# On the scalar $\pi K$ form factor beyond the elastic region


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Universität Bern
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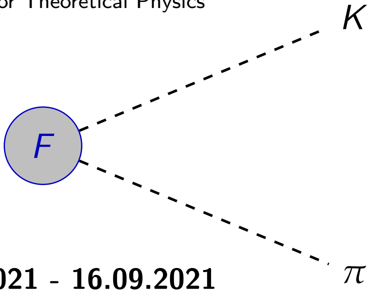



**14.09.2021 - 16.09.2021**

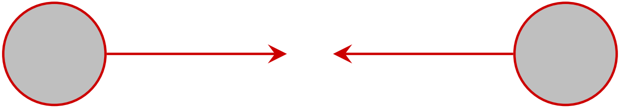
2nd STRONG-2020 online workshop

based on

[Von Detten, Noël, Hanhart, Hoferichter and Kubis, 2021, Eur. Phys. J. C 81.5, p. 420]

# Frontiers in Particle Physics

Search for "new" physics via particle accelerators/colliders and detectors



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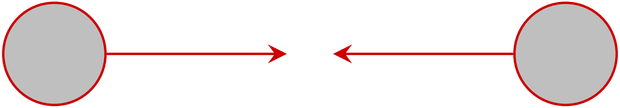

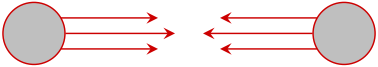

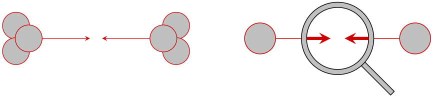

**energy frontier**

- more energy
- more phase space
- more (massive) particles

**intensity/precision frontier**

- more luminosity/measuring time
- more events/statistics
- better/more precise detectors
- more precise quantities

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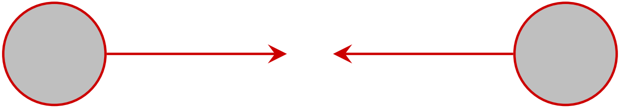

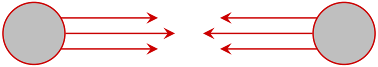

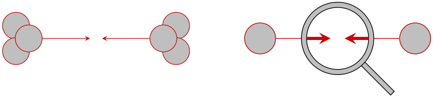

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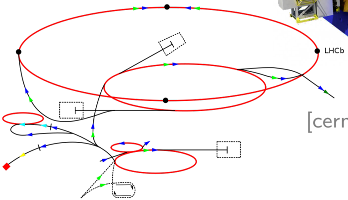
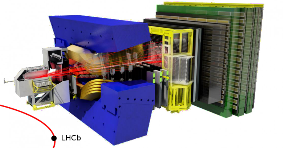
# Precision Frontier - LHCb and Belle



**LHCb**

[cern.ch;wikimedia]

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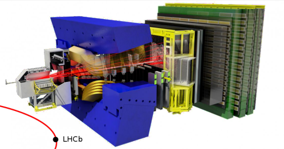

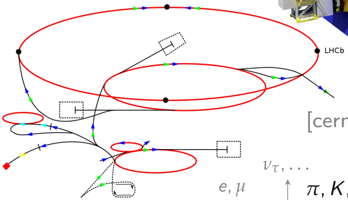


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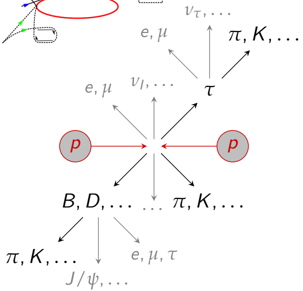

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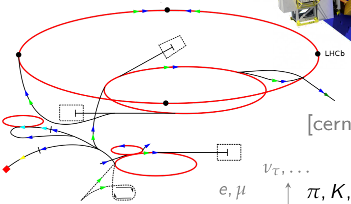


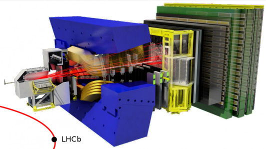

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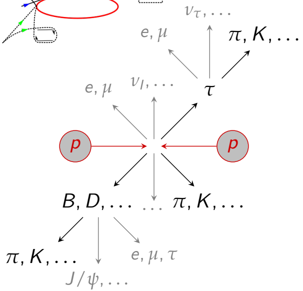


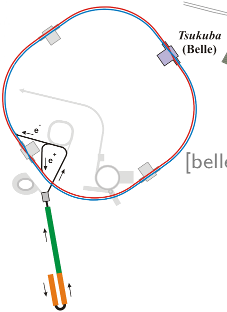


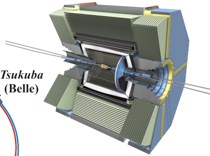

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[belle2.org;wikimedia]

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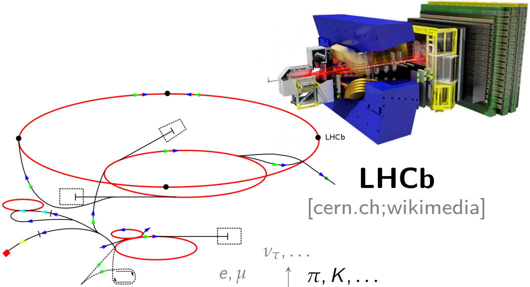


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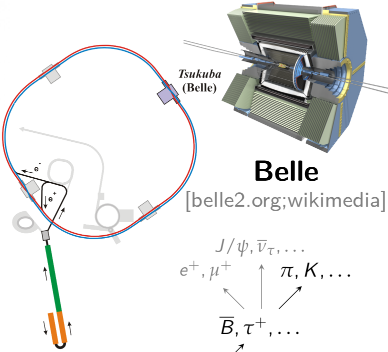


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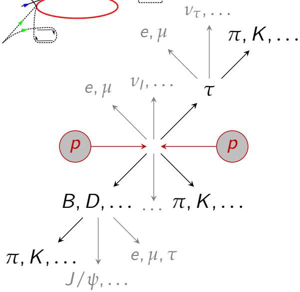


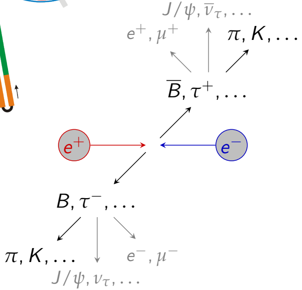

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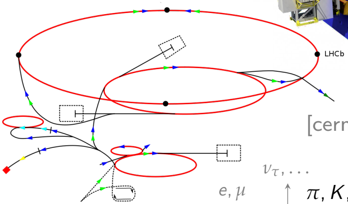
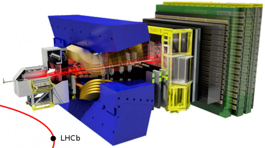

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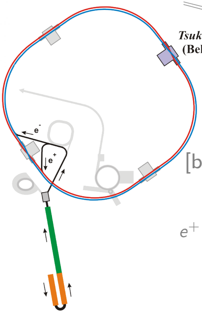

