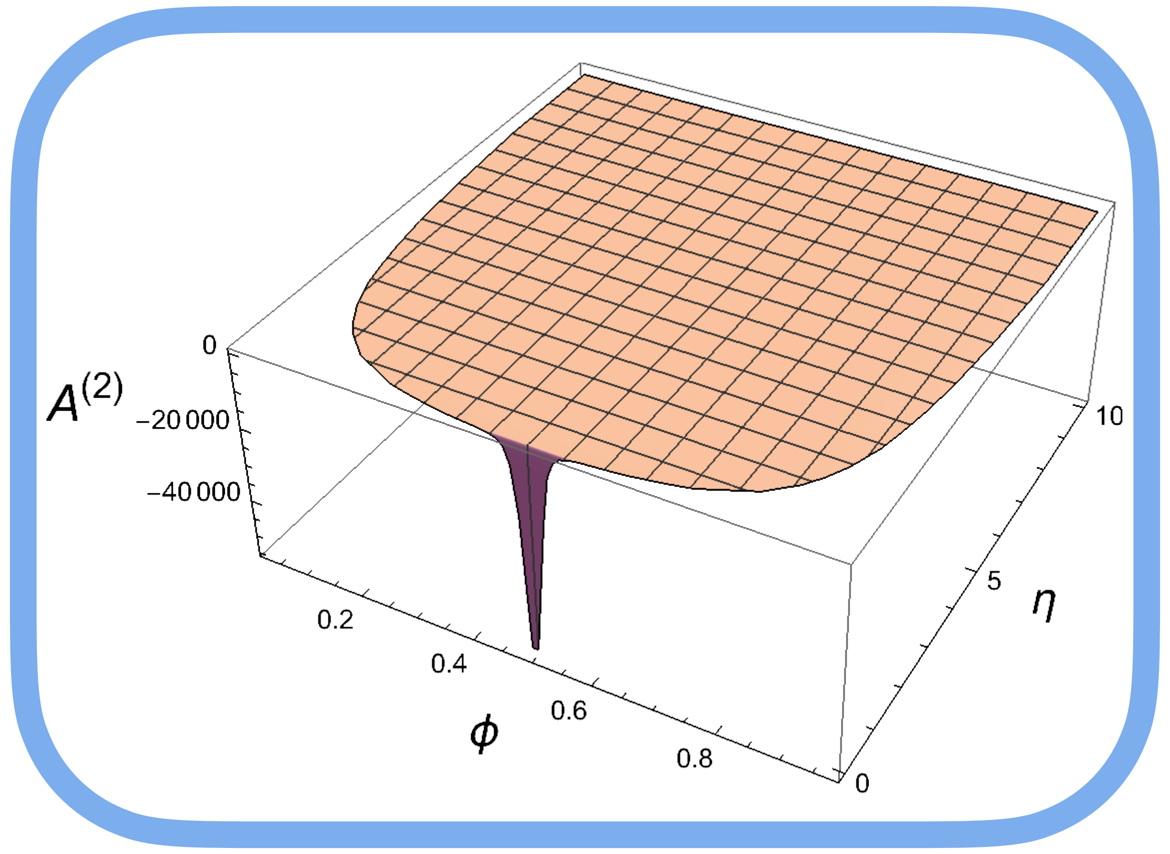


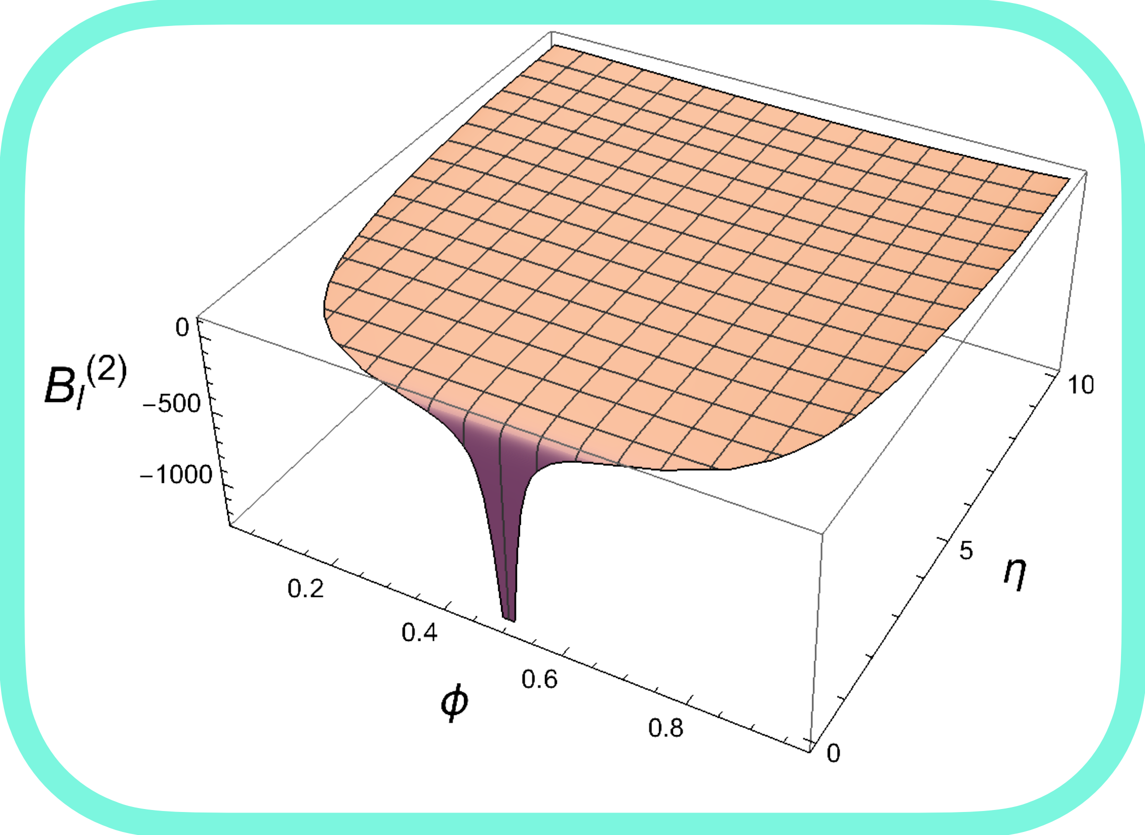


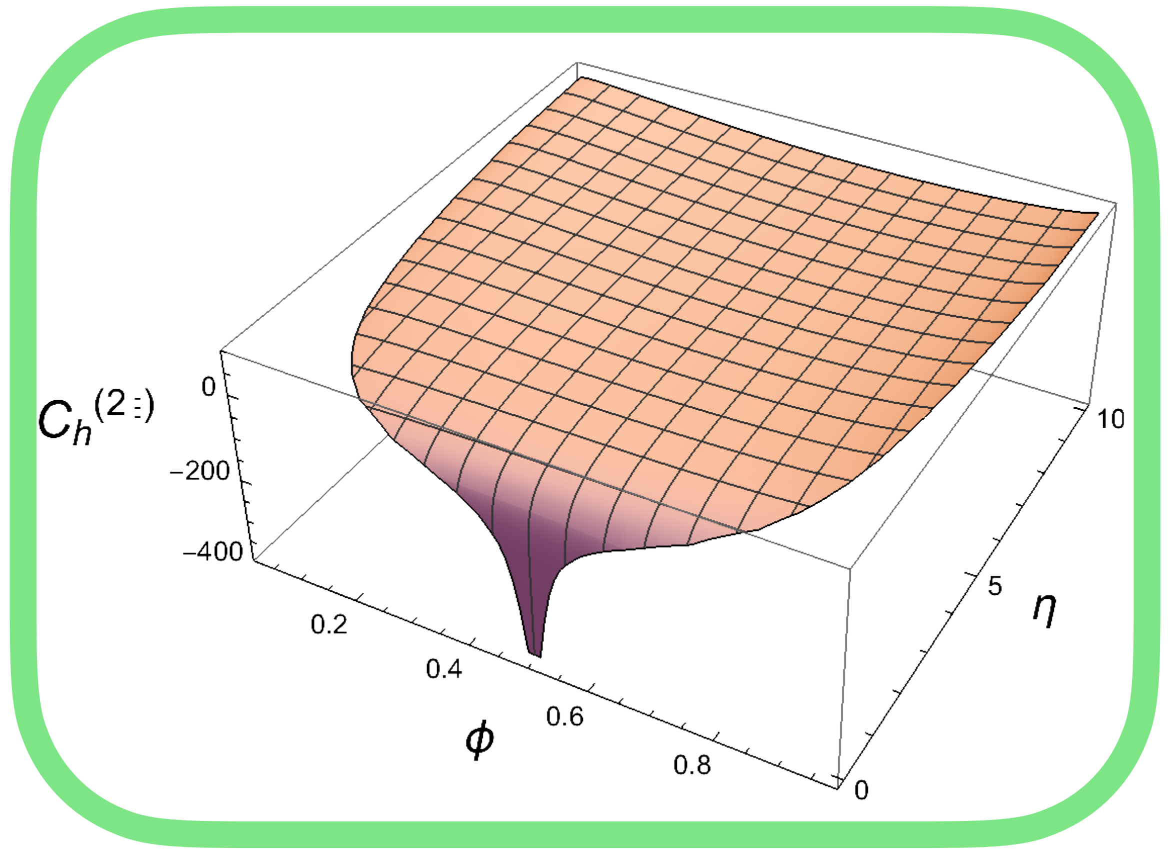
Results: Two Loop

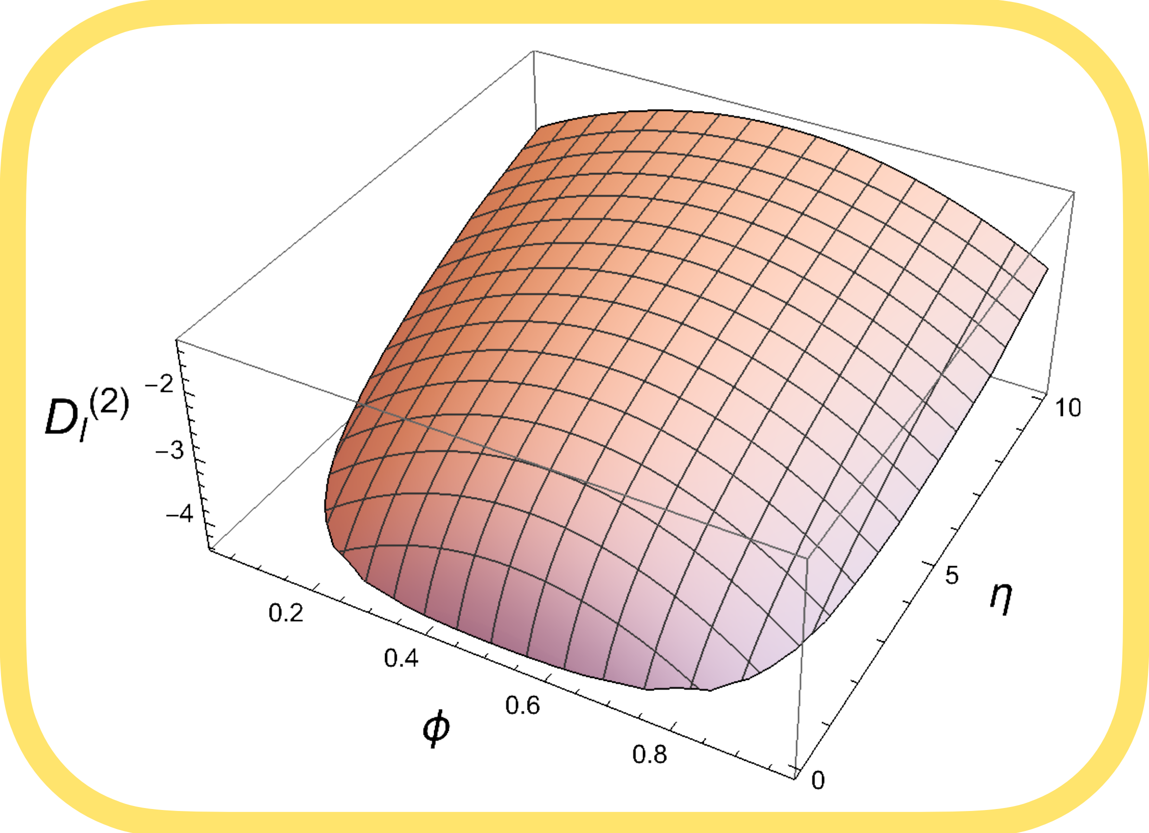
$$\mathcal{M}^{(2)} = A^{(2)} + n_l\, B_l^{(2)} + n_h\, C_h^{(2)} + n_l^2