6 Exercises

6.1 Testing hypothesis via the Likelihood Ratio Test

Suppose our instrument has collected $N = 1000$ observations which appear to suggest that an excess of events at 125 GeV is present. For simplicity, we assume that the background was completely resolved and the goal is to understand what is the exact location of these signal events. The mass distribution is assumed to be a Gaussian with mean μ and variance σ^2 (both unknown). We want to test the hypotheses

$$H_0 : \mu = 125 \qquad \text{versus} \qquad H_1 : \mu \neq 125$$

For this scenario, specify the following.

1. **The (profile) log-likelihood function.**

$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$\ell(\mu;\hat{\sigma}^2_\mu;y) = \log \prod_{i=1}^N f(y_i;\mu,\hat{\sigma}^2_\mu) = \sum_{i=1}^N \log\left\{ \quad \right\}$$

$$= -\frac{N}{2}\log 2\pi\hat{\sigma}^2_\mu - \frac{1}{2\hat{\sigma}^2_\mu}\sum_{i=1}^N (y_i-\mu)^2$$

2. **A formula for $\Lambda(125)$.**

$$X \sim N(\mu,\sigma^2) \qquad \hat{\mu} = \bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$$

$$\hat{\sigma}^2_{\hat{\mu}} = \frac{1}{N}\sum_{i=1}^N (x_i-\hat{\mu})^2$$

$$\Lambda(125) = -2\left[\ell(125,\hat{\sigma}^2_{125},\underline{y}) - \ell(\hat{\mu};\hat{\sigma}^2_{\hat{\mu}},\underline{y})\right]$$

$$= N\log(2\pi\hat{\sigma}^2_{125}) + \frac{1}{\hat{\sigma}^2_{125}}\sum_{i=1}^N (y_i-125)^2 - N\log(2\pi\hat{\sigma}^2_{\hat{\mu}}) - \frac{1}{\hat{\sigma}^2_{\hat{\mu}}}\sum_{i=1}^N (y_i-\hat{\mu})^2$$

$$= N\log\frac{\hat{\sigma}^2_{125}}{\hat{\sigma}^2_{\hat{\mu}}} + \sum_{i=1}^N \frac{(y_i-125)^2}{\hat{\sigma}^2_{125}} - \sum_{i=1}^N \frac{(y_i-\hat{\mu})^2}{\hat{\sigma}^2_{\hat{\mu}}}$$

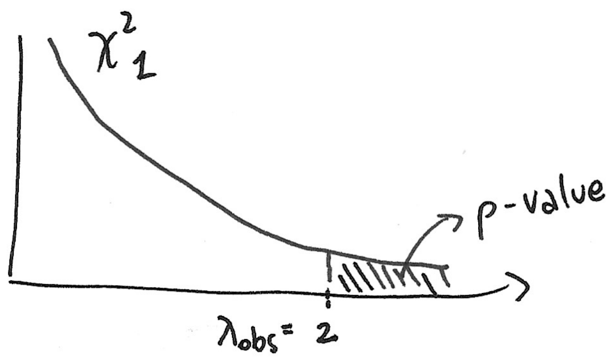
$$= N\log\frac{\hat{\sigma}^2_{125}}{\hat{\sigma}^2_{\hat{\mu}}}$$

3. **The asymptotic distribution of $\Lambda(125)$, when H_0 is true.**

$$\Lambda(125) \sim \chi^2_1$$

4. **Suppose $\Lambda_{obs} = 2$, specify a formula for the p-value.**

$$\text{p-value} = P(\chi^2_1 > 2) = 0.157$$



5. **The p-value is 0.157, can we claim that $\mu = 125$GeV is likely to be the correct location of the signal?**

we fail to reject H_0

$\alpha = 0.05$ (p-value $> \alpha$)

6.2 Performing Goodness-of-Fit via Pearson and Likelihood Ratio

Suppose we are given a sample of events over the energy range 1-50GeV, and the goal is to assess if a power law distribution is a reliable model for the expected number of counts in each bin. Moreover, we assume that the data is Poisson, i.e.,

$$Y_i \sim \text{Poisson}\Big(\int_{x_i-\frac{\Delta}{2}}^{x_i+\frac{\Delta}{2}} \frac{\tau}{x^\theta}\mathrm{d}x\Big).$$

For the moment, we assume that $\theta = 0.5$, $\tau = 500$ and that the data has been divided into $n = 10$ equally spaced bins, each of length $\Delta = 5$. The expected number of events in bin i is $m(x_i, \theta, \tau) = \int_{x_i-\frac{\Delta}{2}}^{x_i+\frac{\Delta}{2}} \frac{\tau}{x^\theta}\mathrm{d}x$.

Bin i	x_i	# of expected events $m(x_i,\theta,\tau)$	# of observed events y_i
[1,5)	2.5	1236.1	1249
[5,10)	7.5	926.2	926
[10,15)	12.5	710.7	712
[15,20)	17.5	599.2	615
[20,25)	22.5	527.9	530
[25,30)	27.5	477.2	482
[30,35)	32.5	438.9	437
[35,40)	37.5	408.5	430
[40,45)	42.5	383.7	363
[45,50]	47.5	362.9	370

The values of our test statistics observed on these data are:

- Pearson: X^2_{obs}= 2.995
- Likelihood Ratio: G^2_{obs}=45.791

A few questions for you...

1. **What is H_0 here?**

$$H_0: E[Y_i] = m(x_i;\theta,\tau) = \int_{bin\ i} \frac{\tau}{x^\theta}dx$$

$$H_1: E[Y_i] \neq m(x_i;\theta,\tau)$$

2. **How were X^2_{obs} and G^2_{obs} calculated?**

$$X^2_{obs} = \left[\frac{(1249-1236.1)^2}{1236.1} + \ldots\ldots\right]$$

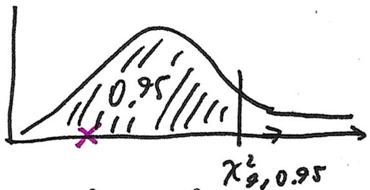
$$G^2_{obs} = 2\left[\left\{1249 \cdot \log\frac{1249}{1236.1} - (1249 - 1236.1)\right\} + \ldots\right]$$

3. **What is the asymptotic distribution of X^2 and G^2 under H_0?**

$$\chi^2_{9 = 10-1}$$

4. **Knowing that the quantile of order 95% of a χ^2_ν with $\nu = 9$ is 16.9 can we conclude that our power-law model fits the data well?**

$$\chi^2_{9,0.95} = 16.9$$



$$X^2_{obs} = 2.995 < \chi^2_{9,0.95}$$ Fail to reject H_0

$$G^2_{obs} = 45.791 > \chi^2_{9,0.95}$$ reject H_0

at significance level of $\alpha = 0.05$

5. **For which test between X^2_{obs} and G^2_{obs} would you expect the χ^2 approximation to be more reliable?**

X^2

6. **Specify the formula of the** asymptotic **p-values to test H_0 using X^2 and G^2.**

p-value $= P(\chi^2_9 > X^2_{obs})$

p-value $= P(\chi^2_9 > G^2_{obs})$

7. **Suppose θ and τ are unknown. If we were to estimate them, would you expect the asymptotic distribution of X^2 and G^2 under H_0 to stay the same?**

$$\chi^2_{10-2-1} = \chi^2_7$$

10: # bins

2: # of parameters being estimated