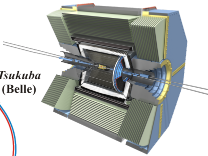

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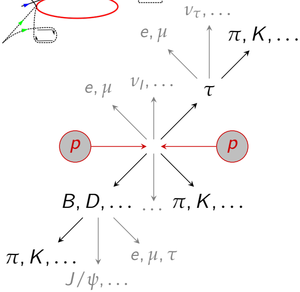
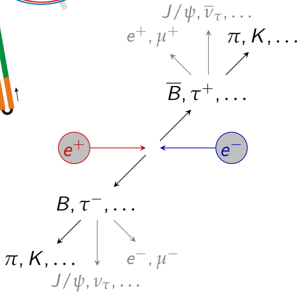
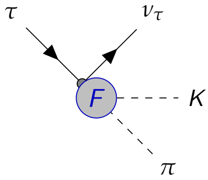



Hadrons in final states are everywhere, especially $\pi$ and $K$

# Observables including $\pi K$ final states

semi-leptonic decays




- BSM CP violation requires inelasticity [Cirigliano et al., 2018]

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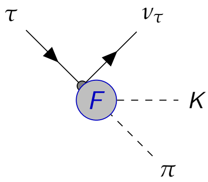


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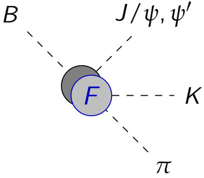
<u>$B$ & $D$ decays</u>

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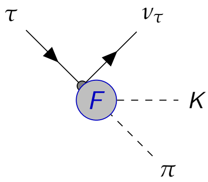


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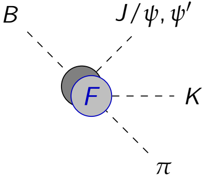


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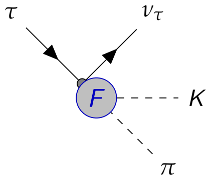


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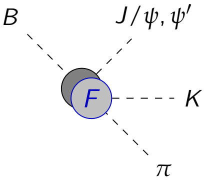


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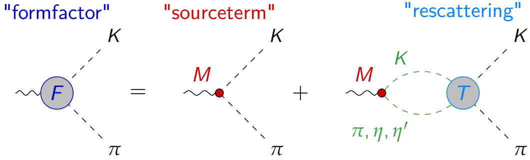

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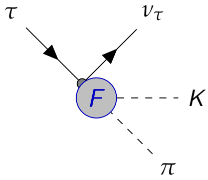


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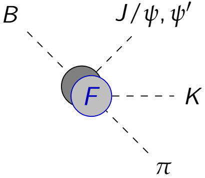


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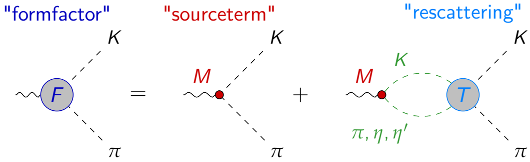
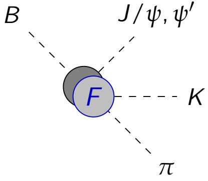



- requires scattering matrix $T$ with elastic $\pi K$ and inelastic $\eta K$, $\eta' K$

# Exotica in B decays



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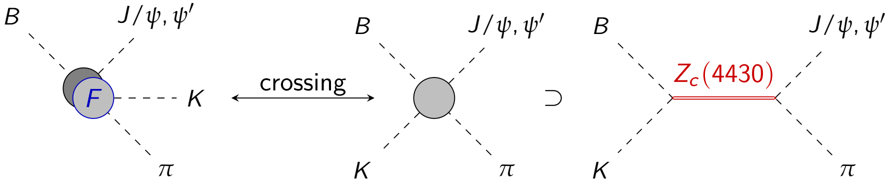


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  - $Z_c^-(4430)$, $J^P = 1^+$
  - supposedly: $c\bar{c}d\bar{u}$

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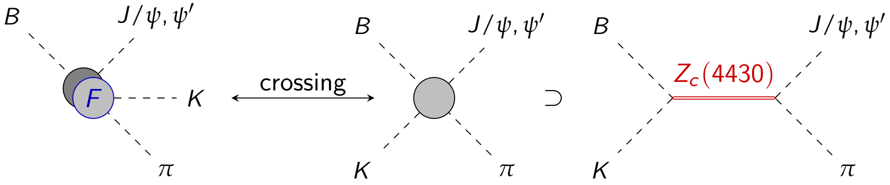


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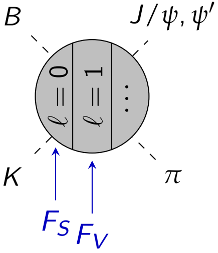

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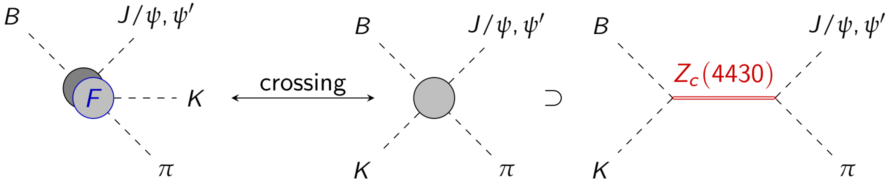


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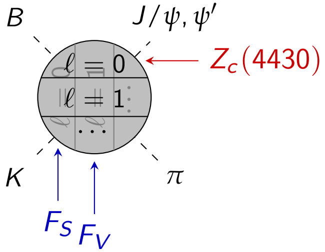

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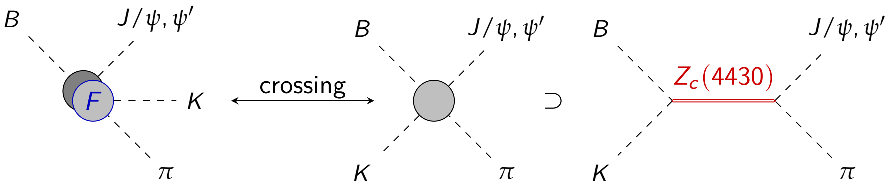
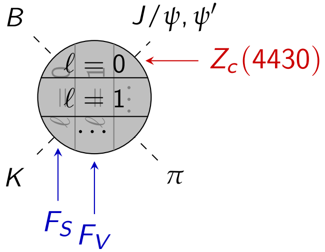
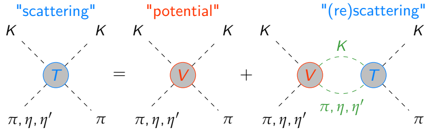



- exotic state in crossed channel:
  - $Z_c^-(4430)$, $J^P = 1^+$
  - supposedly: $c\bar{c}d\bar{u}$
- different $\pi K$ partial waves interfere
- no proper phase behaviour for e.g. Breit-Wigner models by construction




Proper phases of $F_S$ and $F_V$ are essential to investigate these exotica

# Formalism - $T$-matrix




- Bethe-Salpeter (BS) equation:

  $$T = V + V\,G\,T$$

- fulfils unitarity by construction if $V_{ij} \in \mathbb{R}$, $\text{Im}\, G_{ij} = \sigma_i \delta_{ij}$

