Space-like method for hadronic vacuum polarization contributions to muon g-2

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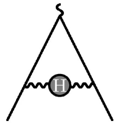
- Very brief summary of time-like method for LO hadronic vacuum polarization contribution to muon g-2
- Space-like method for LO hadronic vacuum polarization contribution to muon g-2
- NLO hadronic vacuum polarization contributions

Reminder: Time-like method



leading order (LO) hadronic vacuum polarization contribution to muon g-2.

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi^2}\int_{s_0=m_\pi^2}^{\infty} \frac{ds}{s}\, K^{(2)}(s/m^2)\,{\rm Im}\Pi(s) \qquad = 6931(40)\times 10^{-11}\ ({\rm WP20})$$

$$\text{optical theorem} \to {\rm Im}\Pi(s) = \frac{\alpha}{3}R(s) \qquad R(s) = \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{4\pi\alpha^2/(3s)}$$

- $R(s)$ fluctuating at low energy due to resonance and particle production threshold effects
- $K^{(2)}(s/m^2)$: 1-loop QED g-2 contribution with a massive photon of mass $\sqrt{s}$

$$K^{(2)}(s/m^2) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right)$$

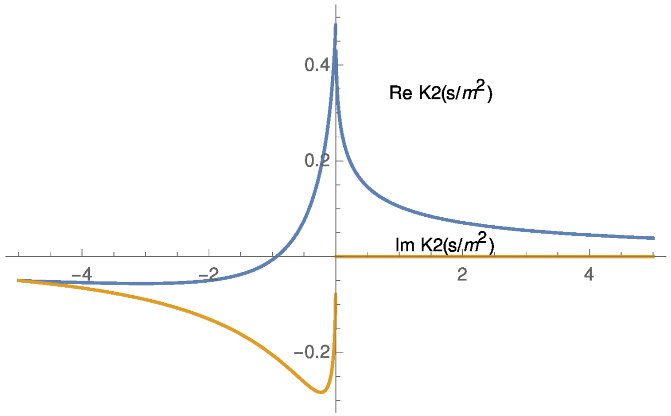
$$z = \frac{s}{m^2} \quad \to \quad \text{(conformal) rationalizing variable} \quad y(z) = \frac{\sqrt{z}-\sqrt{z-4}}{\sqrt{z}+\sqrt{z-4}}$$

$$K^{(2)}(z) = \frac{1}{\pi} \int_{-\infty}^{0} dz' \frac{\mathrm{Im} K^{(2)}(z')}{z'-z}, \; z > 0 \qquad \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{s} \frac{\mathrm{Im}\Pi(s)}{s-q^2} = \frac{\Pi(q^2)}{q^2}\,, \quad q^2 < 0$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{LO}) = -\frac{\alpha}{\pi^2} \int_{-\infty}^{0} \frac{dt}{t} \Pi(t) \mathrm{Im} K^{(2)}(t/m^2)$$

we can evaluate the correct expression of the imaginary part for $z < 0$ from the exact $K^{(2)}(z)$ with the replacement $y(z) \to 1/y(z)$

$$\mathrm{Im} K^{(2)}(z + i\epsilon) = \pi\theta(-z) \left[\frac{z^2}{2} - z + \frac{z - 2z^2 + \frac{z^3}{2}}{\sqrt{z(z-4)}} \right]$$



$$K^{(2)}(0) = 1/2\,, \qquad K^{(2)}(z) \to 1/(3z) \quad z \to \infty$$

$\mathrm{Im}K^{(2)}$ espressed in terms of $y(z)$ is simpler

$$\mathrm{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)F^{(2)}(y(z)) \; , \qquad F^{(2)}(y) = \frac{y+1}{y-1}y^2$$

changing again variable in the dispersive integral $t \to y \to x$ $(t<0 \to -1<y<0 \to 0<x<1)$