\, D_l^{(2)} + n_h\, n_l\, E_{hl}^{(2)} + n_h^2\, F_h^{(2)}$$

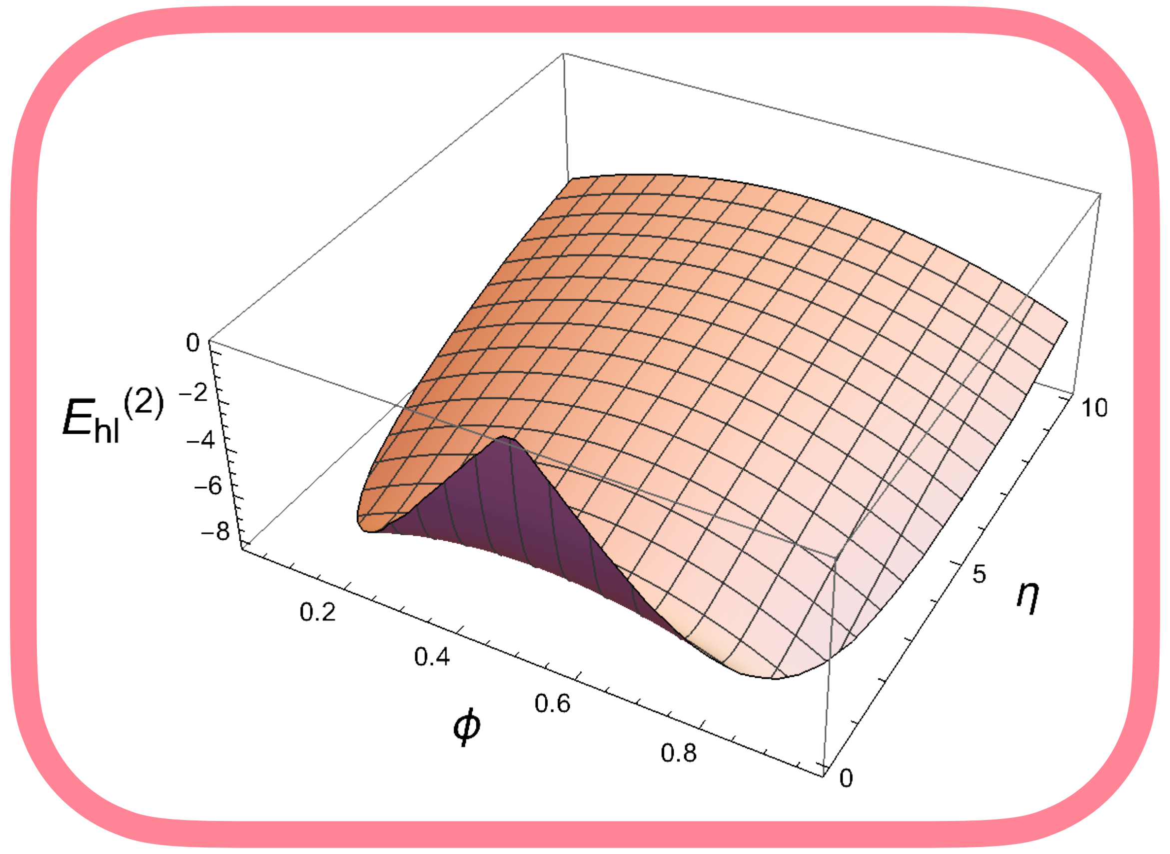
$$\mathcal{M}^{(2)} = \frac{\mathcal{M}_{-4}^{(2)}}{\epsilon^4} + \ldots + \frac{\mathcal{M}_{-1}^{(2)}}{\epsilon} + \mathcal{M}_0^{(2)}$$