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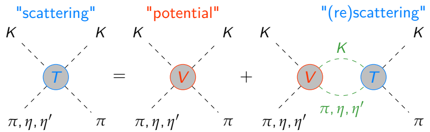


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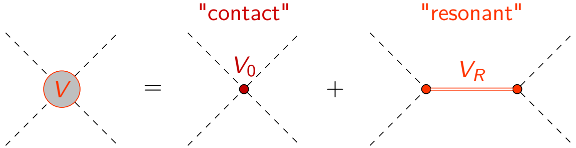

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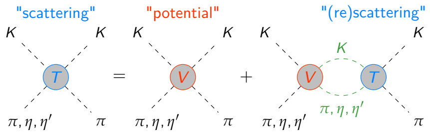


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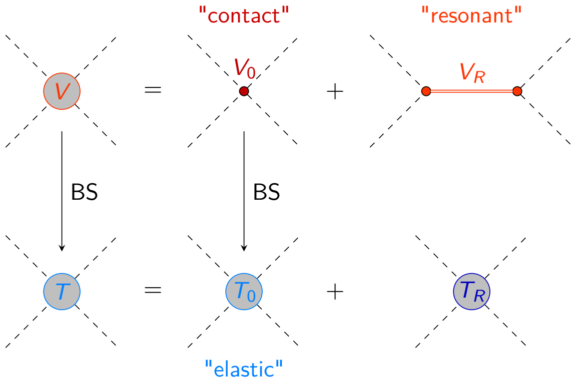

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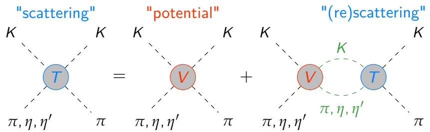


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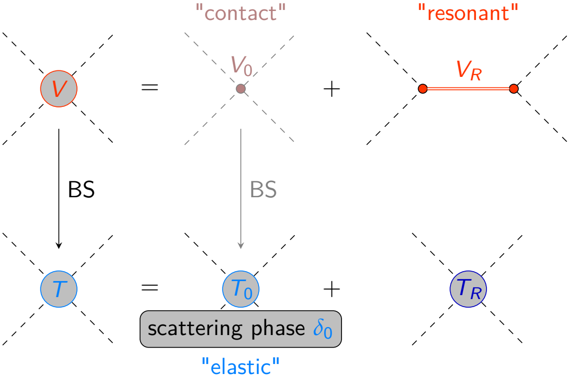

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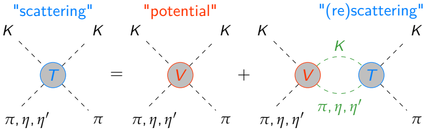


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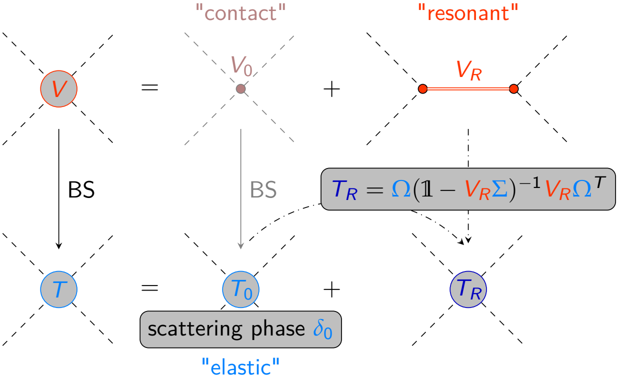


dispersive quantities:
$\Omega[\delta_0]$: Omnès function
$\Sigma[\Omega, \mathrm{Im}\, G]$: Selfenergy

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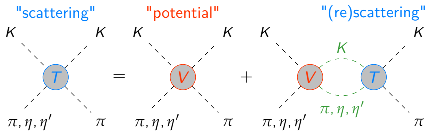


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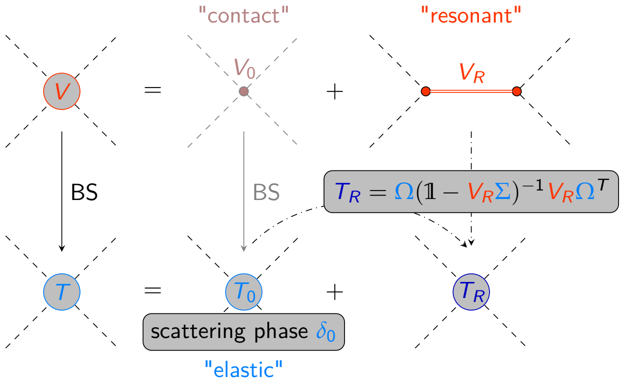


parametrise $V_R$
as subtracted
resonance potential

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# Fit to $\pi K$ scattering data

application to $\pi K \to \pi K$:

- $S$-wave ($\ell = 0$)
- crossed channel $\eta' K$
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  - $\kappa / K_0^*(700)$
  - $K_0^*(1430)$
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- input phase $\delta_0$ as non-resonant part of [Peláez and Rodas, 2016]: $\kappa / K_0^*(700)$ implicitly
- resonance potential $V_R$: $K_0^*(1430)$ & $K_0^*(1950)$ explicitly

# Fit to $\pi K$ scattering data

application to $\pi K \to \pi K$:

- $S$-wave ($\ell = 0$)
- crossed channel $\eta' K$
- present resonances:
  - $\kappa / K_0^*(700)$
  - $K_0^*(1430)$
  - $K_0^*(1950)$
- input phase $\delta_0$ as non-resonant part of [Peláez and Rodas, 2016]: $\kappa / K_0^*(700)$ implicitly
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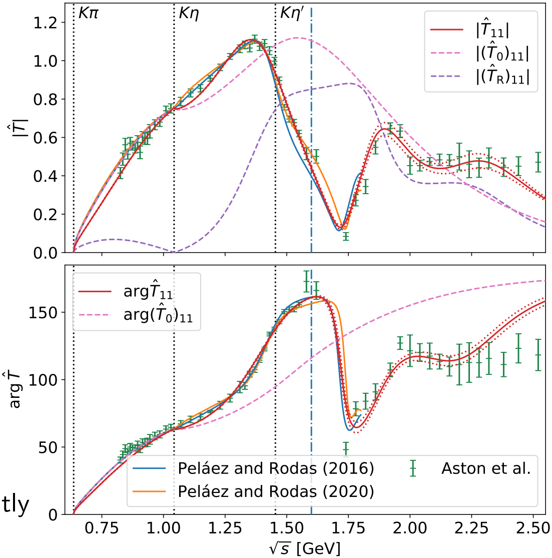
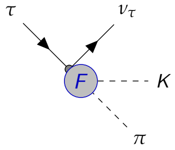



[Aston et al., 1988; Peláez and Rodas, 2016; Peláez and Rodas, 2020]

# Fit to $\tau$ decay rate to $\pi K$




application to the decay $\tau \to \nu_\tau K \pi$:

