The muon g-2 and Lattice QCD

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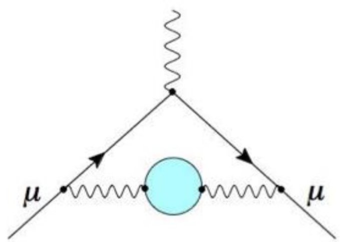
INFN

g_μ-2 and LQCD

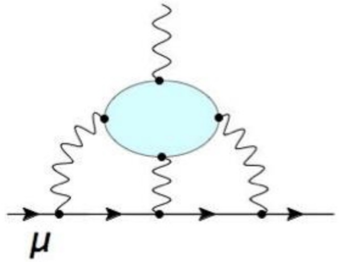
The hadronic contributions

There are 2 relevant hadronic contributions to the muon anomalous magnetic moment



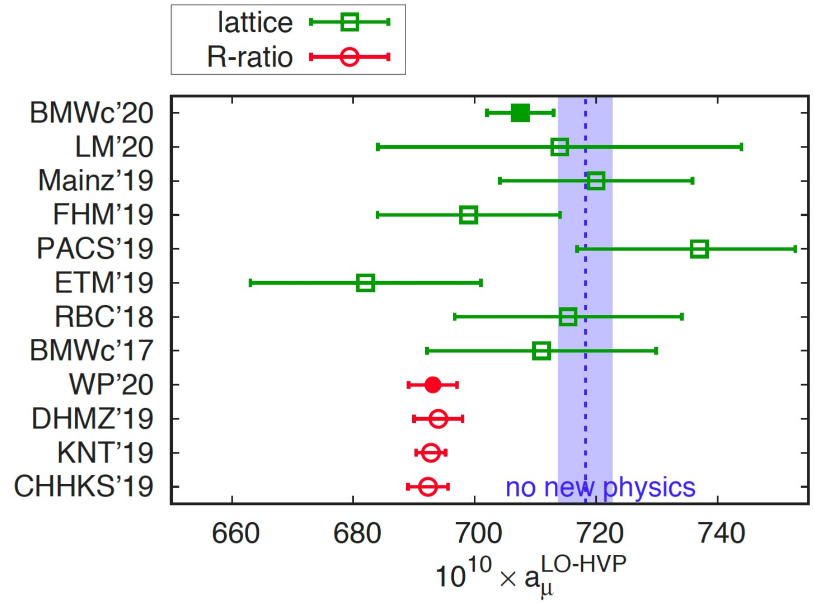
Hadronic Vacuum Polarization (HVP)

α^2



Hadronic Light-by-Light scattering (HLbL)

α^3



The aim of this talk is to illustrate the lattice approach to the calculation of the hadronic contributions to $(g\text{-}2)_\mu$

g_μ-2 and LQCD

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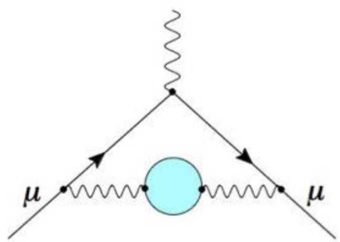

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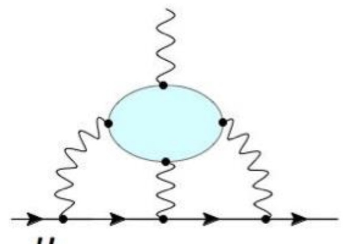
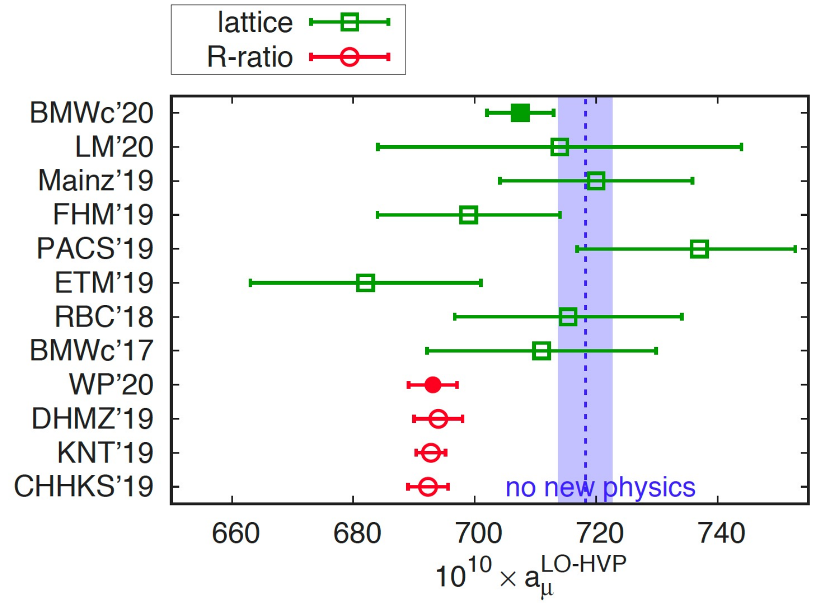


MOST OF THE TALK

(HLbL)

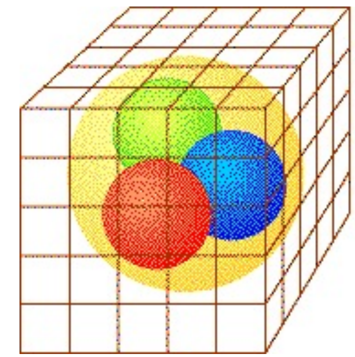
α^3



The aim of this talk is to illustrate the lattice approach to the calculation of the hadronic contributions to $(g\text{-}2)_\mu$

Overview

- The Hadronic Vacuum Polarization
  - The electromagnetic current correlator and the time-momentum representation
- Lattice QCD and numerical simulations
  - The path integral and the lattice regularization
  - The lattice QCD action
  - Montecarlo methods and importance sampling



- The HVP correlation function
  - Connected and disconnected diagrams
  - Statistical errors and noise reduction
  - Scale setting and continuum extrapolation
  - Finite size effects

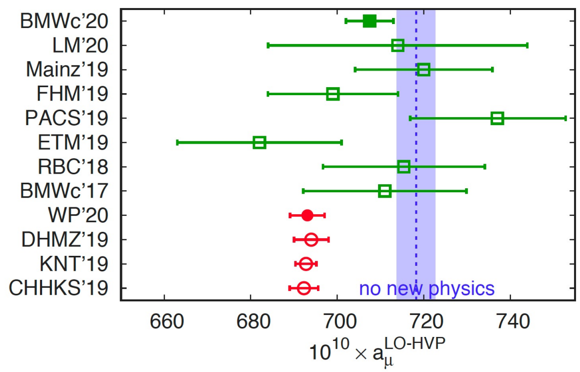
Main challenges when aiming at sub-percent accuracy

Overview

- Isospin breaking effects
  - Strong and electromagnetic isospin breaking
  - The expansion in $m_d - m_u$ and α

} Crucial for sub-percent accuracy

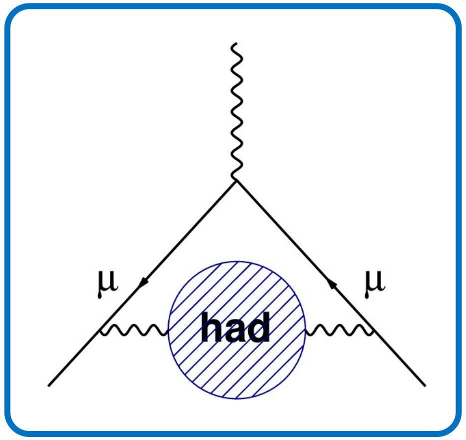
- HVP: lattice results
  - The various contributions
  - Results and comparisons
  - Crosschecks: the window observable



- The Hadronic Light-by-Light
- Conclusions

An excellent reference is the White Paper 2020:
Phys.Rept. 887 (2020), 1-166. e-Print: 2006.04822 [hep-ph]

The Hadronic Vacuum Polarization



The basic ingredient of any lattice calculation of the HVP contribution to the muon anomalous magnetic moment is

$$C_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle$$

the electromagnetic current correlator

where

$$j_\mu(x) = \frac{2}{3}\left(\bar{u}\gamma_\mu u\right)(x) - \frac{1}{3}\left(\bar{d}\gamma_\mu d\right)(x) - \frac{1}{3}\left(\bar{s}\gamma_\mu s\right)(x) + \ldots$$

Its Fourier transform is the Vacuum Polarization tensor

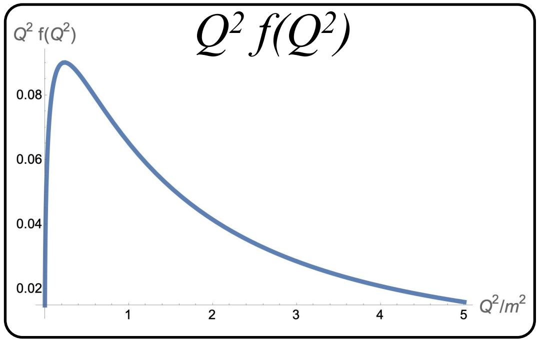
$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x}\, C_{\mu\nu}(x)$$

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right)\Pi(Q^2)$$

The Hadronic Vacuum Polarization

- Traditionally, the leading hadronic contribution to the muon anomalous magnetic moment is expressed as

$$a_\mu^{HVP,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2)\hat{\Pi}(Q^2)$$



where $\hat{\Pi}(Q^2) \equiv 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$ and $f(Q^2)$ is a known function.

The Hadronic Vacuum Polarization

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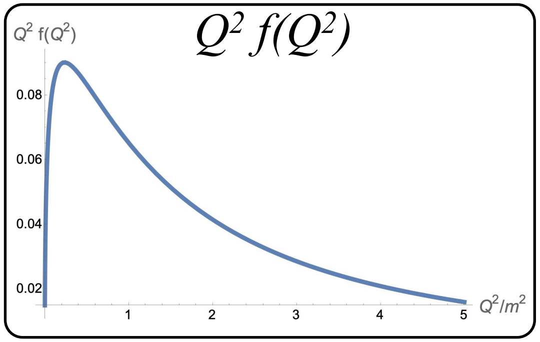


where $\hat{\Pi}(Q^2) \equiv 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$ and $f(Q^2)$ is a known function.

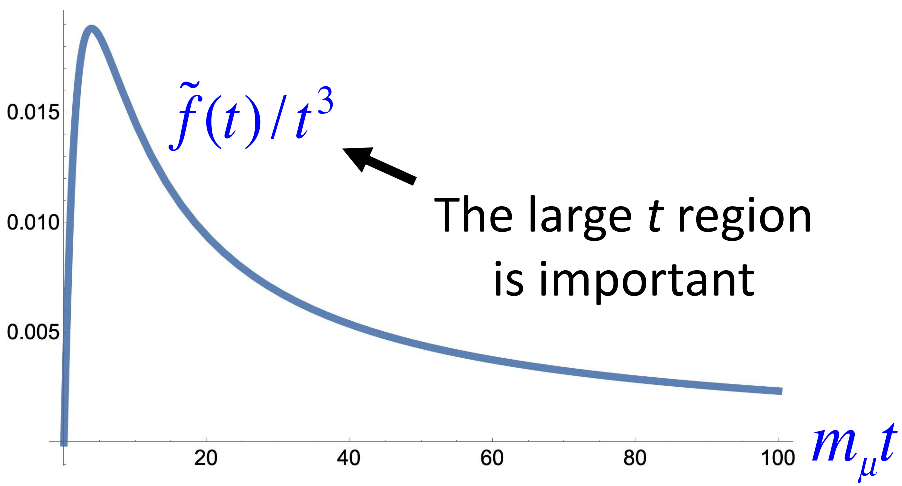
- Most of recent lattice calculations of $a_\mu^{HVP,LO}$, however, start from the so-called time-momentum representation. Choosing $Q_\mu = (\omega, 0, 0, 0)$, the order of the integrals in d^4x e dQ^2 is inverted, and one finds:

The time-momentum representation

$$a_\mu^{HVP,LO} = 8\alpha^2 \int_0^\infty dt\, \tilde{f}(t)\, C(t)$$

$$C(t) = -\frac{1}{3}\sum_{k=1}^{3}\int_{-\infty}^{+\infty} d^3x \left\langle j_k(\vec{x},t)\, j_k(0)\right\rangle$$

$$\tilde{f}(t) = t^2 \int_0^\infty \frac{d\omega}{\sqrt{4+\omega^2}}\left(\frac{\sqrt{4+\omega^2}-\omega}{\sqrt{4+\omega^2}+\omega}\right)^2\left[1-\frac{\sin^2(m_\mu \omega t/2)}{(m_\mu \omega t/2)^2}\right]$$