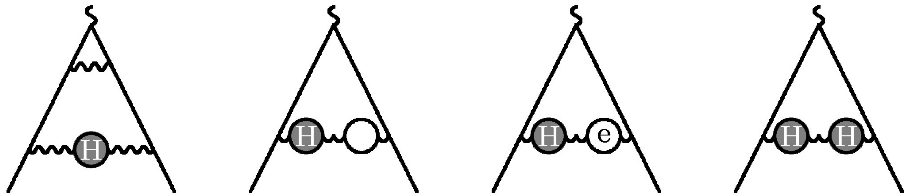
$$t(x) = m^2 \frac{x^2}{1-x} \; , \qquad x = 1+y$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{LO}) = \frac{\alpha}{\pi}\int_0^1 dx \; \kappa^{(2)}(x)\Delta\alpha_{\mathrm{had}}(t(x))$$

Lautrup Peterman de Rafael 1975

- $\kappa^{(2)}(x) = 1-x$ simple space-like kernel
- $\Delta\alpha_{\mathrm{had}}(t) = -\Pi(t)$ hadronic contribution to the running of the effective fine-structure constant in the space-like region

The above expression was proposed for the first time (Carloni Calame Passera Trentadue Venanzoni 2015) to determine a_μ^{HVP} measuring the electromagnetic effective coupling in the space-like region through scattering data.



(2a) (2a) (2b) (2c)

- Class a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

$$a_\mu^{\rm HVP}({\rm NLO};a) = -209.0 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO};b) = +106.8 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO};c) = +3.5 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO};total) = -98.7(9) \times 10^{-11}$$

(Hagiwara Martin Nomura Toebner 2004, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)

We write the time-like expression

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};a) = \frac{\alpha}{\pi^2}\int_{m_\pi^2}^{\infty} \frac{ds}{s}\, 2K^{(4)}(s/m^2)\,\mathrm{Im}\Pi(s)$$

$2K^{(4)}$ is the anomaly from all 2-loop QED diagrams with 1 photon massless and 1 photon of mass $\sqrt{s}$

(factor 2 due to normalization chosen, as in the limit $z \to 0$ the anomaly is the $2\times$ the 2-loop QED g-2 anomaly)

The analitical expression is lengthy

NLO class a: Analytic expression of 2-loop $K^{(4)}(z)$

$$
\begin{aligned}
K^{(4)}(z) = {} & \left(\frac{z^2}{2} - \frac{7z}{6} + \frac{1}{2}\right)\Big[-3\mathrm{Li}_3(-y(z)) - 6\mathrm{Li}_3(y(z)) + 2\left(\mathrm{Li}_2(-y(z)) + 2\mathrm{Li}_2(y(z))\right)\ln(y(z)) \\
& + \ln(1-y(z))\ln^2(y(z)) + \frac{1}{2}\left(\ln^2(y(z)) + \pi^2\right)\ln(y(z)+1)\Big] \\
& + \frac{\left(-\frac{z^3}{6} + \frac{z^2}{4} - \frac{7z}{6} - \frac{4}{z-4} + \frac{13}{3}\right)\left(\mathrm{Li}_2(-y(z)) + \frac{\ln^2(y(z))}{4} + \frac{\pi^2}{12}\right)}{\sqrt{(z-4)z}} \\
& + \frac{\left(-\frac{7z^3}{12} + \frac{17z^2}{6} - 2z\right)\left(\mathrm{Li}_2(y(z)) - \frac{1}{4}\ln^2(y(z)) + \ln(1-y(z))\ln(y(z)) - \frac{\pi^2}{6}\right)}{\sqrt{(z-4)z}} \\
& + \left(-\frac{29z^2}{96} + \frac{53z}{48} + \frac{2}{z-4} - \frac{1}{3z} + \frac{19}{24}\right)\ln^2(y(z)) + \frac{\left(\frac{23z^3}{144} - \frac{115z^2}{72} + \frac{127z}{36} - \frac{4}{3}\right)\ln(y(z))}{\sqrt{(z-4)z}} \\
& + \frac{\left(-\frac{7z^3}{48} + \frac{17z^2}{24} - \frac{z}{2}\right)\ln(y(z))\ln(z)}{\sqrt{(z-4)z}} + \frac{1}{6}\pi^2\left(-\frac{z^2}{2} + \frac{5z}{24} - \frac{2}{z} + \frac{9}{4}\right) + \frac{5}{96}z^2\ln^2(z) \\
& + \left(\frac{23z^2}{144} - \frac{7z}{36} + \frac{1}{z-4} + \frac{19}{12}\right)\ln(z) + \frac{115z}{72} - \frac{139}{144}
\end{aligned}
$$