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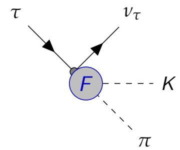


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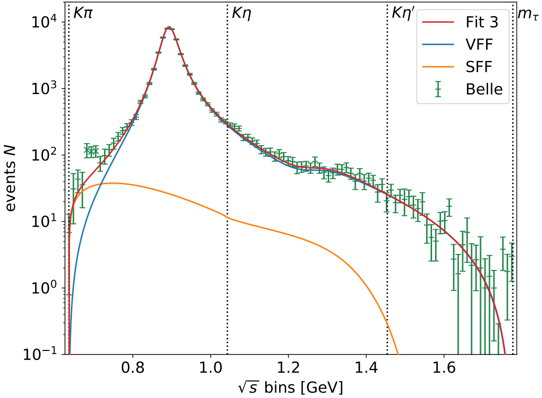


(Belle = [Epifanov et al., 2007])

- *S*-wave (SFF) parametrised by our formalism
- *P*-wave (VFF) parametrised by conventional RChPT model [Antonelli et al., 2013]

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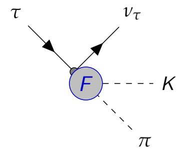
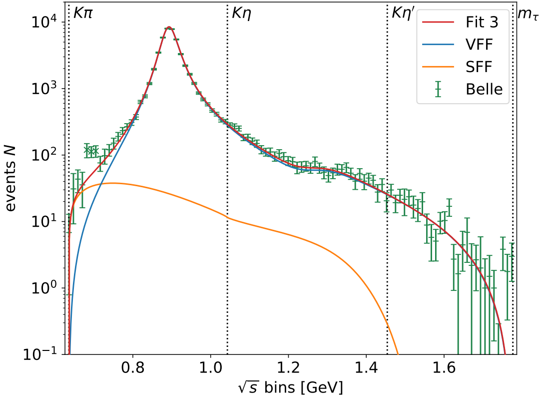



application to the decay $\tau \to \nu_\tau K\pi$:

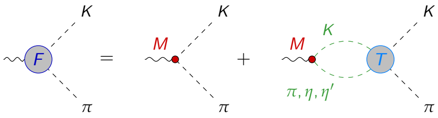



(Belle = [Epifanov et al., 2007])

- *S*-wave (SFF) parametrised by our formalism
- *P*-wave (VFF) parametrised by conventional RChPT model [Antonelli et al., 2013]
- fit difficulty:
  overlapping $K_0^*(1430)$ and $K^*(1410)$ resonances
- four fit scenarios : red. $\chi^2 \in [0.98, 1.07]$

# Scalar form factor

- scalar form factor $F$ fixed by $T$ up to source term $M$:



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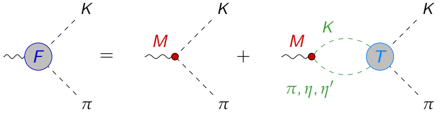


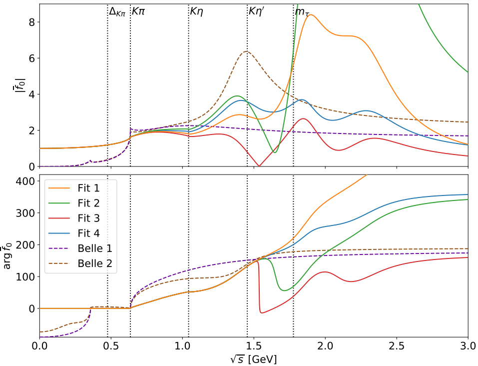

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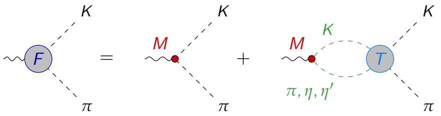


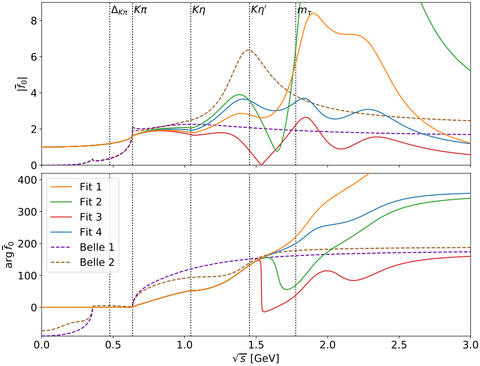


- correct phase by construction
- fulfils unitarity & analyticity
- correct normalisation
- in contrast to Breit-Wigner
- further distinction requires additional data

# Results

- Branching Ratio: $\mathrm{BR}(\tau \to K_S \pi \nu_\tau) = 4.35(10) \times 10^{-3}$
  agrees at $1.5\sigma$ with PDG [Zyla et al., 2020] and [Ryu et al., 2014]

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$$m_{K_0^*(1430)} = 1408(4)(47)\ \mathrm{MeV}, \qquad m_{K_0^*(1950)} = 1863(11)(4)\ \mathrm{MeV},$$

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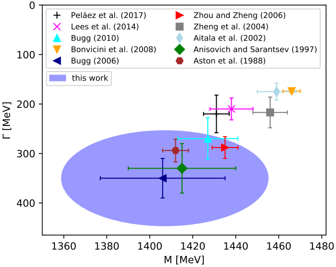


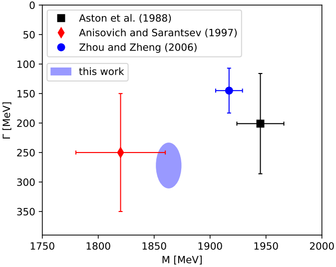

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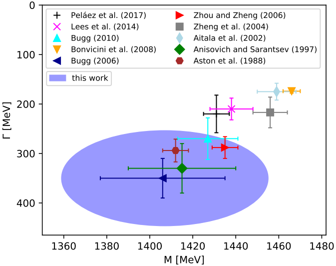


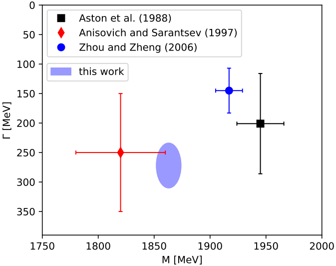


- extracted physical couplings via residues:
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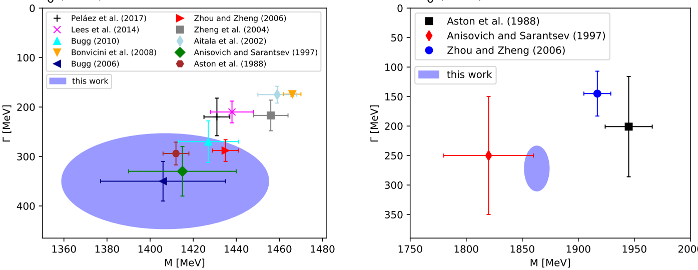