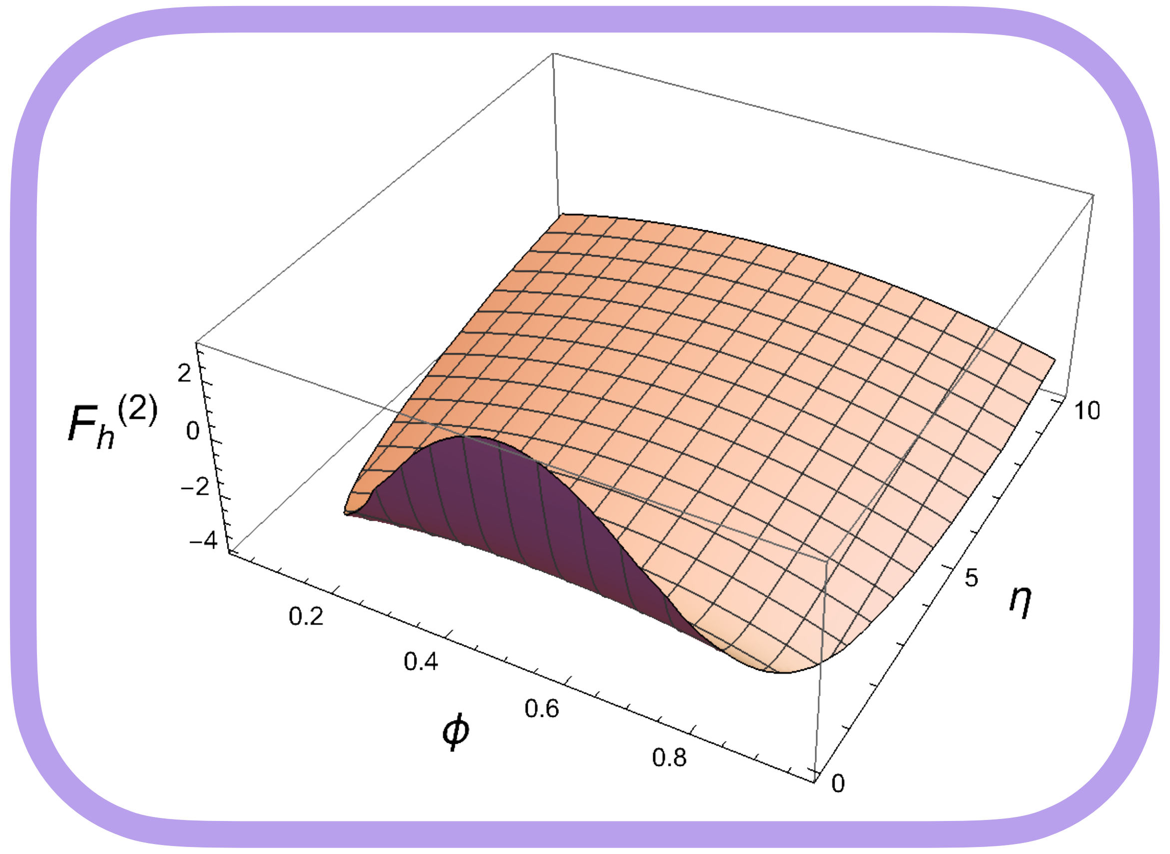


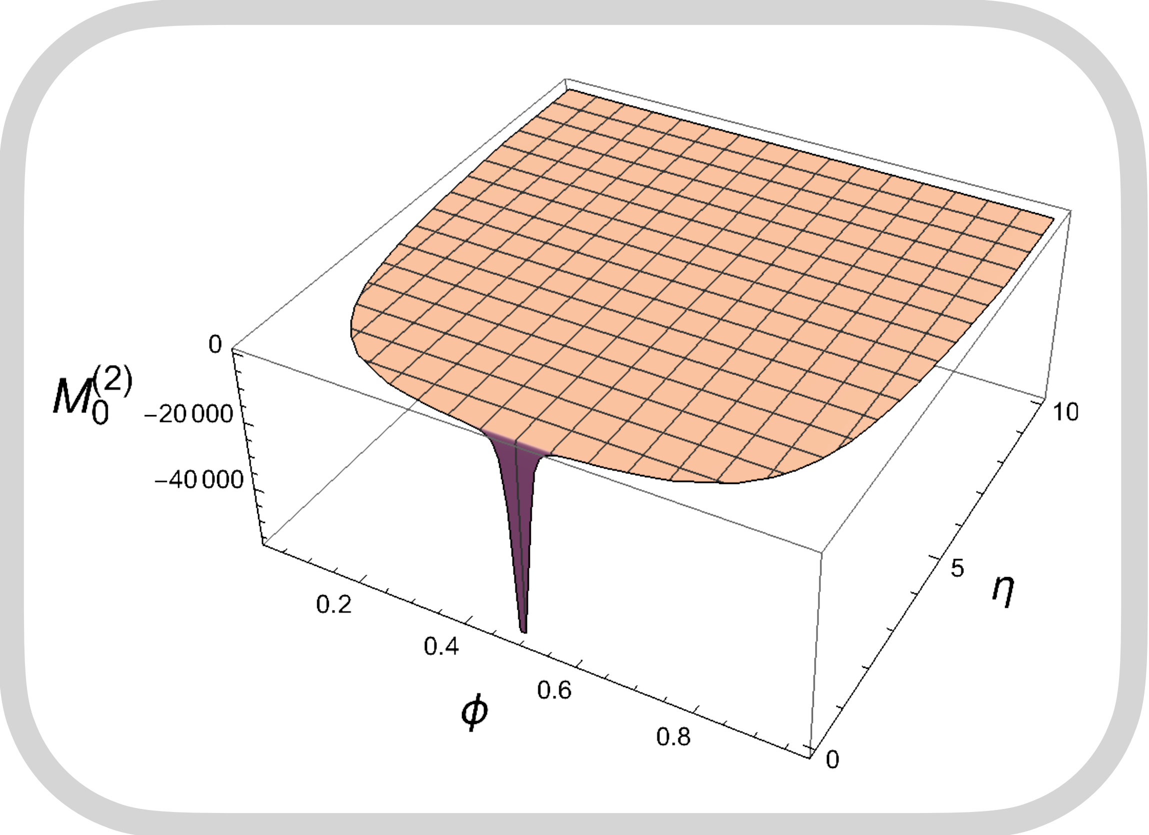












Conclusion and Outlook

- ☑ First complete analytic 2-Loop amplitude for four fermion scattering in QED with a pair of massive leptons
- ☑ The computation of the 2-Loop amplitude is done within the framework of AIDA [Automated]
- ☑ Complete agreement with the universal IR poles predicted by SCET
- ☑ We have created a grid of 10500 points for the 2-Loop amplitude
- ☑ One crucial input for the theory initiative at MUonE

- Inclusion of the effects of the mass of the Muon   Mitov, Moch (2006)   Becher, Melnikov (2007)   Engel, Gnendiger, Signer, Ulrich (2019)   Heller (2021)
- Possible synergy with the MC efforts to include this matrix element   Pavia and PSI Group
- Roundtable discussion: To discuss / understand how the grids of the $e\mu$ amplitude should be prepared to felicitate the inclusion to MC generators   Pavia and PSI Group
- All the ingredients are available for the full analytic evaluation of the 2-Loop amplitude of $q\bar{q} \to t\bar{t}$