analytic: Barbieri Remiddi 1975

As in the LO case we write the dispersive relation for $K^{(4)}(z)$

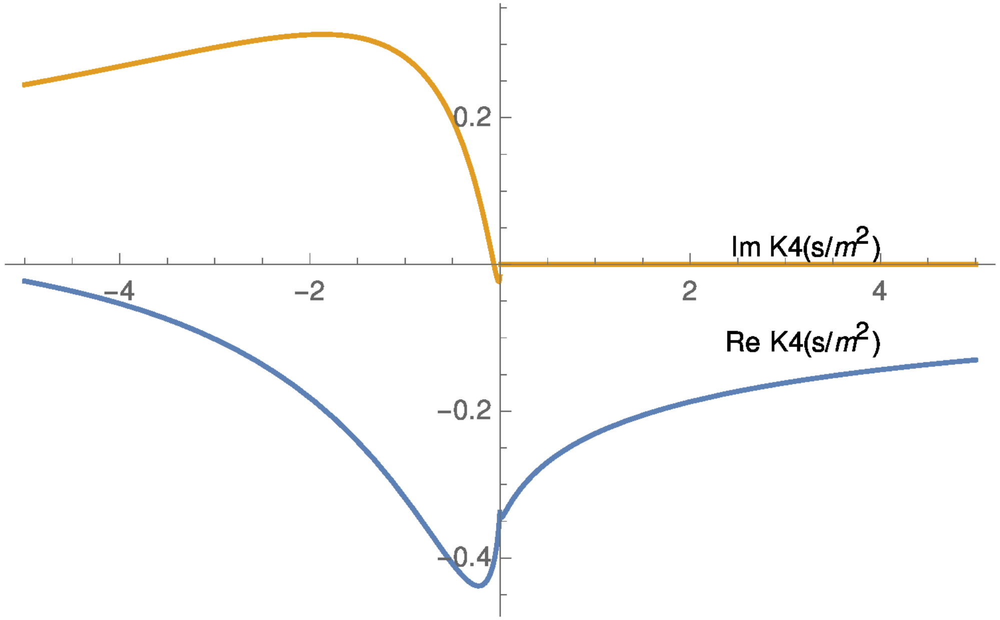
$$K^{(4)}(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\mathrm{Im}K^{(4)}(z')}{z'-z}, z > 0$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO;a}) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0} \frac{dt}{t}\Pi(t)\mathrm{Im}K^{(4)}\left(\frac{t}{m^2}\right)$$

As in the LO case, the correct imaginary part is obtained from $K^{(4)}(z)$ replacing $y(z)$ with $1/y(z)$ for $z < 0$ The expression is more compact if written in terms of y instead of z:

$$\mathrm{Im}K^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(y(z))$$

$$F^{(4)}(y) = \frac{\left(-3y^4-5y^3-7y^2-5y-3\right)\left(2\mathrm{Li}_2(-y)+4\mathrm{Li}_2(y)+\ln(-y)\ln\left((1-y)^2(y+1)\right)\right)}{6y^2}$$
$$+\frac{(y+1)\left(-y^3+7y^2+8y+6\right)\ln(y+1)}{12y^2}+\frac{\left(-7y^4-8y^3+8y+7\right)\ln(1-y)}{12y^2}$$
$$+\frac{23y^6-37y^5+124y^4-86y^3-57y^2+99y+78}{72(y-1)^2y(y+1)}+\frac{\left(12y^8-11y^7-78y^6+21y^5+4y^4-15y^3+13y+6\right)\ln(-y)}{12(y-1)^3y(y+1)^2}\;,y<0$$