- extracted physical couplings via residues: scalar resonance coupling to $\bar{s}\gamma^\mu u$ gives $\mathrm{BR}(\tau \to K_0^*(1430)\nu_\tau) < 1.6 \times 10^{-4}$
- estimate on BSM CP asymmetry supports [Cirigliano et al., 2018]

$$A_{CP}^{\tau,\mathrm{BSM}} = -0.034(14)\ \mathrm{Im}\, c_T \quad (c_T,\ \text{Tensor Wilson coefficient})$$

# Conclusion/Outlook

Conclusion:

- $\pi K$ form factor to describe $\pi K$ final states in
  - semi-leptonic decays analysing BSM CP violation
  - $B$ and $D$ decays searching for exotica
- formalism combining low energy elastic description with high energy resonance exchange
- successful application to scalar form factor in $\tau \to \nu_\tau K \pi$

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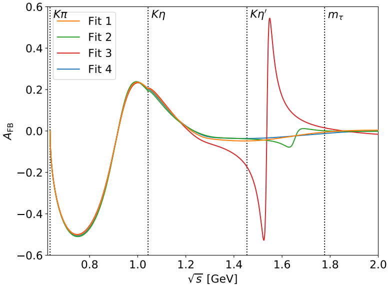
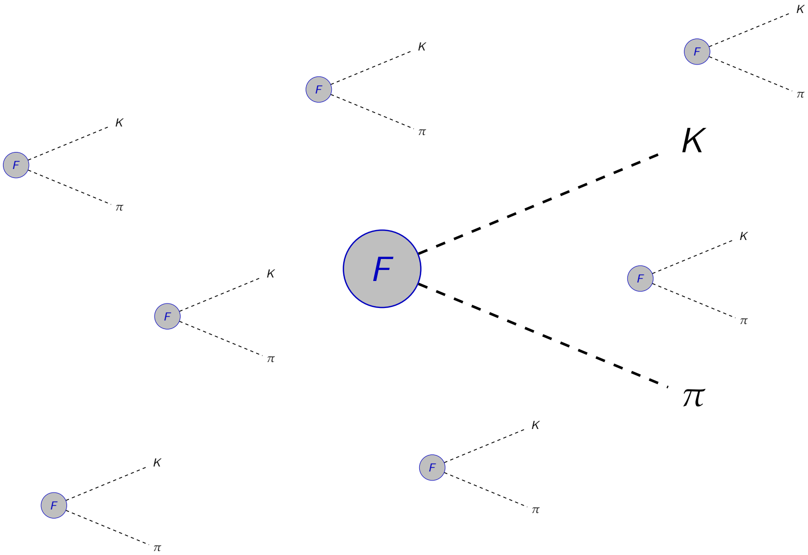
- formalism combining low energy elastic description with high energy resonance exchange
- successful application to scalar form factor in $\tau \to \nu_\tau K\pi$

Outlook:

- application to other partial waves (i.e. $P$-wave)
- forward-backward asymmetry as potentially measured by Belle II $\to$ separates $S$ & $P$-wave
- exotica in $B$ and $D$ decays
- other semi-leptonic decays



# Thank you for your attention!



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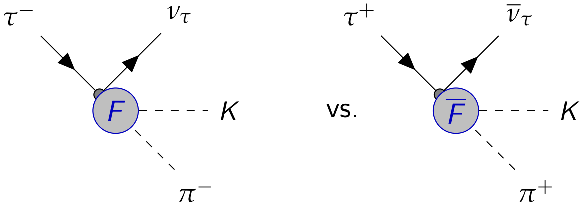
# Backup-Slides

# Comparison to other parametrisations/works

- as we have seen Breit-Wigners are a no-go, especially for $S$-wave
- Single-channel Omnès-function (e.g. [Moussallam, 2008; Boito et al., 2009; Boito et al., 2010; Bernard et al., 2011; Antonelli et al., 2013,...])
  - phase input either ...
    - ... the pure scattering phase $\rightarrow$ no inelastic channels
    - ... the argument of a non-unitary model (e.g. RChPT) $\rightarrow$ no consistency with scattering
- Coupled-channel Omnès-matrix (e.g. [El-Bennich et al., 2009])
  - in principle does fulfil all constraints (unitarity, analyticity)
  - needs data for all scattering channels (only $\pi K \rightarrow \pi K$ atm)
  - can compensate this by using chiral constraints
  - but: would need more constraints to extend the model
- our model is fully consistent with unitarity and easier expandable

# CP violation in $\tau$ decays




- BSM CP violation only via Tensor-Vector interference:

$$A_{CP}^{BSM} \sim \text{Re}\, F_V F_T^* - \text{Re}\, \overline{F}_V \overline{F}_T^* \sim \sin(\phi_V - \phi_T)$$

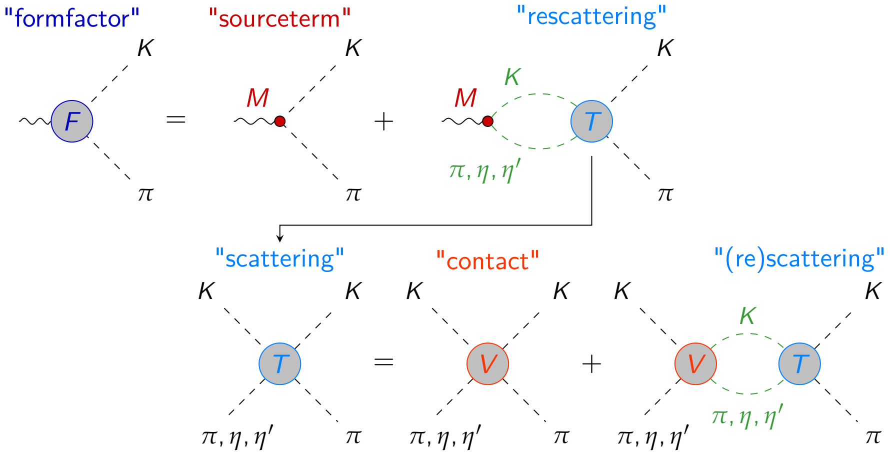
  (requires interference with weak phase)
- Watson's theorem states for elastic interactions:

$$\phi_V = \delta_{\ell=1}^{\pi K} \quad \text{and} \quad \phi_T = \delta_{\ell=1}^{\pi K} \qquad (\delta_{\ell=1}^{\pi K} \text{ from elastic scattering})$$

- $\phi_T - \phi_V \neq 0$ needs inelastic effects [Cirigliano et al., 2018]

Inelastic effects are mandatory to give rise to BSM CP violation

# The $\pi K$ form factor and scattering matrix



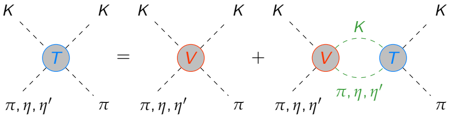

**Bethe-Salpeter equation**

Proper description of $\pi K$ form factor $F$
requires proper description of scattering matrix $T$