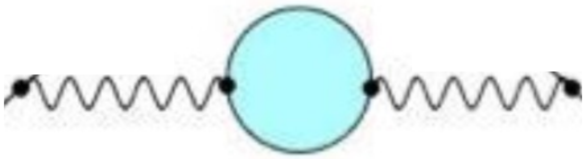


This is the input from lattice QCD

Lattice QCD and numerical simulations

We now discuss how a QCD correlation functions, like

$$C(t) = -\frac{1}{3}\sum_{k=1}^{3}\int_{-\infty}^{+\infty} d^3x \, \langle j_k(\vec{x},t)\, j_k(0) \rangle$$



which is intrinsically non-perturbative at long distance,
is computed with a

Lattice QCD
numerical simulation

The path integral

The theoretical framework for Lattice QCD calculations is the path integral approach in Euclidean time:

$$G(x_1,\ldots,x_n) = \langle \varphi(x_1)\ldots\varphi(x_x) \rangle = \frac{1}{Z}\int [d\varphi]\,\varphi(x_1)\ldots\varphi(x_x)\,e^{-S(\varphi)}$$

with

$$Z = \int [d\varphi]\,e^{-S(\varphi)}$$

Wick rotation: $t \to -i\,t_E$

Essential for the numerical evaluation of the path integral: $e^{iS(\varphi)} \to e^{-S(\varphi)}$

The path integral

The theoretical framework for Lattice QCD calculations is the path integral approach in Euclidean time:

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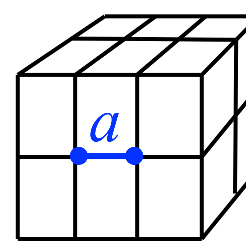
with $Z=\int[d\varphi]\,e^{-S(\varphi)}$

Wick rotation: $t\rightarrow -i\,t_E$

- A mathematical definition of the path integral is obtained by discretizing the space-time on a 4-dimensional lattice:

$$\varphi(x)\rightarrow\varphi_n\equiv\varphi(x_n)\quad,\quad[d\varphi]\rightarrow\Pi_n d\varphi_n$$

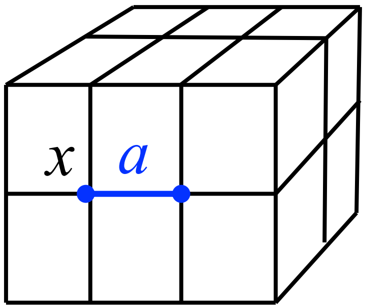


The path integral then reduces to an ordinary multidimensional integral

The lattice regularization

- The fields are defined on a (hypercubic) 4-dimensional lattice

$$\varphi(x) \to \varphi(an) \qquad n = (n_x, n_y, n_z, n_t)$$



- Derivatives are replaced by finite differences

$$\partial_\mu \varphi(x) \to \nabla_\mu \varphi(x) \equiv \frac{\varphi(x + a\hat{\mu}) - \varphi(x)}{a}$$

- Several discretizations are possible:

$$\nabla^*_\mu \varphi(x) \equiv \frac{\varphi(x) - \varphi(x - a\hat{\mu})}{a} \quad , \quad \tilde{\nabla}_\mu \varphi(x) \equiv \frac{\varphi(x + a\hat{\mu}) - \varphi(x - a\hat{\mu})}{2a}$$

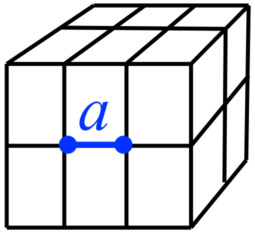
- In the continuum limit a→0: $\nabla_\mu q \to \partial_\mu q + O(a) \quad , \quad \tilde{\nabla}_\mu q \to \partial_\mu q + O(a^2)$

The lattice regularization

The functional integral is only a formal definition, because of ultraviolet and infrared divergences.

The lattice provides both an ultraviolet and an infrared cutoff

1) The ultraviolet cutoff



$$\tilde{\varphi}(p) = \sum_x \varphi(x)\, e^{ip\cdot x} = \sum_n \varphi(an)\, e^{iap\cdot n} \;\longrightarrow\; \tilde{\varphi}\left(p + \frac{2\pi}{a}\right) = \tilde{\varphi}(p)$$

$x = a\,n$

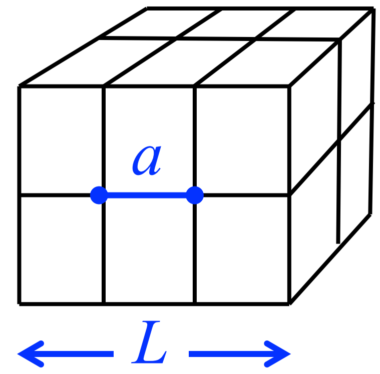
⇨ The momentum p is cutoff at the first Brillouin zone

$$|\,p\,| \le \frac{\pi}{a}$$

1/a is the ultraviolet cutoff

2) The infrared cutoff

The lattice is defined in a finite volume



For periodic boundary conditions

$$\phi(x+L)=\phi(x) \quad , \quad \varphi(x)=\frac{a^3}{L^3}\sum_p \tilde{\varphi}(p)\,e^{-ip\cdot x}$$

$$\Rightarrow \quad p_i=\frac{2\pi}{L}k_i \quad \rightarrow \quad p_i=0,\ \frac{2\pi}{L},\ \frac{4\pi}{L},\ \ldots,\ \frac{2\pi}{L}(L/a-1)$$

$$k_i=0,\ 1,\ 2,\ \ldots,\ L/a-1$$

$p=0$ is just a single point, it can be removed

The lattice regularization

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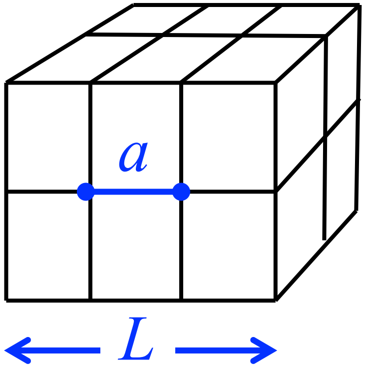

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The physical theory is obtained in the limits

a → 0 Continuum limit , L → ∞ Thermodynamic limit

g_μ-2 and LQCD

The lattice regularization

The simplest example: the scalar free theory on the lattice

The Euclidean free action in the continuum is:

$$S_0 = \int d^4x \, \frac{1}{2}\left[\left(\partial_\mu \varphi\right)^2 + m^2\varphi^2\right] = \int d^4x \, \frac{1}{2}\varphi\left[-\partial^2 + m^2\right]\varphi$$

The scalar propagator is the inverse of the action operator:

$$\left[-\partial^2 + m^2\right]S(x,0) = \delta(x) \quad , \quad S(x) = \int \frac{d^4p}{(2\pi)^4} S(p)\, e^{-ip\cdot x}$$

$$\Rightarrow \quad S(p) = \left(p^2 + m^2\right)^{-1}$$

The lattice regularization

- On the lattice

In the action, derivatives are replaced by discretized derivatives:

$$\left[-\partial^2+m^2\right]S(x,0)=\delta(x) \quad \longrightarrow \quad \boxed{\left[-\nabla^2+m^2\right]S(x,0)=\delta_{x,0}}$$

$$\rightarrow \quad \left[-\nabla^2+m^2\right]S(x)=-\sum_\mu\left[\frac{S(x+a\hat{\mu})-2S(x)+S(x-a\hat{\mu})}{a^2}\right]+m^2S(x)=\delta_{x,0}$$

$$S(x)=\frac{1}{V}\sum_p S(p)\,e^{-ip\cdot x} \quad \Rightarrow \quad \boxed{S_{latt}(p)=\left(\tilde{p}^2+m^2\right)^{-1}} \quad , \quad \boxed{\tilde{p}_\mu\equiv\frac{2}{a}\operatorname{sen}\left(\frac{ap_\mu}{2}\right)}$$

Compare with continuum propagator: $S_{cont}(p)=\left(p^2+m^2\right)^{-1}$

The lattice QCD action

In lattice QCD, as in any other regularization of QCD,
gauge invariance must be exactly preserved

The lattice QCD action

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- In the continuum, gauge invariance is preserved introducing the covariant derivative:

$$\bar{q}(x)\partial_\mu q(x) \;\rightarrow\; \bar{q}(x)D_\mu q(x)$$

$$=\bar{q}(x)\left(\partial_\mu + igA_\mu\right)q(x)$$

$$q(x)\rightarrow V(x)q(x)$$

$$D_\mu q(x)\rightarrow V(x)D_\mu q(x)$$

$$V(x)\in SU(3)$$

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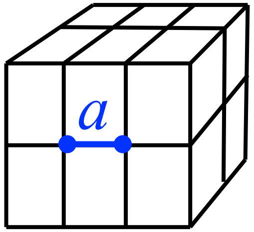
- On the lattice, finite differences are involved:

$$\bar{q}(x) \nabla_\mu q(x) =$$

$$= \frac{1}{a} \bar{q}(x) \left[q(x + a\hat{\mu}) - q(x) \right]$$

It must be rendered gauge invariant



The lattice QCD action

- The solution is known: introduce a "parallel transporter"

$$\bar{q}(x)\,q(y) \;\rightarrow\; \bar{q}(x)\,\underbrace{P(x,y)\,q(y)}$$

- Under gauge transformation: $\bar{q}(x)\,V(x)^{+}\left[V(x)\,P(x,y)\,V(y)^{+}\right]V(y)\,q(y)$

- The explicit form of $P(x,y)$ can be derived by observing that, for infinitesimal transport, it defines the covariant derivative:

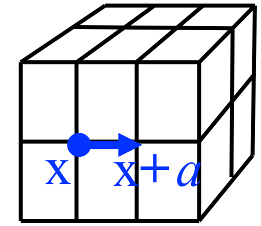
$$\lim_{a\to 0}\frac{1}{a}\left[P(x,x+a\hat{\mu})\,q(x+a\hat{\mu})-q(x)\right]=D_\mu q(x)=\left(\partial_\mu+igA_\mu(x)\right)q(x) \;\rightarrow$$

$$\lim_{a\to 0}\frac{1}{a}\left[P(x,x+a\hat{\mu})-1\right]=igA_\mu(x) \;\rightarrow\; P(x,x+a\hat{\mu})\simeq e^{iagA_\mu(x)}$$

The lattice QCD action

- On the lattice, the parallel transporter on a single lattice site, called link, is the dynamical gauge variable

$$U_\mu(x) = e^{iagA_\mu(x)}$$



- Using the links one can define the lattice covariant derivatives

$$\nabla_\mu q(x) \equiv \frac{1}{a}\left[U_\mu(x)\, q(x+a\hat{\mu}) - q(x)\right]$$

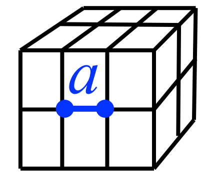
$$\nabla^*_\mu q(x) \equiv \frac{1}{a}\left[q(x) - U_\mu(x)^+\, q(x-a\hat{\mu})\right]$$

$$\tilde{\nabla}_\mu q(x) \equiv \frac{1}{2}\left(\nabla_\mu + \nabla^*_\mu\right) q(x) = \frac{1}{2a}\left[U_\mu(x)\, q(x+a\hat{\mu}) - U_\mu(x)^+\, q(x-a\hat{\mu})\right]$$

- In the continuum limit a→0: $\nabla_\mu q \to D_\mu q + O(a)$, $\tilde{\nabla}_\mu q \to D_\mu q + O(a^2)$

The quark action

- By having a definition of the lattice covariant derivative, the construction of a gauge invariant fermionic action is "almost" straightforward:



Continuum → **Lattice**

$$S_F = \int \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \overleftrightarrow{D}_\mu + m\right]q(x) \longrightarrow S_F = a^4 \sum_x \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \tilde{\nabla}_\mu + m\right]q(x)$$

$\tilde{\nabla}_\mu$: lattice covariant derivative

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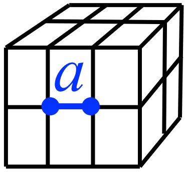


Continuum

$$S_F = \int \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \overleftrightarrow{D}_\mu + m\right]q(x)$$

→

Lattice

$$S_F = a^4 \sum_x \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \tilde{\nabla}_\mu + m\right]q(x)$$

lattice covariant derivative

- The previous form of the action is incorrect, however, because of the doublers problem (not discussed here). The correct form is

$$S_F = a^4 \sum_x \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \tilde{\nabla}_\mu - \frac{ar}{2}\nabla^2 + m\right]q(x)$$

Wilson term

K. Wilson 1975

The pure gauge action

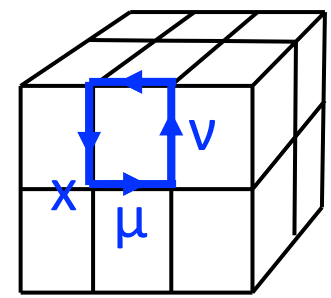
- In the continuum, the gauge action is written in terms of the field tensor

$$S_G = \frac{1}{2}\int d^4x\, Tr\left(G_{\mu\nu}G_{\mu\nu}\right)$$

$$G_{\mu\nu}(x) \rightarrow V(x)G_{\mu\nu}(x)V(x)^+$$

- On the lattice, the simplest pure gauge operator which transforms as $G_{\mu\nu}$ is the plaquette, i.e. the smallest closed loop

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu(x+a\nu)^+ U_\nu(x)^+$$



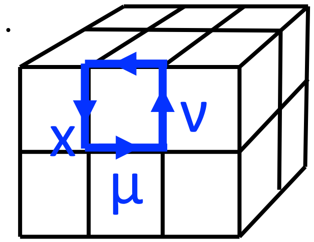
$$P_{\mu\nu}(x) \rightarrow V(x)P_{\mu\nu}(x)V(x)^+ \quad \longrightarrow \quad Tr\left[P_{\mu\nu}(x)\right] \rightarrow Tr\left[P_{\mu\nu}(x)\right]$$