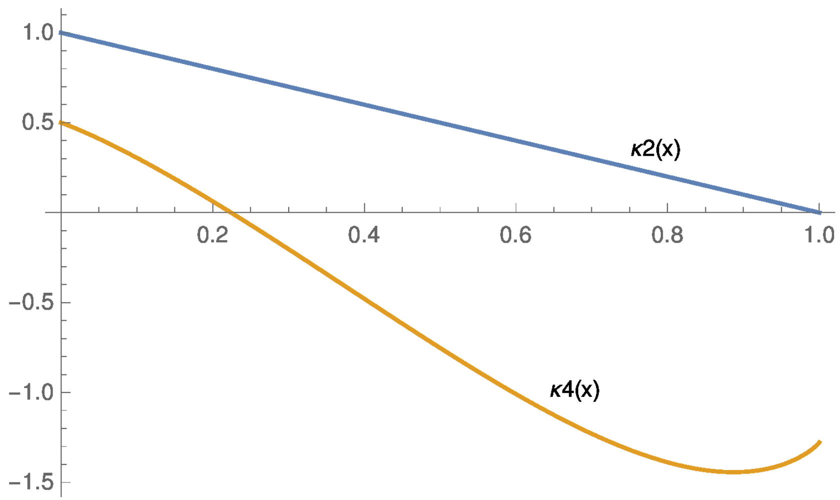


$K^{(4)}(0) = \frac{197}{144}+\frac{1}{12}\pi^2-\frac{1}{2}\pi^2\ln 2+\frac{3}{4}\zeta(3) = -0.328479$ 2-loop g-2 $\qquad K^{(4)}(z) \to \frac{1}{z}\left(-\frac{23\ln(z)}{36}-\frac{\pi^2}{3}+\frac{223}{54}\right)$

$$a_\mu^{\rm HVP}(\text{NLO;a}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx\ \kappa^{(4)}(x)\Delta\alpha_{\rm had}(t(x))$$

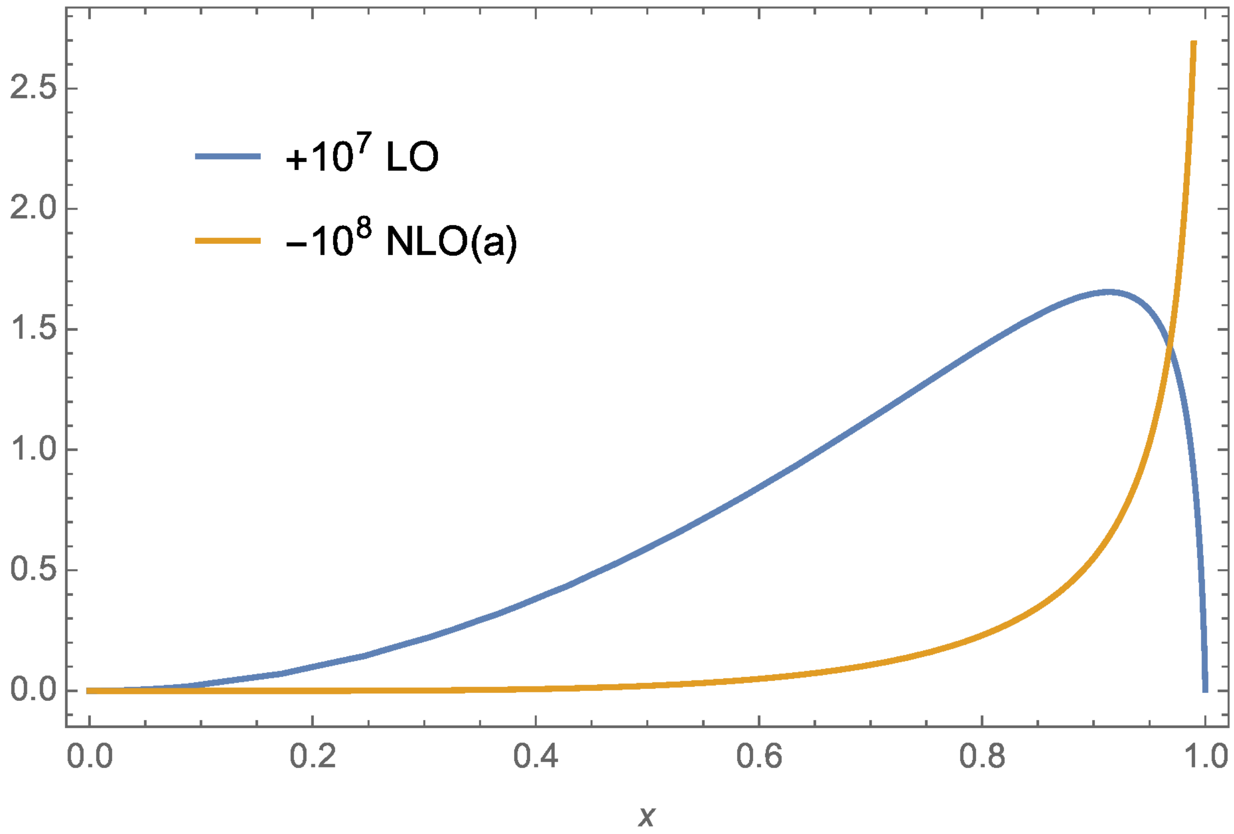
space-like kernel $\kappa^{(4)}(x)$:

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$



$$\kappa^{(4)}(1) = -\tfrac{23}{18}$$

- $\kappa^{(4)}(x)$ provides stronger weight a large $q^2 < 0$ $(x \to 1)$ than $\kappa^{(2)}(x)$



the integrands $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\text{had}}(t(x))$ (blue) $(\alpha/\pi)^2\kappa^{(4)}(x)\Delta\alpha_{\text{had}}(t(x))$ (orange)

- LO integrand has a peak at $x \approx 0.914$
- NLO has an (integrable) logarithmic singularity at $x \to 1$

Asymptotic expansion of $K^{(4)}(z)$ for large z (Lautrup 1997)

$$K^{(4)}(z) = \frac{1}{z}\left(-\frac{23\ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right) + \frac{1}{z^2}\left(\frac{19\ln^2(z)}{144} - \frac{367\ln(z)}{216} - \frac{37\pi^2}{48} + \frac{8785}{1152}\right) + \frac{1}{z^3}\left(\frac{141\ln^2(z)}{80} - \frac{10079\ln(z)}{3600} - \frac{883\pi^2}{240} + \frac{13072841}{432000}\right) + \ldots$$

from these expansions it is possible to derive some approximations to the space-like kernel $\kappa^{(4)}(x)$

[Chakraborty Davies Kobonen Lepage VandeWater 2018] uses expansions for small $1/z$ up to $1/z^4$, with the ansatz from [Groote Körner Pivovarov 2002]

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dw \left[(a_0 + a_1 w + a_2 w^2 + a_3 w^3)\,\Pi\left(\frac{m^2}{w}\right) + \frac{b_0 + b_1 w + b_2 w^2 + b_3 w^3}{w}\,\Pi\left(m^2 w\right)\right]$$

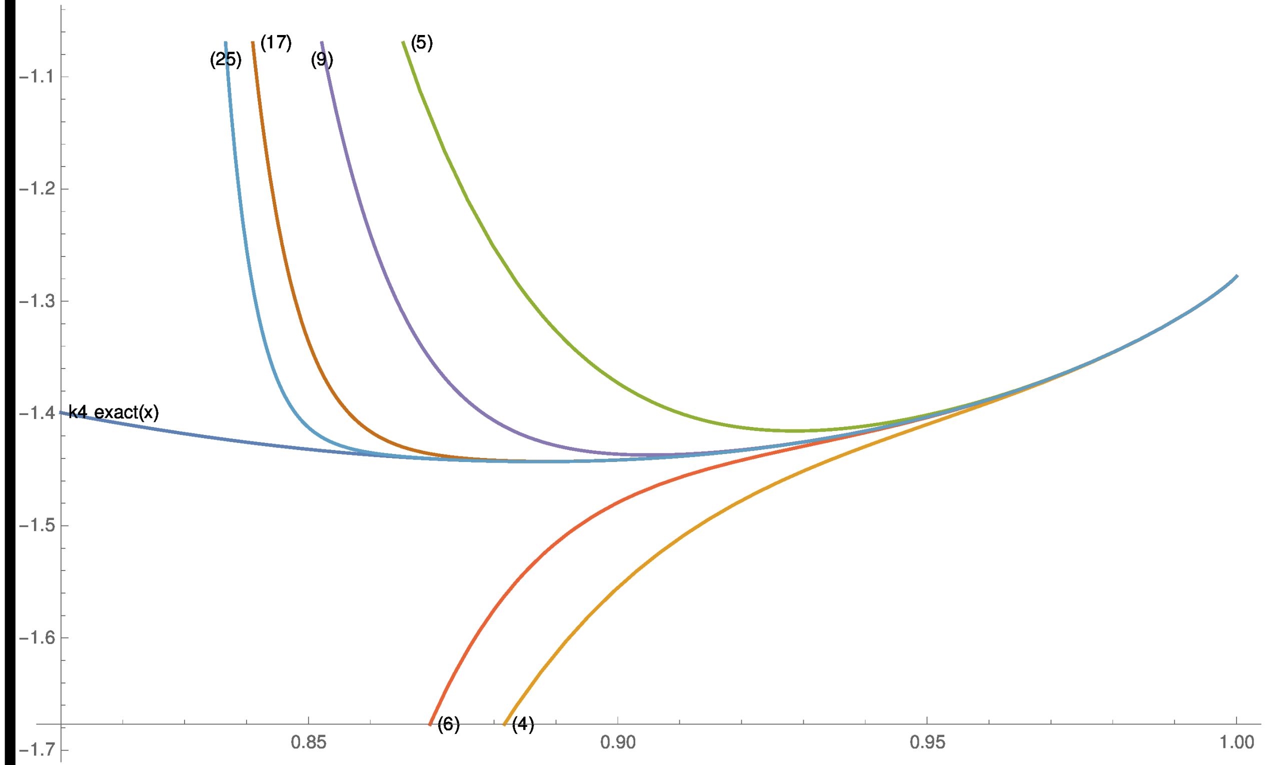
Unknown coefficients are found matching the terms of large z asymptotic expansion: $\{a_i\}$ match terms $(\ln z)/z^n$, $\{b_i\}$ match terms $1/z^n$.