# Formalism - $T$-matrix

- describe $\pi K \to \pi K$ scattering $T$ to describe $\pi K$ form factor $F$
- Bethe-Salpeter (BS) equation:

$$T = V + VGT$$




- fulfils unitarity by construction if $V_{ij} \in \mathbb{R}, \mathrm{Im}\, G_{ij} = \sigma_i \delta_{ij}$
- two-potential formalism: [Nakano, 1982; Hanhart, 2012; Ropertz et al., 2018]

$$V = V_0 + V_R \;\to\; T = T_0 + T_R$$

- $T_0$ elastic, given by elastic scattering phase $\delta_0$
- Inelasticity driven by resonance exchange via $V_R$
- $T_R$ given implicitly in terms of $T_0$ and $V_R$ via BS equation

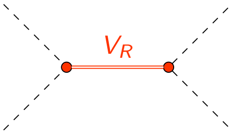
$$T_R = \Omega(\mathbb{1} - V_R\Sigma)^{-1} V_R \Omega^T$$

- $\Omega$ - Omnès function, given via dispersion relation from $\delta_0$
- $\Sigma$ - Selfenergy, given via dispersion relation from $\Omega$ and Im $G$

# Formalism - $T$-matrix

- parametrisation of the resonance potential $V_R$

$$(V_R)_{ij} = \sum_r g_i^{(r)} \frac{s - s_0}{(s - m_{(r)}^2)(s_0 - m_{(r)}^2)} g_j^{(r)}$$



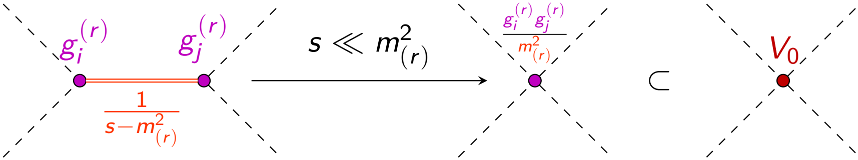

- subtracted at $\eta K$ threshold $s_0 = (m_\eta + m_K)^2$
- parameters: bare couplings $g_i^{(r)}$, bare resonance masses $m_{(r)}$

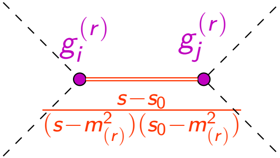
application to $\pi K \to \pi K$:

- $S$-wave ($\ell = 0$) with crossed channel $\eta' K$, $\eta K$ nearly elastic
- present resonances: $\kappa / K_0^*(700)$ , $K_0^*(1430)$ , $K_0^*(1950)$
- input phase $\delta_0$ - non-resonant part of [Peláez and Rodas, 2016]
  $\to$ includes $\kappa / K_0^*(700)$ implicitly
- two resonances explicitly via $V_R$: $K_0^*(1430)$ and $K_0^*(1950)$

# Formalism - resonance potential

- parametrisation of the resonance potential $V_R$:

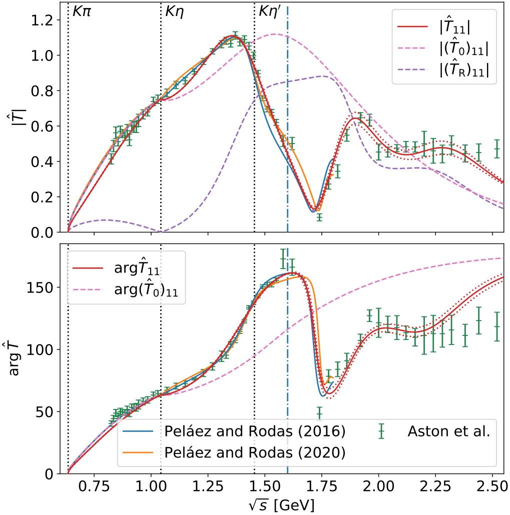



$$(V_R)_{ij} = \widetilde{V}_R(s)_{ij} - \widetilde{V}_R(s_0)_{ij}$$
$$= \sum_r g_i^{(r)} \frac{s - s_0}{(s - m_{(r)}^2)(s_0 - m_{(r)}^2)} g_j^{(r)}$$

- subtracted at $\eta K$ threshold $s_0 = (m_\eta + m_K)^2$:
- parameters: bare couplings $g_i^{(r)}$, bare resonance masses $m_{(r)}$

# Fit to $\pi K$ scattering data

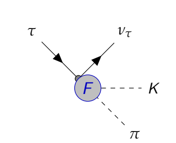



red. $\chi^2$: $\frac{370}{112-6} \approx 3.5$

- contradiction between different data sets [Estabrooks et al., 1978]
- systematic uncertainties underestimated
- covers the data till $\approx 2.3\,\text{GeV}$

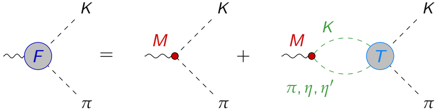
The result is sufficient considering the modest data quality

# Formalism - formfactor $F$




- application to the decay $\tau \to \nu_\tau K \pi$:
- need form factor:

$$F = M + MGT$$




- with the parametrisation of $T$ follows:

$$F = \Omega(\mathbb{1} - V_R \Sigma)^{-1} M$$

- $M$ - reparametrised source term

$$M_i = \sum_k c_i^{(k)} s^k - \sum_r g_i^{(r)} \frac{s - s_0}{(s - m_{(r)}^2)(s_0 - m_{(r)}^2)} \alpha^{(r)}$$

- (additional) parameters: coefficients $c_i^{(k)}$, source couplings $\alpha^{(r)}$

# Application to $\tau \to \nu_\tau K\pi$

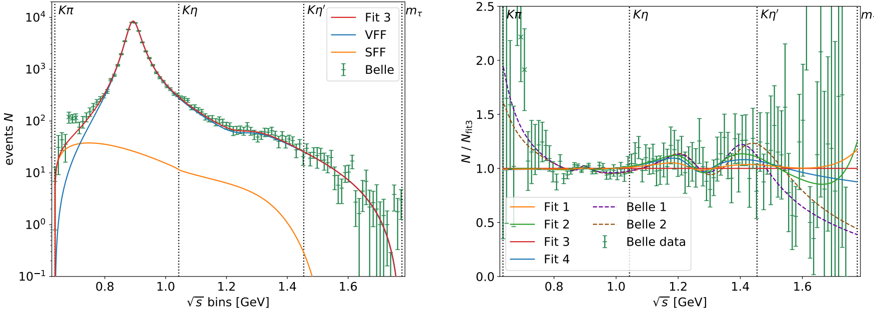
- $S$-wave parametrised as just explained
- $P$-wave parametrised by conventional RChPT model [Antonelli et al., 2013]
- further constraints:
  - normalisation of $F_S^{\pi K}(0), F_S^{\eta' K}(0)$ fixates $c_1^{(0)}$, $c_2^{(0)}$
  - Callan-Treiman (CT) point: $F_S(\Delta_{\pi K}) = \frac{F_K}{F_\pi} + \Delta_{CT}$
- further considerations:
  - $K_0^*(1950)$ lies outside $\tau$ decay region, $\alpha^{(2)}$ poorly constrained
  - higher polynomial terms (i.e. $c_1^{(1)}$) change high-energy behaviour
- four fit scenarios:

$$\alpha^{(2)} = \begin{cases} \text{free} & \text{, fit 1 \& 2} \\ 0 \text{ (fixed)} & \text{, fit 3 \& 4} \end{cases}, \qquad c_1^{(1)} = \begin{cases} \text{free} & \text{, fit 2 \& 4} \\ 0 \text{ (fixed)} & \text{, fit 1 \& 3} \end{cases}$$