Same as $G_{\mu\nu}$

The trace of the plaquette is gauge invariant

The pure gauge action

- Expanding the plaquette up to $O(a^2)$, using $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$



$$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu(x+a\nu)^+ U_\nu(x)^+ =$$

$$= e^{iagA_\mu(x)} e^{iagA_\nu(x+a\hat{\mu})} e^{-iagA_\mu(x+a\hat{\nu})} e^{-iagA_\nu(x)} = e^{ia^2 g\left(\partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right] + O(a)\right)}$$

$$\rightarrow \quad \boxed{P_{\mu\nu}(x) = e^{i a^2 g \left(G_{\mu\nu}(x) + O(a)\right)}} \quad \rightarrow \quad \mathrm{Re}\left(P_{\mu\nu}\right) \simeq 1 - \frac{1}{2} a^4 g^2 G_{\mu\nu} G_{\mu\nu} + \ldots$$

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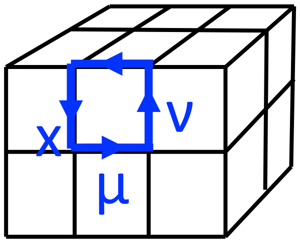

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$$\rightarrow \quad P_{\mu\nu}(x) = e^{i a^2 g\left(G_{\mu\nu}(x) + O(a)\right)} \quad \rightarrow \quad \mathrm{Re}\left(P_{\mu\nu}\right) \simeq 1 - \frac{1}{2} a^4 g^2 G_{\mu\nu} G_{\mu\nu} + \dots$$

$$\Rightarrow \quad S_G = \beta \sum_{x,\,\mu>\nu} \left[1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}\left(P_{\mu\nu}(x)\right)\right]$$

$$\beta = \frac{2N}{g^2}$$

K. Wilson 1974

The lattice QCD action

- Summarizing: the lattice QCD Wilson action is $S = S_G + S_F$ with:

$$S_G = \beta \sum_P \left[1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}\left(P\right) \right]$$

$$S_F = a^4 \sum_x \bar{q}(x) \left[\frac{1}{2} \gamma_\mu \tilde{\nabla}_\mu - \frac{ar}{2} \nabla^2 + m \right] q(x)$$

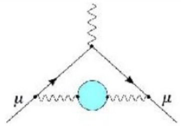
In the limit $a \to 0$:

$$S_G = \frac{1}{2} \int d^4x \; Tr\left(G_{\mu\nu}(x) G_{\mu\nu}(x) \right)$$

$$S_F = \int \bar{q}(x) \left[\frac{1}{2} \gamma_\mu \overleftrightarrow{D}_\mu + m \right] q(x)$$

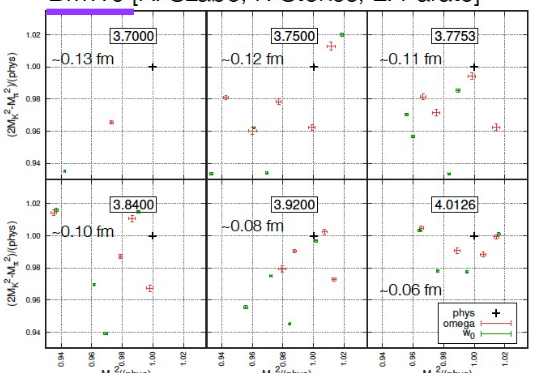
We may define an infinite number of lattice actions which all converge to the same continuum action but with different rate and symmetries

Wilson-clover, Twisted mass, Staggered, Domain Wall, Overlap …



Lattice HVP: Ensemble parameters

BMWc [K. Szabo, F. Stokes, L. Parato]



- Stout-smeared staggered fermions 2+1+1
- $L \sim 6 - 11$ fm

ETMc [D. Giusti]

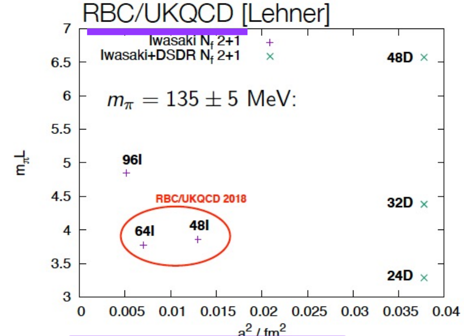
Pion masses in the range 220 - 490 MeV

4 volumes @ $M_\pi \simeq 320$ MeV and $a \simeq 0.09$ fm

$M_\pi L \simeq 3.0 \div 5.8$

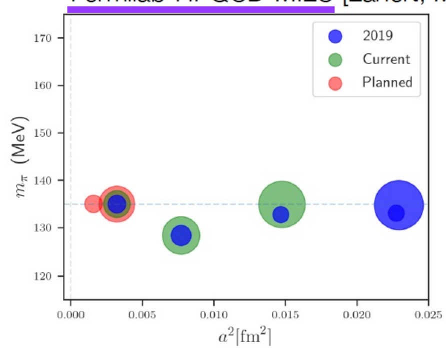
- Twisted-mass Wilson fermions, 2+1+1
- Plan to include phys. mass ensemble in future

RBC/UKQCD [Lehner]

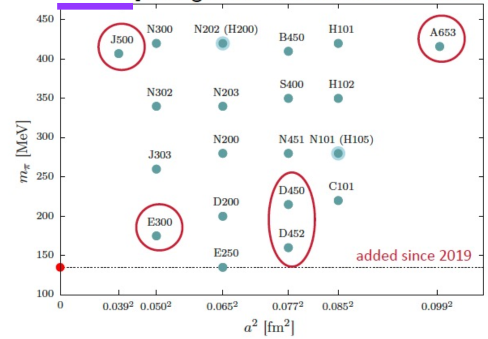


- Domain Wall fermions

Fermilab-HPQCD-MILC [Lahert, McNeile]



Mainz [Wittig, Risch, Miura, San Jose, Chao]



- Wilson-clover fermions, 2+1
- $L \sim 5 - 6$ fm

- Highly Improved Staggered Quarks (HISQ) 2+1+1+1
- $L \sim 5 - 6$ fm
- Subset also used by Aubin et al [Aubin]

Numerical simulations

The QCD path integral on the lattice is reduced to an ordinary multidimensional integral

$$\langle O(U,q,\bar{q})\rangle = \frac{1}{Z}\int [dU][dq][d\bar{q}]\, O(U,q,\bar{q})\, e^{-S_G[U]-S_F[U,q,\bar{q}]}$$

It can be evaluated, numerically, with

Montecarlo methods

Integral over fermionic variables

- Fermions in the path integral are represented by anticommuting Grassmann variables.
- The action is quadratic in the quark fields and the integral over the Grassmann variables can be performed analytically. For example:

$$\langle q(x)\bar{q}(y)\rangle = Z^{-1}\int [dU][dq][d\bar{q}]\, q(x)\bar{q}(y)\, e^{-S_G[U]-\sum_{x',y'}\bar{q}_{x'}M[U]_{x'y'}q_{y'}} =$$

$$= Z^{-1}\int [dU]\, M[U]^{-1}_{x\,y}\, \det M[U]\, e^{-S_G[U]}$$

Effective action

$$\Rightarrow \quad \langle q(x)\bar{q}(y)\rangle = \frac{1}{Z}\int [dU]\, M[U]^{-1}_{x\,y}\, e^{-S_G[U]+\mathrm{Tr}\log M[U]}$$

- For products of more quark fields the Wick theorem holds

The Montecarlo method

- The remaining integral is over gauge fields:

$$\langle O(x,y\ldots)\rangle = \frac{1}{Z}\int [dU]\, O[U](x,y\ldots)\, e^{-S[U]}$$

This is like a statistical Boltzmann system with $S = \beta H$

- Its dimension is L^4 for each scalar field (for $L=32 \rightarrow L^4 \approx 10^6$).

It can be only evaluated with a **Montecarlo method**

For a 1–D function:

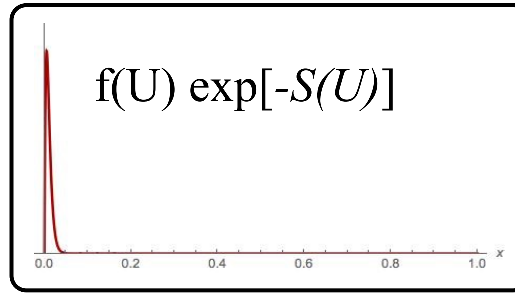
$$I = \int_0^1 dx\, f(x) \simeq \frac{1}{N}\sum_{k=1}^{N} f(x_k) \quad , \quad x_k = \mathrm{Random}[0,1] \longrightarrow$$

$$\langle O\rangle = \frac{1}{Z}\int [dU]\, O(U) e^{-S(U)} \simeq \frac{1}{N_C}\sum_{C=1}^{N_C} O(U_C) e^{-S(U_C)}$$

The fields are extracted with uniform weight

The importance sampling

- Because of the Boltzmann weight exp[-$S(U)$], only a small region of the configuration space gives a relevant contribution to the functional integral. The procedure is highly inefficient



⇨ Use the importance sampling technique

$$\langle O \rangle = \frac{1}{Z}\int [dU]\, O(U) e^{-S(U)} \simeq \frac{1}{N_C}\sum_{c=1}^{N_C} O(U_C) e^{-S(U_C)}$$

The fields are extracted with uniform weight

$$\langle O \rangle = \frac{1}{Z}\int [dU]\, O(U) e^{-S(U)} \simeq \frac{1}{N_C}\sum_{C=1}^{N_C} O(U_C)$$

The fields are extracted with weight ~ exp[-$S(U)$]

Statistical errors

The multidimensional integral is replaced by a sum over a finite number of field configurations sampled with weight exp[-S]:

$$\langle O \rangle = \frac{1}{Z}\int [dU]\, O(U) e^{-S(U)} \simeq \frac{1}{N_C}\sum_{C=1}^{N_C} O(U_C)$$

The estimate is affected by statistical error which decrease like $1/\sqrt{N_C}$:

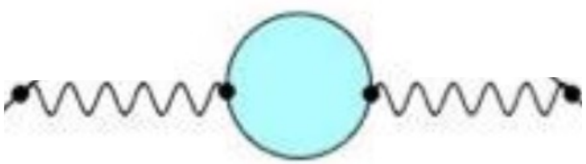
$$\langle \Delta O^2 \rangle = \frac{1}{N_C}\sum_{C=1}^{N_C}\left(O(U_C) - \langle O \rangle\right)^2$$

In order to ensure enough statistical precision, many gauge field configurations must be generated

The HVP correlation function

We now have all the ingredients required to discuss the numerical calculation of the

HVP correlation function in Lattice QCD



$$C(t) = -\sum_{\vec{x}} \langle j_k(\vec{x},t)\, j_k(0) \rangle$$

$$j_k(x) = \sum_f Q_f \left(\bar{q}_f \gamma_k q_f \right)(x)$$

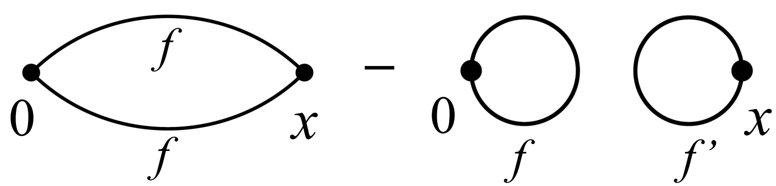
Connected and disconnected diagrams

1 The integral over fermionic fields is evaluated analytically

Performing the Wick contractions:

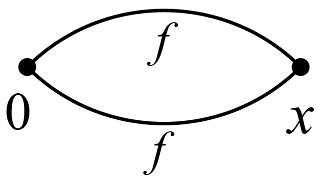
$$C(t) = -\sum_{\vec{x}} \langle j_k(x) j_k(0) \rangle = -\sum_{f,f'} Q_f Q_{f'} \sum_{\vec{x}} \left\langle \left(\bar{q}_f \gamma_k q_f \right)(x) \left(\bar{q}_{f'} \gamma_k q_{f'} \right)(0) \right\rangle =$$

$$= \sum_{f,f'} Q_f Q_{f'} \sum_{\vec{x}} \left[\delta_{ff'} \left\langle \mathrm{Tr}\left(S_f(x,0)\gamma_k S_f(0,x)\gamma_k \right) \right\rangle_U - \left\langle \mathrm{Tr}\left(S_f(x,x)\gamma_k \right) \mathrm{Tr}\left(S_f(0,0)\gamma_k \right) \right\rangle_U \right]$$

$=$ 

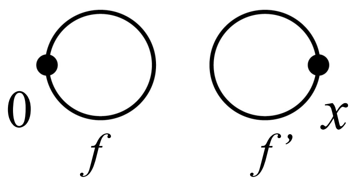
where $\langle \ldots \rangle_U$ denotes the average over gauge fields.