This approximation gives an error of 6% on $a_\mu^{\rm HVP}$(NLO;a). The reason is that this ansatz does not fit terms with $\ln^2 z/z^n$.

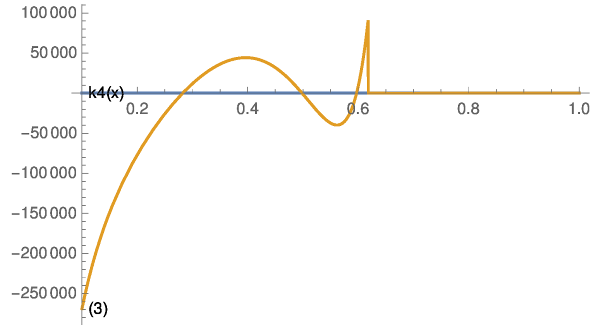
In order to fit all terms, we added to the ansatz a logarithmic term like
$+\ln w (c_0 + c_1 w + c_2 c^2 + c_3 w^3)\,\Pi\left(\frac{m^2}{w}\right)$.

(E.Balzani, S.L. and M.Passera, in preparation)

Expansion of analytical $K_{(4)}(z) \to \bar{\kappa}_n^{(4)}(x)$ for $n_{terms} = 4$ to $n_{terms} = 25$.

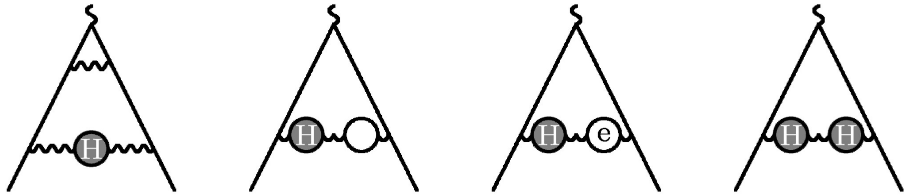


$n = 4$



- $\bar{\kappa}_n^{(4)}(x)$: good approximation for x near 1;
- Discontinuity for $x = (\sqrt{5} - 1)/2 \approx 0.618$
- Wild oscillations for small x, worse for large n.
- For $n = 25$ up to $\sim \pm 10^{30}$! But the integral reproduces the exact result with error 10^{-20} $\to$ deep numerical cancellations!.
- Large n not necessary, $n = 4$ reproduces $a_\mu^{\text{HVP}}(\text{NLO;a})$ with error $\lesssim 0.1\%$

NLO class b and c



(2a) (2a) (2b) (2c)