# Fit to $\tau \to \nu_\tau K\pi$

- data from Belle [Epifanov et al., 2007]
- compare to Breit-Wigner parametrisation from Belle

red. $\chi^2$: $[0.98, 1.07]$




- $\tau \to \nu_\tau K\pi$ dominated by $K^*(892)$ ($P$-wave)
- overlapping $K_0^*(1430)$ ($S$-wave) and $K^*(1410)$ ($P$-wave)

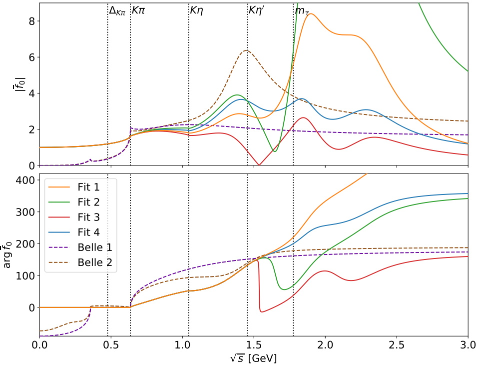
All four fits are equally suitable, only very slight variations

# Scalar form factor

- scalar form factor $F$ fixed by scattering up to source term $M$:

$$M_i = \sum_k c_i^{(k)} s^k - \sum_r g_i^{(r)} \frac{s - s_0}{(s - m_{(r)}^2)(s_0 - m_{(r)}^2)} \alpha^{(r)}$$

- different fit scenarios: $m_\tau < m_{K_0^*(1950)}$; $M_i \to s^{k_{\max}}$




$$\alpha^{(2)} = \begin{cases} \text{free} & ,1 \;\&\; 2 \\ 0 & ,3 \;\&\; 4 \end{cases}, \quad c_1^{(1)} = \begin{cases} \text{free} & ,2 \;\&\; 4 \\ 0 & ,1 \;\&\; 3 \end{cases}$$

- correct phase by construction
- fulfils unitarity & analyticity
- correct normalisation
- in contrast to Breit-Wigner
- further distinction requires additional data

# Further Results

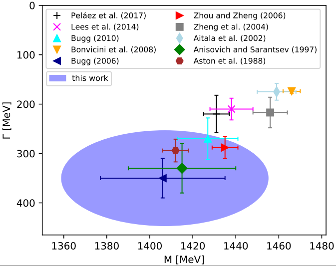
- Branching Ratio: $\mathrm{BR}(\tau \to K_S \pi \nu_\tau) = 4.35(10) \times 10^{-3}$ agrees at $1.5\sigma$ with PDG [Zyla et al., 2020] and [Ryu et al., 2014]
- estimate on BSM CP asymmetry supports [Cirigliano et al., 2018]

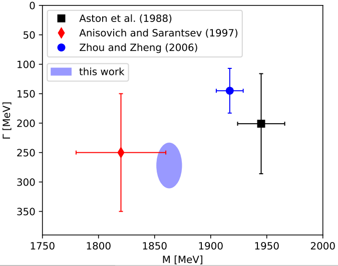
$$A_{CP}^{\tau,\mathrm{BSM}} = -0.034(14)\ \mathrm{Im}\, c_T \quad (c_T,\ \text{Tensor Wilson coefficient})$$

- pole extraction via Padé approximants:

$$m_{K_0^*(1430)} = 1408(4)(47)\ \mathrm{MeV}, \qquad \Gamma_{K_0^*(1430)} = 360(14)(96)\ \mathrm{MeV}$$

$$m_{K_0^*(1950)} = 1863(11)(4)\ \mathrm{MeV}, \qquad \Gamma_{K_0^*(1950)} = 272(38)(8)\ \mathrm{MeV}$$

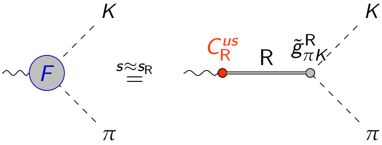


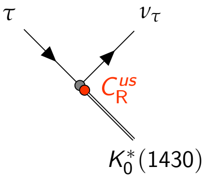
# Further Results

- coupling of scalar resonance $C_R^{us}$ via residue:

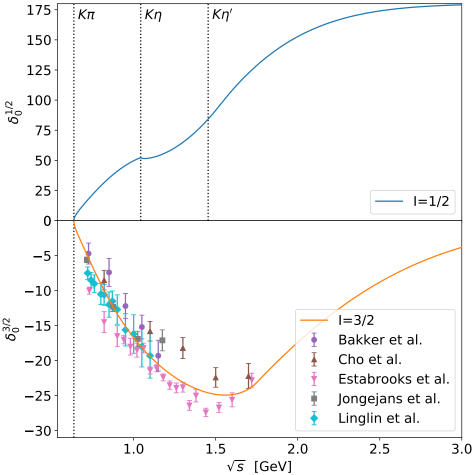
$$F = \sqrt{\frac{2}{3}} \frac{\tilde{g}_{\pi K}^{\mathrm{R}} C_{\mathrm{R}}^{us}}{s_{\mathrm{R}} - s}$$




- $\mathrm{BR}(\tau \to K_0^*(1430)\nu_\tau) \sim \left| C^{us}_{K_0^*(1430)} \right|^2$
- $\mathrm{BR}(\tau \to K_0^*(1430)\nu_\tau) < 1.6 \times 10^{-4}$

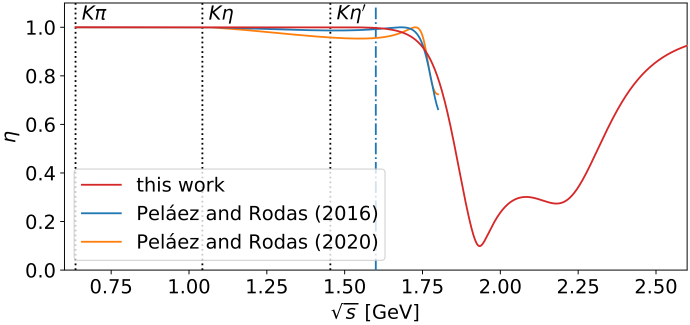


# Plots I



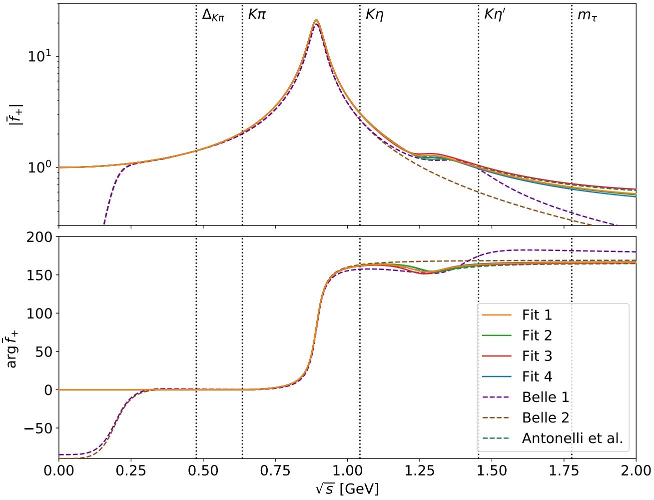
175
150
125
100
75
50
25
0
−5
−10
−15
−20
−25
−30
$\delta_0^{1/2}$
$\delta_0^{3/2}$
$K\pi$
$K\eta$
$K\eta'$
I=1/2
I=3/2
Bakker et al.
Cho et al.
Estabrooks et al.
Jongejans et al.
Linglin et al.
1.0
1.5
2.0
2.5
3.0
$\sqrt{s}$ [GeV]