Connected and disconnected diagrams



$$= \sum_{\vec{x}} \left\langle \mathrm{Tr}\left(S_f(x,0)\,\gamma_k\, S_f(0,x)\,\gamma_k \right) \right\rangle_U$$

Connected diagram



$$= \sum_{\vec{x}} \mathrm{Tr}\left(S_f(x,x)\,\gamma_k \right) \mathrm{Tr}\left(S_{f'}(0,0)\,\gamma_k \right)$$

Disconnected diagram

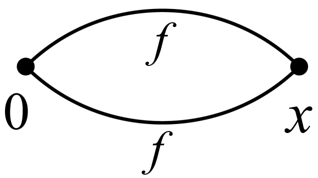
- For the connected diagram only $S_f(x,0)$ is needed. Solve the equation

$$\sum_y \Delta_f(z,y)\, S_f(y,x) = \delta(z,x)$$

for $x = 0$ only. For the disconnected contribution, $S_f(x,x)$ is needed for each x. A factor $\mathrm{F} = \mathrm{L}^4 \simeq 10^7$ more expensive. With stochastic methods: $\mathrm{F} \simeq 10^3$ - 10^4

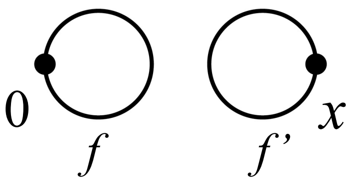
Connected and disconnected diagrams

2 The integral over gauge fields is performed numerically with the Montecarlo method:



$$= \frac{1}{N_C}\sum_{C=1}^{N_C}\sum_{\vec{x}} \mathrm{Tr}\left(S_f(U_C;x,0)\gamma_k S_f(U_C;0,x)\gamma_k\right)$$

Connected



$$= \frac{1}{N_C}\sum_{C=1}^{N_C}\sum_{\vec{x}} \mathrm{Tr}\left(S_f(U_C;x,x)\gamma_k\right)\mathrm{Tr}\left(S_f(U_C;0,0)\gamma_k\right)$$

Disconnected

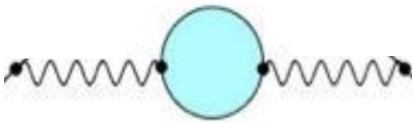
Sum over gauge configurations

The propagator is computed for each gauge configuration

The HVP correlation function

We now discuss the main sources of

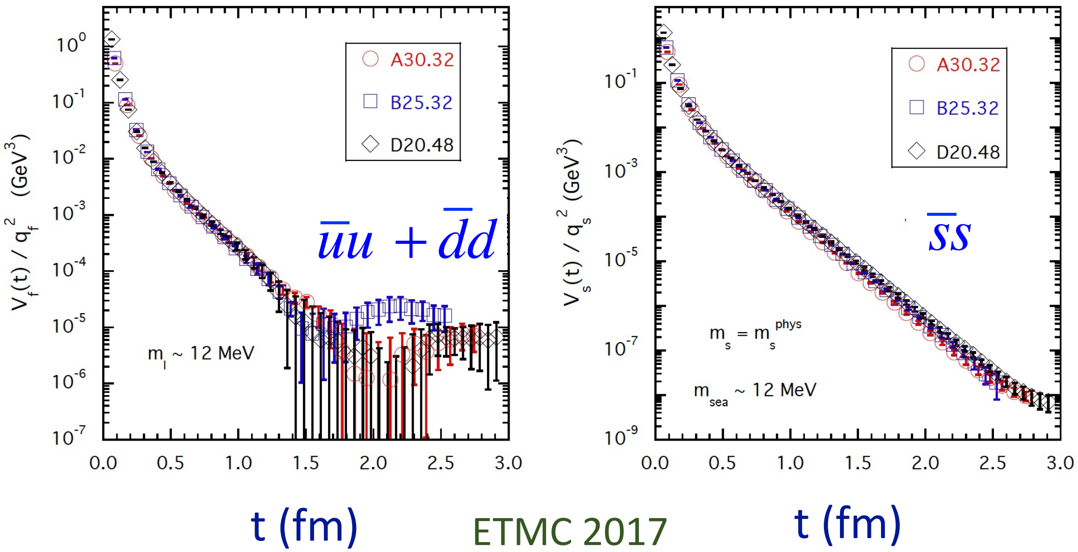
UNCERTAINTIES

in the lattice calculation of the correlation function 

1) Statistical errors and noise reduction
2) Scale setting and continuum extrapolation
3) Finite size effects

Statistical errors and noise reduction

- For light quark (u,d) contribution to the HVP, statistical errors increase rapidly at large times and the signal/noise ratio in $C(t)$ becomes small



ETMC 2017

Integral over all times

$$a_\mu^{HVP,LO} = 8\alpha^2 \int_0^\infty dt\, \tilde{f}(t)\, C(t)$$

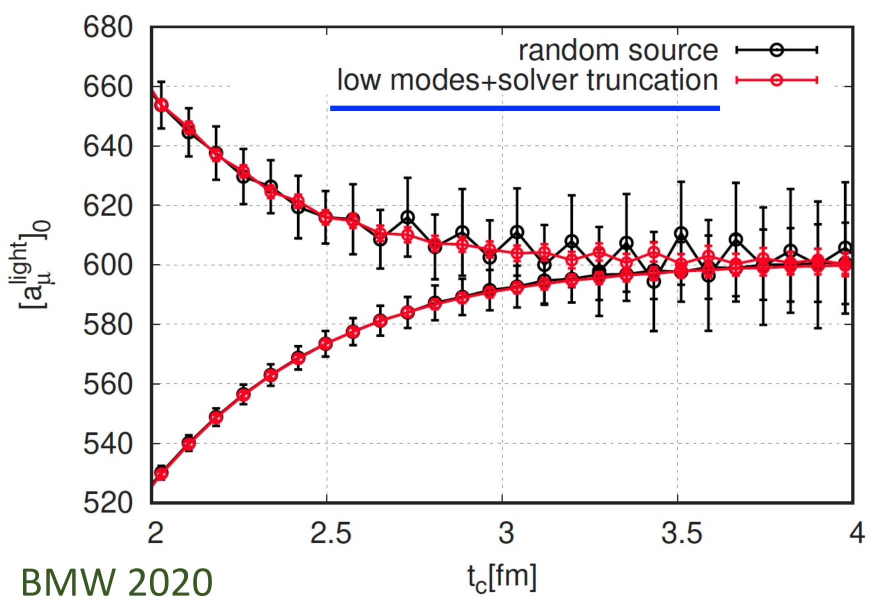
Specific of $a_\mu^{HVP,LO}$

→ Large statistics and dedicated techniques are required

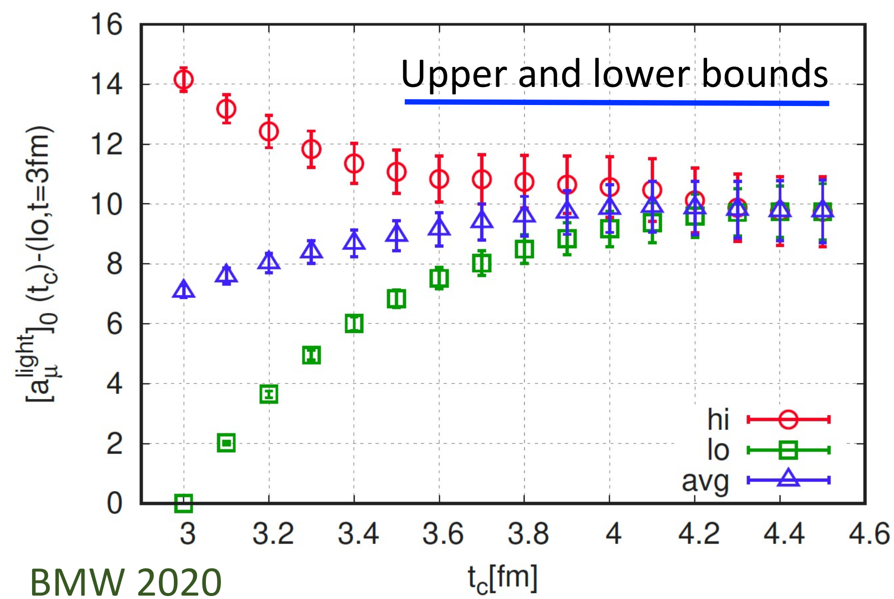
g_μ-2 and LQCD

Statistical errors and noise reduction

An example, from the BMW Collab.



For $t < t_c$

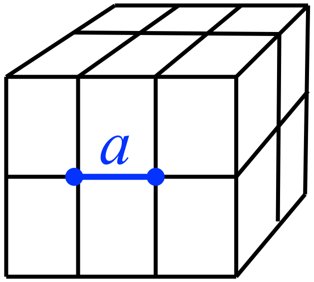


For $t > t_c$

Scale setting and continuum extrapolation

- In a lattice simulation, any dimensionful physical quantity turns out to be expressed in units of the lattice scale. For example:

$$M_h^{latt} = a\, m_h^{phys}$$



→ A (precise) determination of the lattice spacing is thus required.

Scale setting and continuum extrapolation

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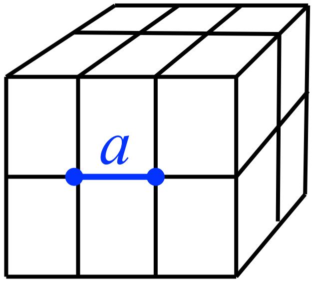

→ A (precise) determination of the lattice spacing is thus required.

- In a mass independent renormalization scheme, the bare coupling is only a function of the UV cutoff:

$$g_0^2 = g_0^2(a) \;\rightarrow\; a = a(g_0^2) \;\Rightarrow_{\text{At fixed } g_0^2}\; a = M_h^{latt} / m_h^{phys}$$

Scale setting and continuum extrapolation

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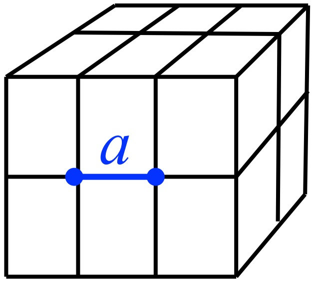
  For example:

$$M_h^{latt} = a\, m_h^{phys}$$



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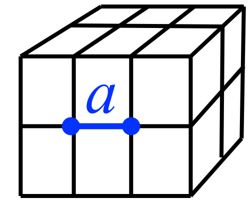
$$g_0^2 = g_0^2(a) \;\rightarrow\; a = a(g_0^2) \;\Rightarrow\; a = M_h^{latt} / m_h^{phys}$$

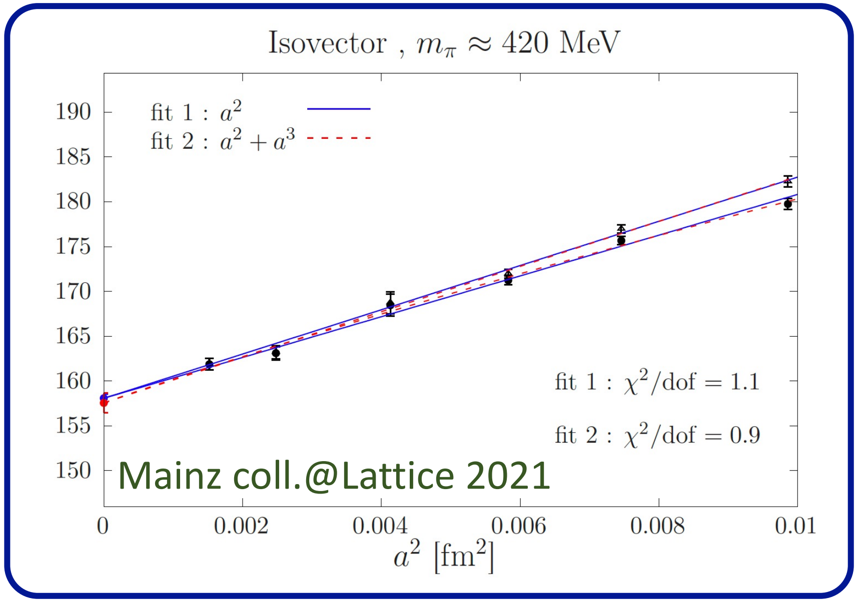
At fixed g_0^2

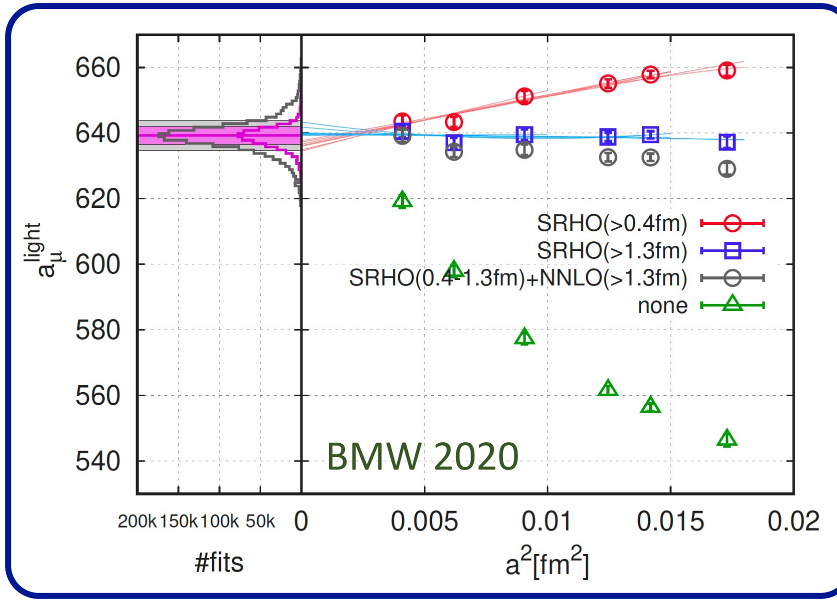
- Note: a_μ is dimensionless, but it depends on a through $a * m_\mu$ in the kernel.
  It is found that $\delta a_\mu^{HVP} / a_\mu^{HVP} \approx 1.8\, \delta a / a$ → High precision on a is required

