INFN School of Statistics 2022
Paestum, Italy

Probability Theory
Excercises / Notebooks

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Outline

1. Exercises
   - 1.3
   - 2.3
2. Notebooks

Outline

Exercise 1.3

Prove

$$P(A) + P(\bar{A}) = 1,$$
$$P(A + B) = P(A) + P(B) - P(AB).$$

Exercise 2.3

Solve

$$\frac{dP_k}{dt} = -qP_k + qP_{k-1} \text{ and show that}$$
$$P_k = \text{Poisson}(k, a) = \frac{e^{-a} a^k}{k!},$$

where the mean count is $a = qt$.

Exercise 1.3

Prove

$$\begin{aligned} P(A) + P(\bar{A}) &= 1, \\ P(A + B) &= P(A) + P(B) - P(AB). \end{aligned}$$

$$A + B = B + A \quad (1)$$
$$A + (BC) = (A + B)(A + C) \quad (2)$$
$$A + \emptyset = A \quad (3)$$
$$A + \bar{A} = \Omega \quad (4)$$

$$AB = BA \quad (5)$$
$$A(B + C) = AB + AC \quad (6)$$
$$A\Omega = A \quad (7)$$
$$A\bar{A} = \emptyset \quad (8)$$

$$P(A) \geq 0 \quad (9)$$
$$P(\Omega) = 1 \quad (10)$$
$$P(A + B) = P(A) + P(B) \quad AB = \emptyset. \quad (11)$$

Lemmas and Theorems

$$A + A = A$$
$$A + \Omega = \Omega$$
$$A\emptyset = \emptyset$$
$$AA = A$$

$$\bar{\emptyset} = \Omega$$
$$\bar{\Omega} = \emptyset$$
$$A + AB = A$$
$$A(A + B) = A$$

De Morgan's Laws

$$\overline{A + B} = \bar{A}\,\bar{B}$$
$$\overline{AB} = \bar{A} + \bar{B}$$

Exercise 1.3

Prove

$$P(A) + P(\bar{A}) = 1.$$

Proof

$$\begin{aligned} A + \bar{A} &= \Omega && \text{(axiom 4)} \\ \therefore\ P(A + \bar{A}) &= P(\Omega) && \\ P(A + \bar{A}) &= 1 && \text{(axiom 10)} \\ A\bar{A} &= \emptyset && \text{(axiom 8)} \\ \therefore\ P(A) + P(\bar{A}) &= 1 && \text{(axiom 11)} \end{aligned}$$

Exercise 1.3

Prove

$$P(A+B) = P(A) + P(B) - P(AB).$$

Proof

$$\begin{aligned} A + B + \overline{A+B} &= \Omega && \text{(axiom 4)} \\ \therefore\ P(A+B) + P(\overline{A+B}) &= P(\Omega) \\ \therefore\ P(A+B) &= 1 - P(\overline{A+B}) && \text{(axiom 10)} \\ P(A+B) &= 1 - P(\bar{A}\bar{B}) && \text{(De Morgan's Theorems)} \end{aligned}$$

$$
\begin{aligned}
P(A+B) &= 1 - P(\bar{A}\bar{B}) \\
P(A+B) &= 1 - P(\bar{B}\bar{A}) && \text{(axiom 5)} \\
&= 1 - P(\bar{B}|\bar{A})\,P(\bar{A}) && \text{(by definition)} \\
&= 1 - [1 - P(B|\bar{A})]\,P(\bar{A}) && \text{(axiom 11)} \\
&= 1 - P(\bar{A}) + P(B\bar{A}) && \text{(by definition)} \\
&= 1 - P(\bar{A}) + P(\bar{A}B) && \text{(axiom 5)} \\
&= P(A) + P(\bar{A}B) && \text{(theorem)} \\
&= P(A) + P(\bar{A}|B)P(B) && \text{(by definition} \\
&= P(A) + [1 - P(A|B)]P(B) && \text{(axiom 11)} \\
&= P(A) + P(B) - P(A|B)P(B) \\
&= P(A) + P(B) - P(AB) && \text{(by definition)}
\end{aligned}
$$

Exercise 2.3

Solve

$$\frac{dP_k}{dt} = -qP_k + qP_{k-1} \text{ and show that}$$

$$P_k = \text{Poisson}(k, a) = \frac{e^{-a}a^k}{k!},$$

where the mean count is $a = qt$.

Proof

$$\frac{dP_k}{da} + P_k = P_{k-1},$$

$$\frac{d}{da}\left(e^a P_k\right) = e^a\, P_{k-1},$$

$$\text{For } k = 0, \quad P_0 = Ae^{-a}.$$

$$\begin{aligned}
\frac{d}{da}\left(e^a P_k\right) &= e^a\, P_{k-1}, \\
P_k &= Ae^{-a} \int e^a\, P_{k-1}\, da, \\
P_1 &= Ae^{-a} \int da \quad = Ae^{-a} a, \\
P_2 &= Ae^{-a} \int a\, da \quad = Ae^{-a} a^2/2, \\
\vdots & \quad \vdots \\
P_n &= Ae^{-a} \int \frac{a^{n-1}}{(n-1)!}\, da \quad = Ae^{-a} a^n/n!,
\end{aligned}$$

$\sum_{n=0}^{\infty} P_n = 1, \implies A = 1.$

Outline

Instructions

It is convenient to use a software environment system, such as Anaconda, or its slimmed-down version miniconda3.

See instructions at

```
https://www.github.com/hbprosper/INFN-SOS
```

Download

```
cd
mkdir -p Projects
git clone https://www.github.com/hbprosper/INFN-SOS
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