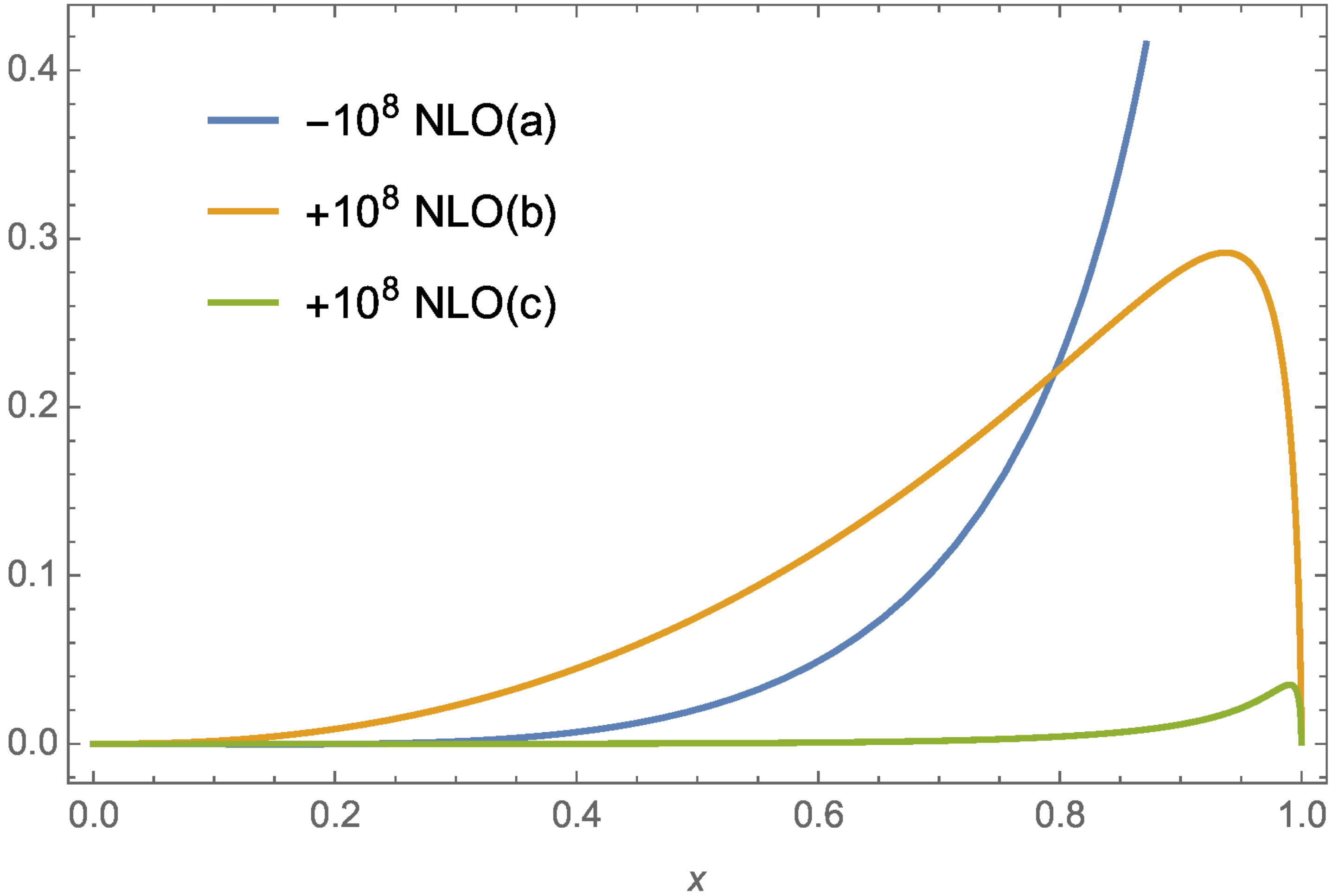
starting from timelike expressions one finds

$$a_\mu^{\rm HVP}(\text{NLO;b}) = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))\ 2\left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right)$$

$$a_\mu^{\rm HVP}(\text{NLO;c}) = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)\left(\Delta\alpha_{\rm had}(t(x))\right)^2$$

$\Pi_l^{(2)}(t) = -\Delta\alpha_{\rm l}(t)$ renormalized QED vacuum polarization function

$$\Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi}\right)\left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l\left(\frac{1}{2} - \frac{\beta_l^2}{6}\right)\ln\frac{\beta_l - 1}{\beta_l + 1}\right] \quad \beta_l = \sqrt{1 - 4m_l^2/t}$$



the 3 NLO integrands

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