Scale setting and continuum extrapolation

- Once the lattice spacing has been determined for each β, the physical result is obtained after a continuum extrapolation:

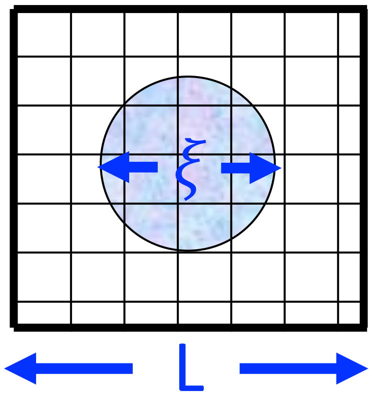






Improved lattice actions and other techniques may help in better controlling the extrapolation

Finite size effects



- In order to keep finite size effects (FSE) small:

$$L \gg \xi \simeq 1/m \quad \longrightarrow \quad mL \gg 1$$

In QCD $m \to m_\pi$, the lightest particle in the spectrum

- For a large class of physical amplitudes, including a_μ^{HVP}, FSE are exponentially small:

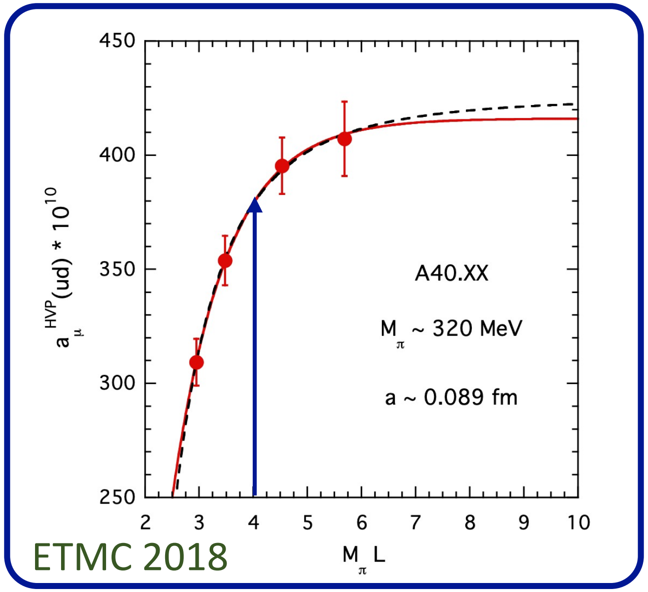
$$[C(L) - C(\infty)] \sim \exp(-m_\pi L) \quad \Rightarrow$$

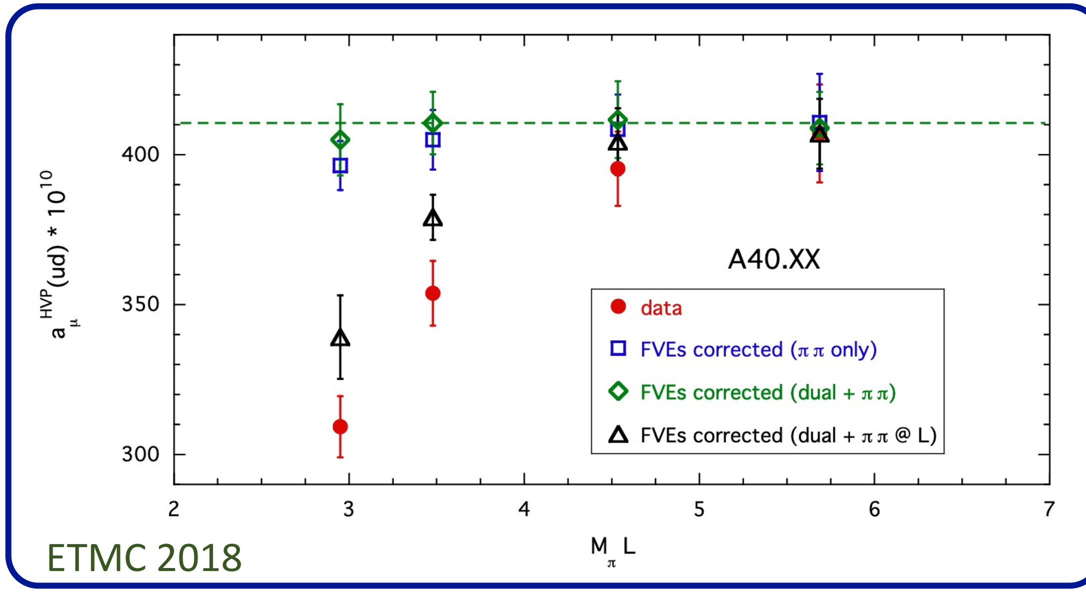
$m_\pi L \sim 4 \longrightarrow L \sim 6 \text{ fm}$ is typically required for percent accuracy

- Finite volume effects are controlled by long distance dynamics
  → Chiral Perturbation Theory (ChPT) may help in their evaluation

Finite size effects

- It turns out that for a_μ^{HVP} the condition $m_\pi L \sim 4$ is not sufficient: FSE are exponentially small, but the prefactor is large



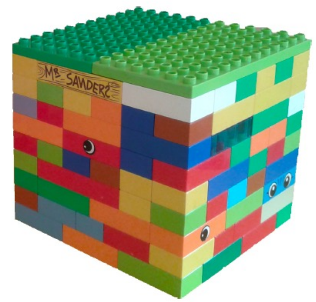


- For FSE in a_μ^{HVP}, NLO ChPT is not sufficient ⟶ N^2LO ChPT and other phenomenological models have been studied and tested with simulations on very large lattices ($L \simeq 11$ fm)

FINITE SIZE EFFECTS

Finite-size effects

- Typical lattice runs use $L \lesssim 6$ fm, earlier model estimates gave $O(2)\%$ FV effect [Aubin *et.al.* '16]

$L_{\text{ref}} = 6.272$ fm  

$L_{\text{big}} = 10.752$ fm

From B. C. Tóth
BMW Collab.
@ Lattice 2021

1. $a_\mu(\text{big}) - a_\mu(\text{ref})$
   - perform numerical simulations in $L_{\text{big}} = 10.752$ fm
   - perform analytical computations to check models

lattice	NLO XPT	NNLO XPT	MLLGS	HP	RHO
$18.1(2.0)_{\text{stat}}(1.4)_{\text{cont}}$	11.6	15.7	17.8	16.7	15.2

[Gounaris & Sakurai '68][Lellouch & Lüscher '01][Bernecker & Meyer '11]
[Hansen & Patella '19, '20] [Chakraborty *et.al.* '17]

2. $a_\mu(\infty) - a_\mu(\text{big})$
   - NNLO XPT: 0.6(0.3) [Aubin *et.al.* '20]

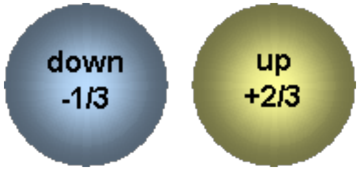
$$a_\mu(\infty) - a_\mu(\text{ref}) = 18.7(2.0)_{\text{stat}}(1.4)_{\text{cont}}(0.3)_{\text{big}}(0.6)_{I=0}(0.1)_{\text{qed}}[2.5]$$

More details: [F. Stokes, Mon 1:00pm EDT]

Isospin breaking effects

In a theoretical determination of a_μ^{HVP} which aims at a sub-percent accuracy, isospin breaking (IB) effects cannot be neglected

Isospin symmetry is an almost exact property of the strong interactions whose breaking effects are induced by



1. $m_u \neq m_d$ — $O[(m_d - m_u)/\Lambda_{QCD}] \simeq 1/100$ — "STRONG"

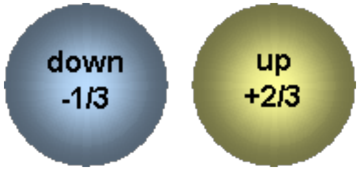
2. $q_u \neq q_d$ — $O(\alpha) \simeq 1/100$ — "ELECTROMAGNETIC"

Since electromagnetic interactions renormalize m_u and m_d in a different way, the two effects are actually intrinsically related

Isospin breaking effects

In a theoretical determination of a_μ^{HVP} which aims at a sub-percent accuracy, isospin breaking (IB) effects cannot be neglected

- A strategy for evaluating IB effect on the lattice consists in treating the IB part of the Lagrangian as a perturbation and



Expand in $m_d - m_u$ **and** α

The "RM123" method
2011-2013

- Since $O(m_d - m_u) \simeq O(\alpha) \simeq 10^{-2}$, only the leading term in the expansion is required when aiming at 0.1-1.0 % accuracy
- This approach has been followed by all present lattice calculations of IB effects in a_μ^{HVP}

STRONG IB EFFECTS: THE m_d-m_u EXPANSION

1) Identify the IB term in the action:

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)(\bar{u}u - \bar{d}d) \right] =$$

$$= \sum_x \left[m_{ud}(\bar{u}u + \bar{d}d) - \Delta m(\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S} \quad , \quad \hat{S} \equiv \sum_x (\bar{u}u - \bar{d}d)$$

2) Expand the functional integral in powers of $\Delta m = m_d - m_u$:

$$\langle O \rangle = \frac{\int D\varphi\, O\, e^{-S_0 + \Delta m \hat{S}}}{\int D\varphi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\varphi\, O\, e^{-S_0}(1 + \Delta m\, \hat{S})}{\int D\varphi\, e^{-S_0}(1 + \Delta m\, \hat{S})} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0}{1 + \Delta m \cancel{\langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0$$

Strong IB effects: the m_d-m_u expansion

$$\langle O \rangle = \langle O \rangle_0 + \Delta m \langle O\,\hat{S} \rangle_0 + \dots$$,

$$\Delta m = \frac{1}{2}(m_d - m_u)$$

$$\hat{S} \equiv \sum_x \left(\bar{u}u - \bar{d}d\right)$$

The small IB parameter is factorized out

Correlation functions are computed with the standard isospin symmetric QCD action

ADVANTAGES

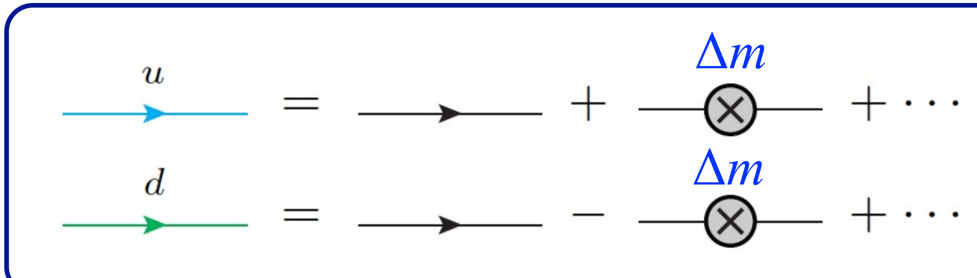
Strong IB effects: the m_d-m_u expansion

$$\langle O\rangle = \langle O\rangle_0 + \Delta m \langle O\,\hat{S}\rangle_0 + \ldots \quad , \quad \Delta m = \frac{1}{2}(m_d - m_u) \qquad \hat{S} \equiv \sum_x \left(\bar{u}u - \bar{d}d\right)$$

- The small IB parameter is factorized out
- Correlation functions are computed with the standard isospin symmetric QCD action

⇐ ADVANTAGES

3) At leading order in Δm, the corrections only appear in the valence quark propagators:



(disconnected contractions of $\bar{u}u$ and $\bar{d}d$ vanish due to isospin symmetry)

QED IB effects: the expansion in α

QED interactions are introduced through a full covariant derivative which contains both QCD and QED links:

$$\nabla_\mu q_f(x) \equiv \frac{1}{a}\left[\left(E_\mu(x)\right)^{e_f} U_\mu(x)\, q(x+a\hat{\mu}) - q(x)\right]$$

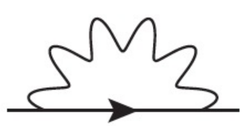
QED → $\left(E_\mu(x)\right)^{e_f}$; QCD → $U_\mu(x)$

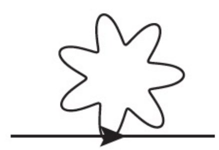
$$E_\mu(x) = e^{-i\,a\,e\,A_\mu(x)}$$

Since

$$E_\mu(x) = e^{-i\,e\,A_\mu(x)} = 1 - i e A_\mu(x) - \frac{1}{2} e^2 A_\mu^2(x) + \ldots$$

the expansion in α for the quark propagator leads to:

$(e_f e)^2$ 

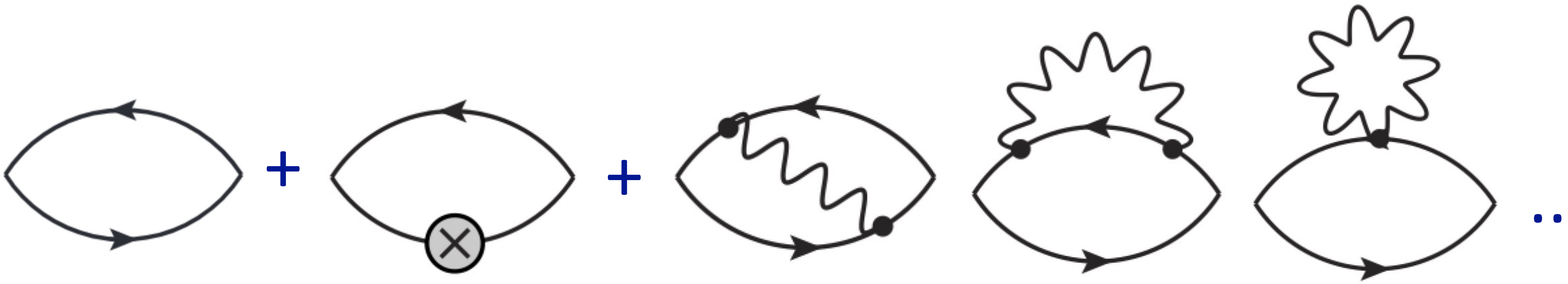
$(e_f e)^2$ 

+ counterterms

Tadpole diagrams are required by gauge invariance (in QCD and QED)

Isospin breaking effects

- For $a_\mu^{HVP,LO}$, the expansion at leading order in Δm and α has the form:



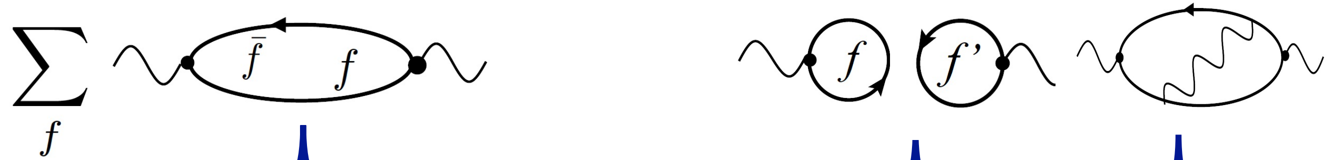
$O(1)$ + $O(m_d - m_u)$ + ... $O(\alpha)$ ...

+ QED sea-quark diagrams + disconnected diagrams ...

These corrections are essentials when aiming at < 1% accuracy

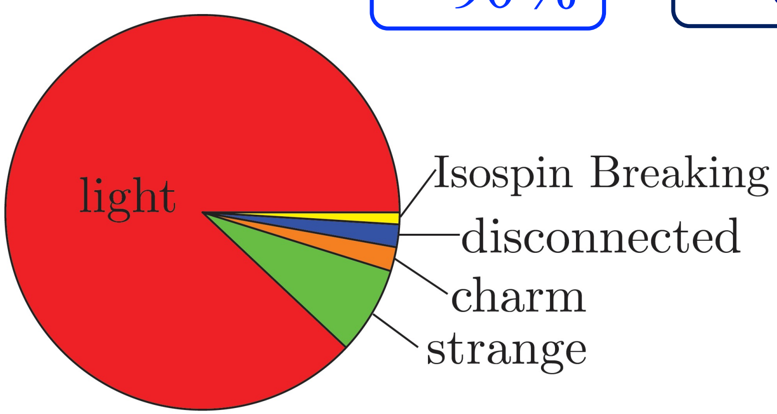
LO-HVP Lattice results

Summing up all the various contributions:



$$a_\mu^{HVP,LO} = a_\mu^{HVP,LO}(ud) + a_\mu^{HVP,LO}(s) + a_\mu^{HVP,LO}(c) + a_{\mu,\,disc}^{HVP,LO} + \delta a_\mu^{HVP,LO}$$

$a_\mu^{HVP,LO}(ud)$	$a_\mu^{HVP,LO}(s)$	$a_\mu^{HVP,LO}(c)$	$a_{\mu,\,disc}^{HVP,LO}$	$\delta a_\mu^{HVP,LO}$
$\sim 90\%$	$\sim 8\%$	$\sim 2\%$	$\sim 2\%$ (< 0)	$\sim 1\%$

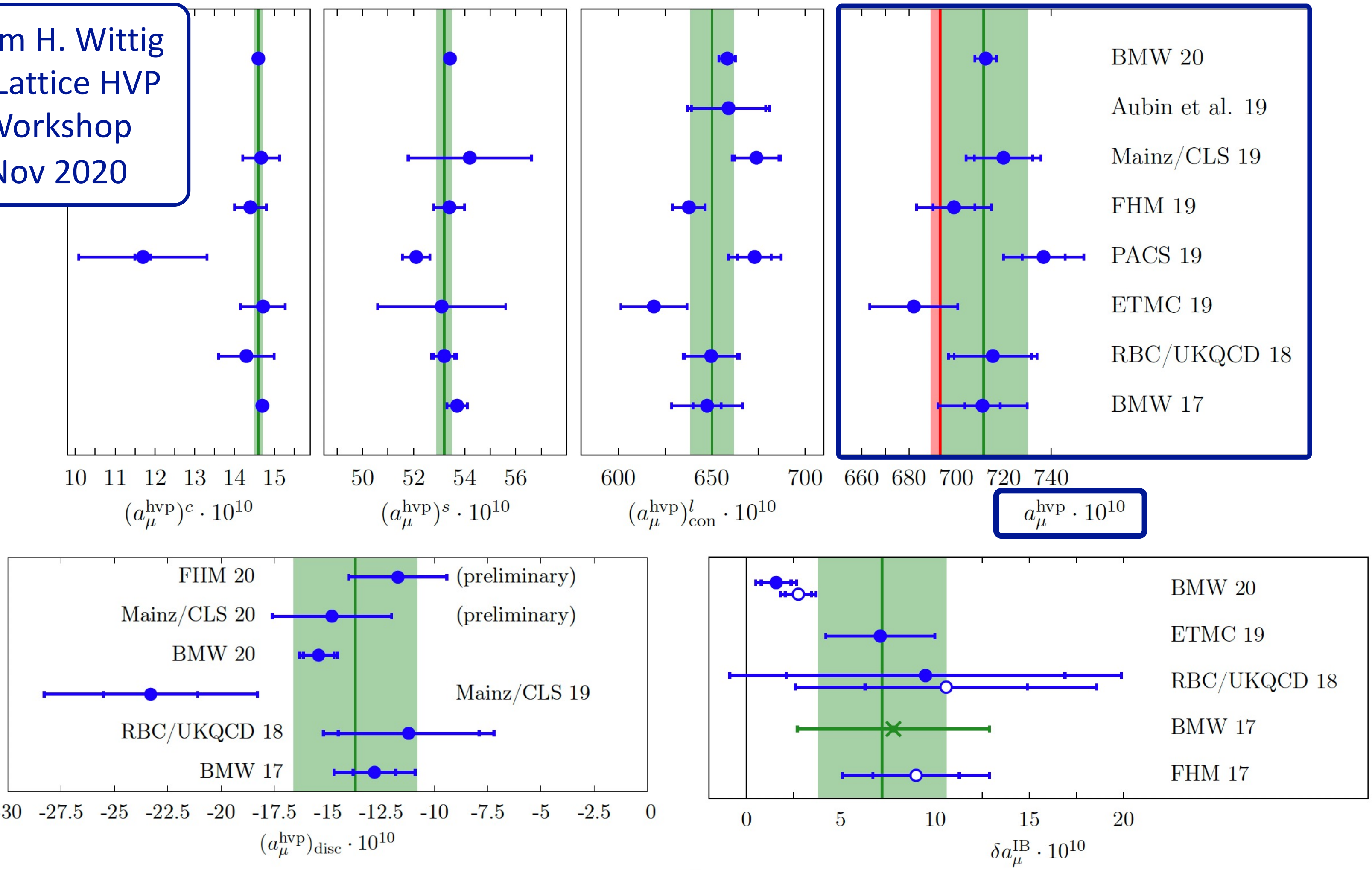


Only in the light-quark connected contribution the sub-percent accuracy is required

g_μ-2 and LQCD

LO-HVP Lattice results

From H. Wittig @ Lattice HVP Workshop Nov 2020



g_μ-2 and LQCD

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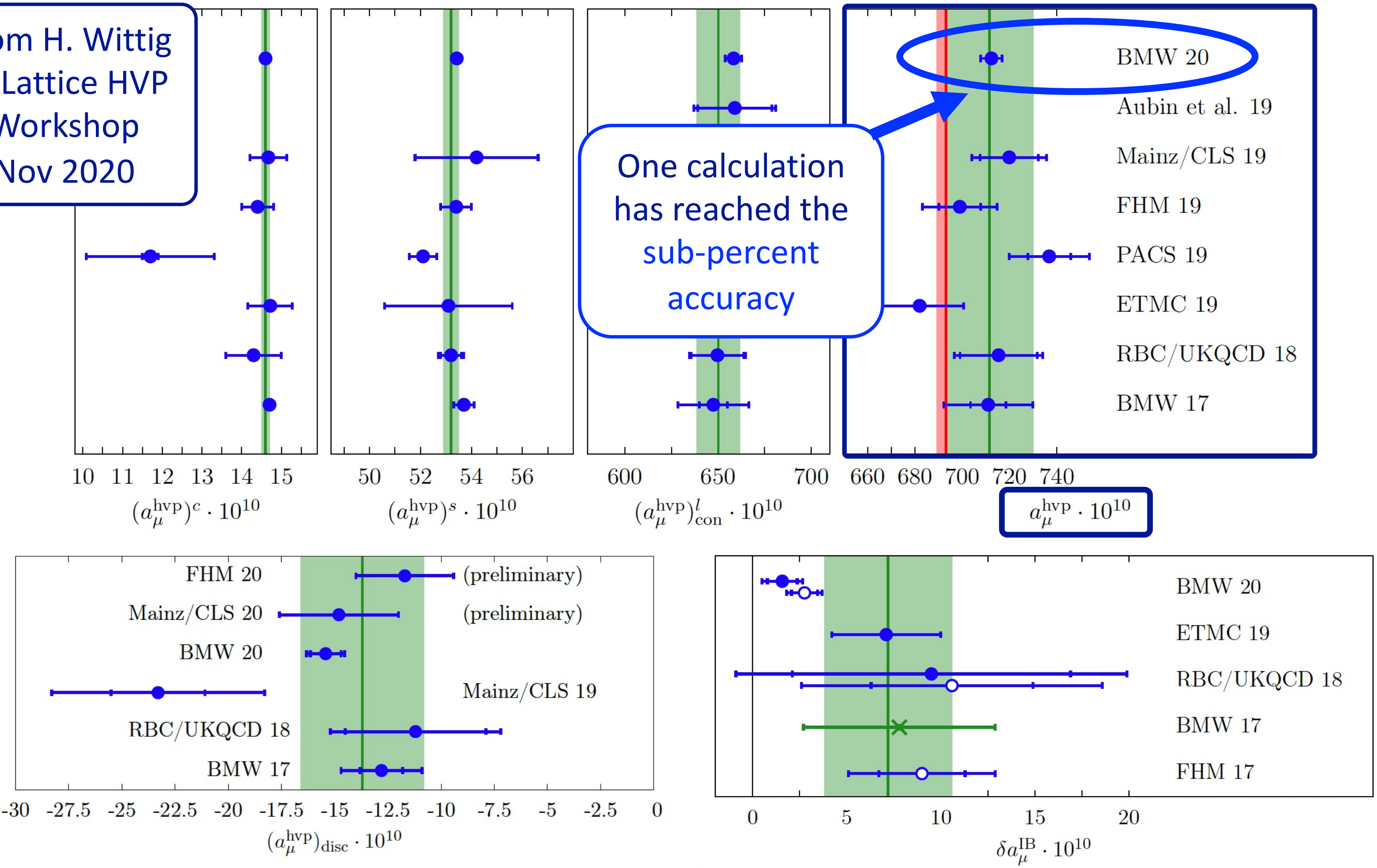
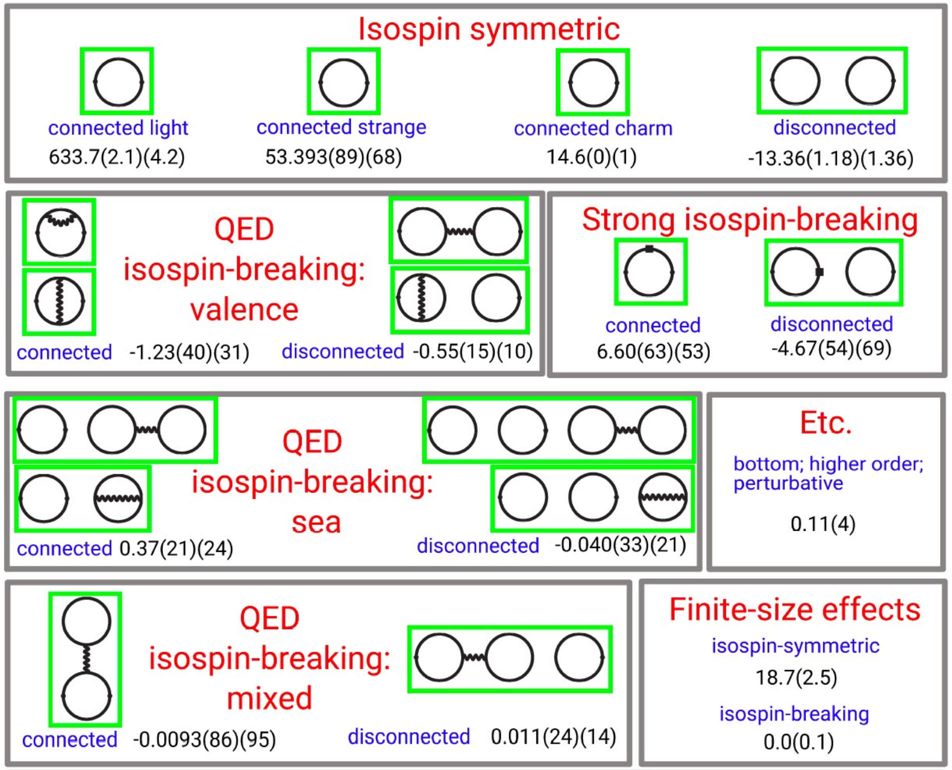


Summary of contributions to $a_\mu^{\text{LO-HVP}}$

From B. C. Tóth
BMW Collab.
@ Lattice 2021

Nature 593 (2021) 7857, 51-55 [arxiv:2002.12347]



Final result →

$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$

g_μ-2 and LQCD

LO-HVP Lattice results

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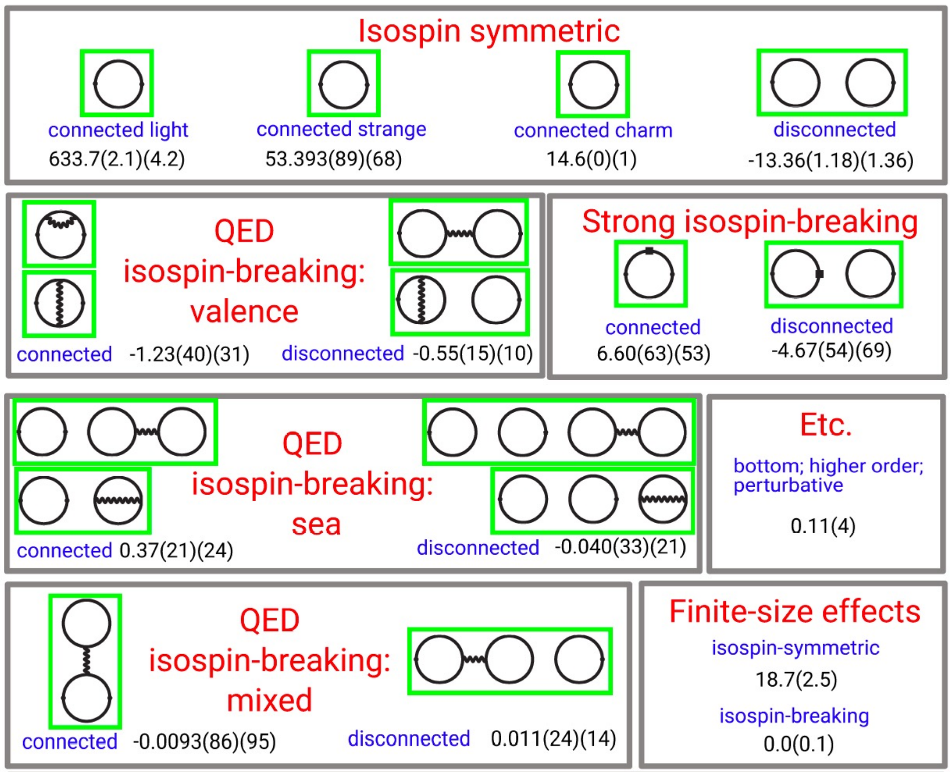


Final result →

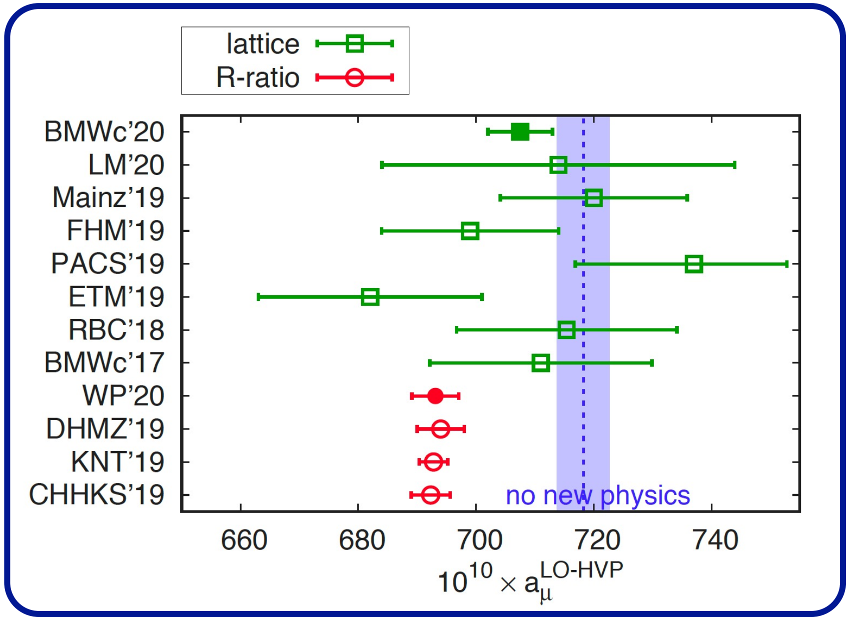
$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$

→ Accuracy 0.8%

First message:

The technology and methods to reach the sub-percent accuracy are available !

LO-HVP Lattice results



The BMW result is:

- Compatible with other lattice results

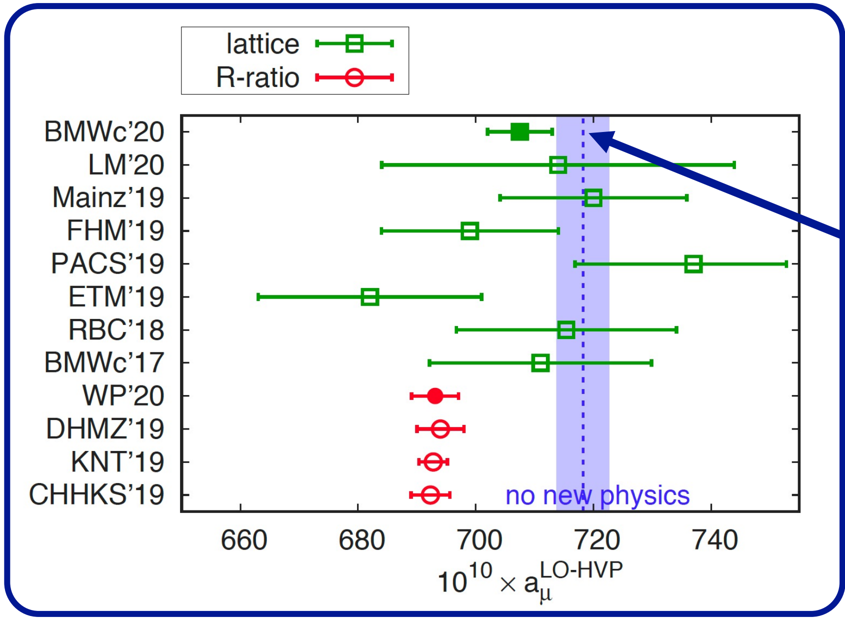
Lattice QCD
BMW Collab.

$$a_\mu^{HVP,LO} = (707.5 \pm 5.5) \cdot 10^{-10}$$

Lattice QCD
WP prev. average

$$a_\mu^{HVP,LO} = (711.6 \pm 18.4) \cdot 10^{-10}$$

LO-HVP Lattice results



The BMW result is:

- Compatible with other lattice results
- Consistent with the experiment within 1.5 σ

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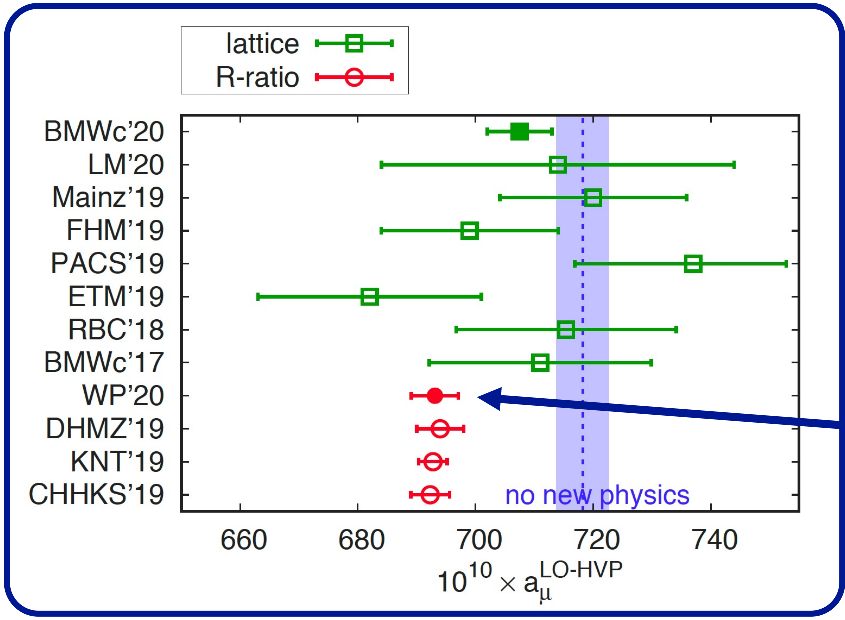

The BMW result is:

- Compatible with other lattice results
- Consistent with the experiment within 1.5 σ
- 2.1 σ higher than the R-ratio value

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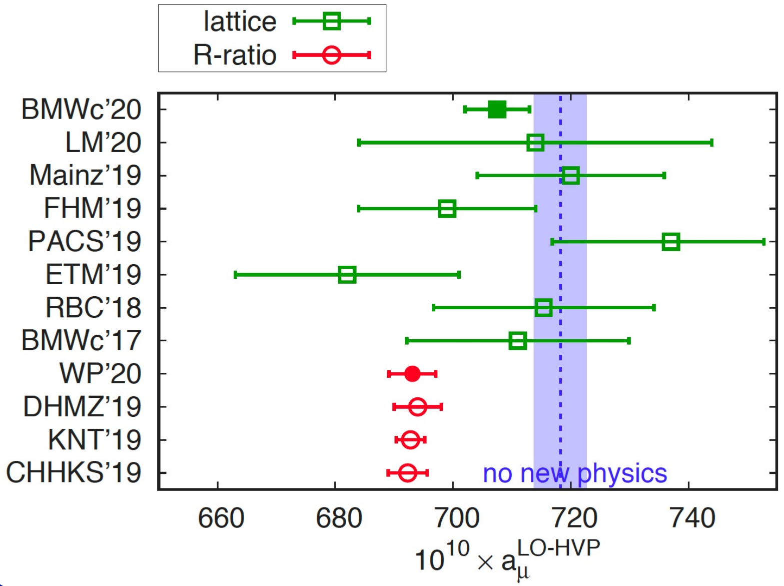
WP prev. average

$$a_\mu^{HVP,LO} = (711.6 \pm 18.4) \cdot 10^{-10}$$

$$a_\mu^{HVP,LO} = (693.1 \pm 4.0) \cdot 10^{-10}$$

R-ratio - WP average

LO-HVP Lattice results



The BMW result is:

- Compatible with other lattice results
- Consistent with the experiment within 1.5 σ
- 2.1 σ higher than the R-ratio value

It is important to test the BMW result with other lattice calculations

Lattice QCD
BMW Collab.

$$a_\mu^{HVP,LO} = (707.5 \pm 5.5) \cdot 10^{-10}$$

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WP prev. average

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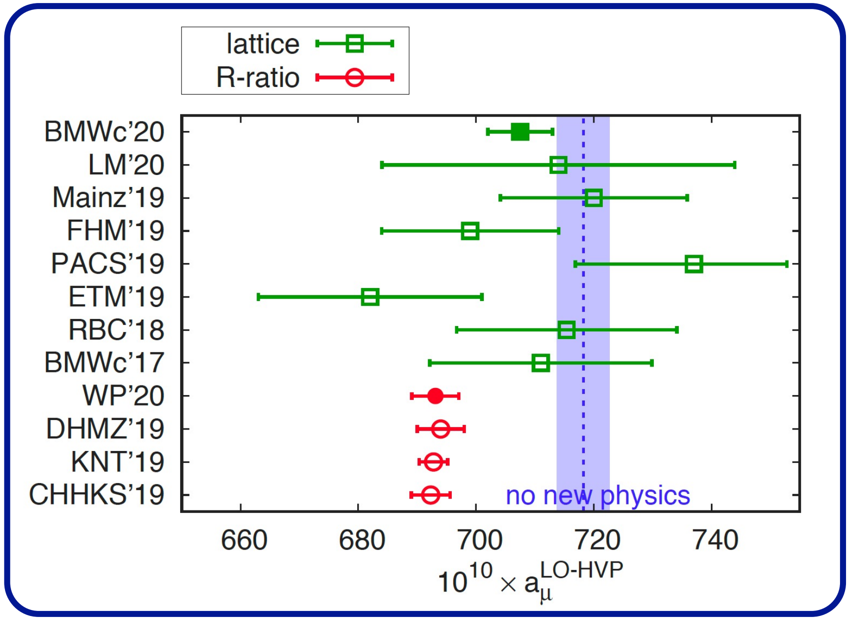
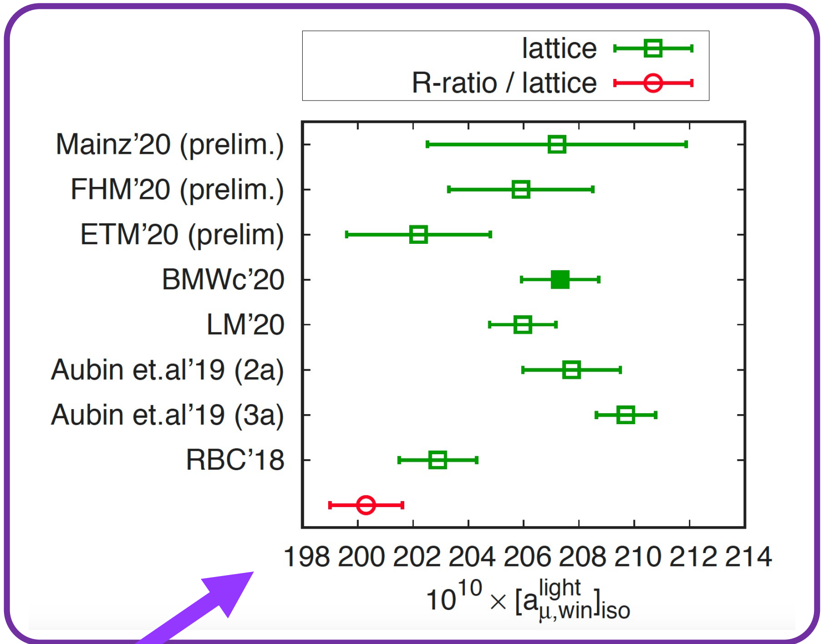
R-ratio - WP average

The window observable





B.C.Tóth @ Lattice 2021

- A very useful crosscheck: the window observable:

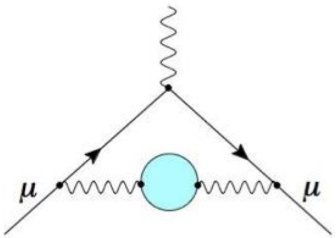
$$a_\mu^{win} = 8\alpha^2 \int_{t_0}^{t_1} dt\, \tilde{f}(t)\, C(t)$$

t_0 = 0.4 fm
t_1 = 1.0 fm
Euclidean time

Less challenging than full $a_\mu^{HVP,LO}$ for lattice calculation (signal/noise, finite size effects, discretization errors)

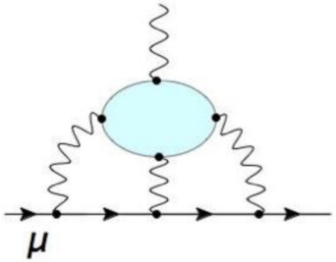
The hadronic light-by-light

There are 2 relevant hadronic contributions to the muon anomalous magnetic moment



Hadronic Vacuum Polarization (HVP)

α^2



Hadronic Light-by-Light scattering (HLbL)

α^3

The hadronic light-by-light

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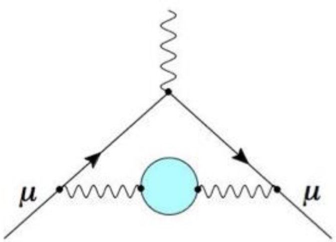
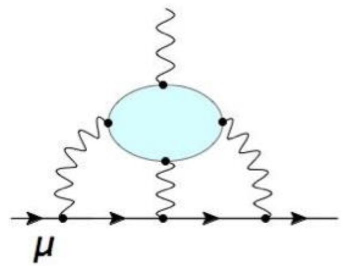


Hadronic Vacuum Polarization (HVP)

α^2



Hadronic Light-by-Light scattering (HLbL)

α^3

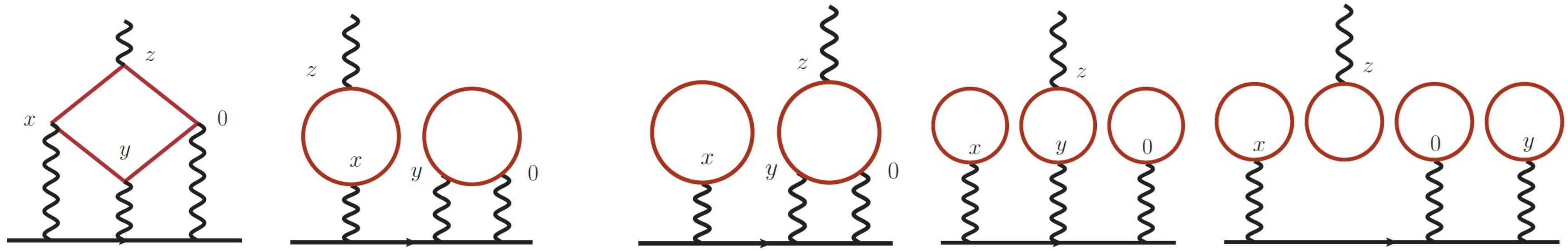
- The lattice calculation of HLbL is essentially analogous to the one discussed for HVP
- The basic ingredient is the 4-point correlator of the e.m. current:

$$\langle j_\mu(x) j_\nu(y) j_\rho(z) j_\sigma(0) \rangle$$

- The calculation is much more challenging. However, being HLbL of $O(\alpha^3)$, an accuracy of $\sim$ 10% is sufficient

The hadronic light-by-light

- The calculation involves both connected and disconnected diagrams



Leading — Sub-leading

and there is a large cancellation between connected and disconnected

- At present, two complete lattice calculations:

RBC 2019 $a_\mu^{HLbL} = (7.87 \pm 3.54) \cdot 10^{-10}$

Mainz 2021 $a_\mu^{HLbL} = (10.68 \pm 1.47) \cdot 10^{-10}$

Dispersive approach

$a_\mu^{HLbL} = (9.2 \pm 1.9) \cdot 10^{-10}$

[WP average]

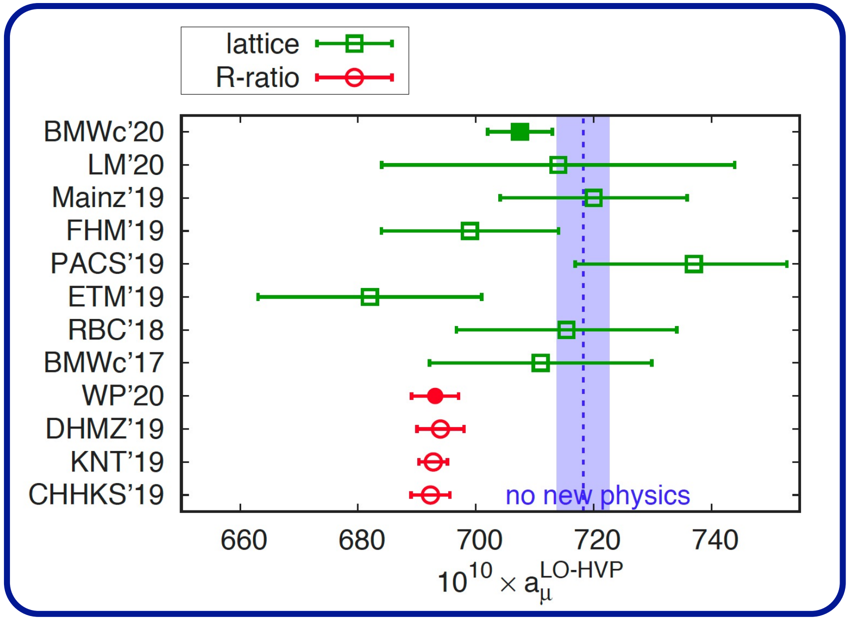
g_μ-2 and LQCD

CONCLUSIONS

- Tremendous progress of lattice calculation of HVP and HLdL in the last few years

- The first calculation of HVP with sub-percent accuracy presented by the BWM collab. It must be checked by other lattice calculations

- Two complete lattice calculations of HLbL are available, in agreement within each other and the data driven result



GRAZIE PER L'ATTENZIONE !

SLIDES DI RISERVA

The Hadronic Vacuum Polarization

Most of recent lattice calculations of $a_\mu^{HVP,LO}$ use the so-called

Time-momentum representation

$$Q_\mu = (\omega, 0, 0, 0) \quad , \quad Q^2 = \omega^2$$

1) $\Pi_{kk}(\omega) = \left(Q_k Q_k - \delta_{kk} Q^2\right)\Pi(Q^2) = -Q^2\,\Pi(Q^2) = -\omega^2\,\Pi(\omega^2) \quad , \quad k = 1,2,3$

2) $\Pi_{kk}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_{-\infty}^{+\infty} d^3x\, C_{kk}(x) = -\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, C(t) \quad , \quad C(t) \equiv -\frac{1}{3}\sum_{k=1}^{3} \int_{-\infty}^{+\infty} d^3x\, C_{kk}(\vec{x}, t)$

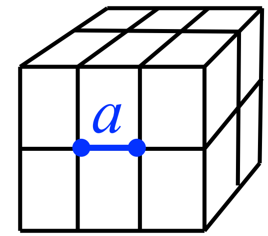
1) + 2) $\longrightarrow$ $\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, C(t) = \omega^2 \Pi\left(\omega^2\right) \quad , \quad a_\mu^{HVP,LO} = 4\alpha_{em}^2 \int_0^\infty d\omega^2 f(\omega^2)\left(\Pi(\omega^2) - \Pi(0)\right)$

It is the Fourier transform in time

The lattice regularization

1) The ultraviolet cutoff

$$|p| \leq \frac{\pi}{a}$$

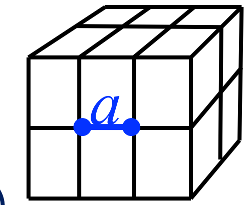


The cutoff can be in conflict with important symmetries of the theory, as for example Lorentz invariance or chiral invariance

This problem is common to all regularizations, like for example Pauli-Villars, dimensional regularization etc.

Some of the symmetries are not recovered in the infinite cutoff limit (continuum limit) ⇒ Anomalies

The quark action



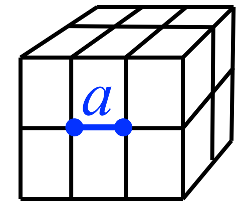
- By having a definition of the lattice covariant derivatives, the construction of a gauge invariant fermionic action seems trivial:

$$S_F = \int \bar{q}(x) \left[\frac{1}{2} \gamma_\mu \overleftrightarrow{D}_\mu + m \right] q(x)$$

$$S_F = a^4 \sum_x \bar{q}(x) \left[\frac{1}{2} \gamma_\mu \tilde{\nabla}_\mu + m \right] q(x)$$

covariant derivative

The quark action



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$$S_F = a^4 \sum_x \bar{q}(x)\left[\frac{1}{2}\gamma_\mu \tilde{\nabla}_\mu + m\right]q(x)$$

covariant derivative

- However, already in the free theory ($g=0$, $U_\mu=1$):

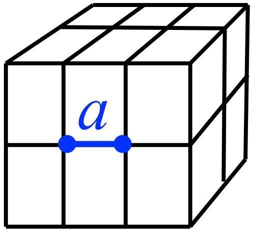
$$\left[\frac{1}{2}\sum_\mu \gamma_\mu \tilde{\nabla}_\mu + m\right]S(x) = \frac{1}{2a}\sum_\mu \gamma_\mu\left[S(x+a\hat{\mu}) - S(x-a\hat{\mu})\right] + mS(x) = \delta(x)$$

$$\Rightarrow \quad S(p) = \frac{i\hat{\not{p}} + m}{\hat{p}^2 + m^2} \quad \text{with} \quad \hat{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$$

The propagator for $m=0$ has 16 poles (doublers) rather than 1 !!

The poles are located at $ap = (0,0,0,0), (\pi,0,0,0), \ldots, (\pi,\pi,0,0), \ldots, (\pi,\pi,\pi,\pi)$.

The quark action



- The solution proposed by K. Wilson, consists in adding to the action a formally irrelevant (d=5) operator, the "Wilson term":

K. Wilson 1975

$$S_F = a^4 \sum_x \bar{q}(x) \left[\frac{1}{2} \gamma_\mu \tilde{\nabla}_\mu - \frac{a r}{2} \nabla^2 + m \right] q(x)$$

- The free quark propagator now is:

$$S(p) = \frac{i\hat{\not{p}} + M(p)}{\hat{p}^2 + M(p)^2}$$

with

$$M(p) = m + \frac{2r}{a} \sum_\mu sen^2\left(\frac{a p_\mu}{2} \right)$$

$\rightarrow$ $M(0,0,0,0) = m$, $M(\pi,0,0,0) = m + \frac{2r}{a}$, $M(\pi,\pi,0,0) = m + \frac{4r}{a}$, ...

The mass of the doublers is at the cutoff scale. They decouple from the theory.