# Plots II



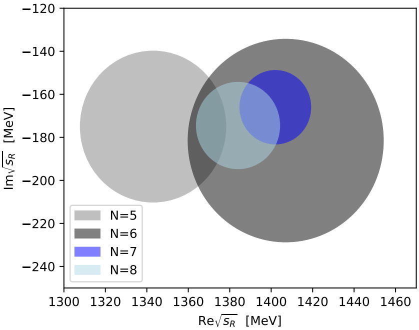
Kπ
Kη
Kη′
1.0
0.8
0.6
0.4
0.2
0.0
η
0.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
√s [GeV]
this work
Peláez and Rodas (2016)
Peláez and Rodas (2020)

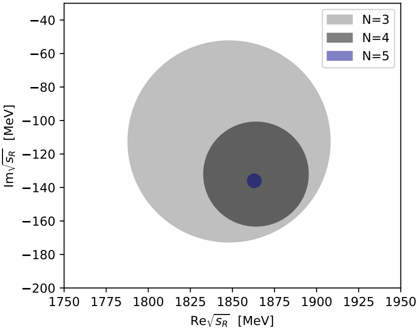
# Plots III



$\Delta_{K\pi}$
$K\pi$
$K\eta$
$K\eta'$
$m_\tau$
$10^1$
$10^0$
$|\bar{f}_+|$
200
150
100
50
0
−50
$\arg \bar{f}_+$
Fit 1
Fit 2
Fit 3
Fit 4
Belle 1
Belle 2
Antonelli et al.
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
$\sqrt{s}$ [GeV]

# Plots IV


−120
−140
−160
−180
−200
−220
−240
Im$\sqrt{s_R}$ [MeV]
1300 1320 1340 1360 1380 1400 1420 1440 1460
Re$\sqrt{s_R}$ [MeV]
N=5
N=6
N=7
N=8


−40
−60
−80
−100
−120
−140
−160
−180
−200
Im$\sqrt{s_R}$ [MeV]
1750 1775 1800 1825 1850 1875 1900 1925 1950
Re$\sqrt{s_R}$ [MeV]
N=3
N=4
N=5

# Tables I

| Parameter | Value |
|---|---|
| $g_1^{(1)}$ [GeV] | 2.898(29) |
| $g_2^{(1)}$ [GeV] | −0.25(35) |
| $g_1^{(2)}$ [GeV] | 2.14(17) |
| $g_2^{(2)}$ [GeV] | 7.70(64) |
| $\widetilde{M}_{(1)}$ [GeV] | 1.5708(33) |
| $\widetilde{M}_{(2)}$ [GeV] | 2.133(36) |
| #data points | 112 |
| #variables | 6 |
| $\chi^2$ | 370.8 |
| $\chi^2$/#d.o.f. | 3.50 |

# Tables II

| Parameter | Fit 1 | Fit 2 | Fit 3 | Fit 4 |
|---|---|---|---|---|
| $\lambda$ | 0.753(11) | 0.7440(94) | 0.7617(87) | 0.746(11) |
| $\alpha^{(1)}$ [GeV] | −0.35(26) | −0.28(21) | 0.035(40) | −0.42(19) |
| $\alpha^{(2)}$ [GeV] | 1.9(1.3) | −4.3(3.9) | 0 (fixed) | 0 (fixed) |
| $c_1^{(1)}$ [GeV$^{-2}$] | 0 (fixed) | −0.65(34) | 0 (fixed) | −0.25(11) |
| $\widetilde{M}_{K^*(892)}$ [MeV] | 943.71(57) | 943.26(53) | 944.04(52) | 943.40(54) |
| $\widetilde{\Gamma}_{K^*(892)}$ [MeV] | 67.15(88) | 66.46(82) | 67.61(80) | 66.69(82) |
| $\widetilde{M}_{K^*(1410)}$ [MeV] | 1355(34) | 1381(39) | 1354(15) | 1357(24) |
| $\widetilde{\Gamma}_{K^*(1410)}$ [MeV] | 205(100) | 205(100) | 229(22) | 176(35) |
| $\beta$ | −0.032(16) | −0.029(12) | −0.0418(48) | −0.0251(75) |
| #data points | 97+1 | 97+1 | 97+1 | 97+1 |
| #variables | 8 | 9 | 7 | 8 |
| $\chi^2$ | 93.1 | 87.4 | 97.7 | 89.4 |
| $\chi^2$/#d.o.f. | 1.03 | 0.98 | 1.07 | 0.99 |

# Tables III

$$
\begin{aligned}
\mathrm{BR}(\tau \to K_S \pi \nu_\tau)\Big|_{\text{Fit 1}} &= 4.334(66)(25) \times 10^{-3}\,, \\
\mathrm{BR}(\tau \to K_S \pi \nu_\tau)\Big|_{\text{Fit 2}} &= 4.390(48)(26) \times 10^{-3}\,, \\
\mathrm{BR}(\tau \to K_S \pi \nu_\tau)\Big|_{\text{Fit 3}} &= 4.284(35)(25) \times 10^{-3}\,, \\
\mathrm{BR}(\tau \to K_S \pi \nu_\tau)\Big|_{\text{Fit 4}} &= 4.377(49)(26) \times 10^{-3}\,, \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{BR}(\tau \to K_S \pi \nu_\tau) &= 4.35(6)_{\text{st}}(3)_{\text{norm}}(7)_{\text{sys}} \times 10^{-3} \\
&= 4.35(10) \times 10^{-3}\,. \qquad (2)
\end{aligned}
$$

# Tables IV

| | |
|---|---|
| $\sqrt{s_R^{(5)}}$ [MeV] | $1863(11)(4) - i\ 136(19)(4)$ |
| $\Delta_{sys}^{(5)}$ [MeV] | 4 |
| $\sqrt{s_0^{(5)}}$ [GeV] | 1.86 |
| $\mathrm{mod}\left(\tilde{g}_{\pi K}^{K_0^*(1950)}\right)$ [GeV] | $4.32(35)(8)$ |
| $\arg\left(\tilde{g}_{\pi K}^{K_0^*(1950)}\right)$ | $-0.20(3)(1)$ |
| $\Gamma_{K_0^*(1950)\to\pi K}$ [MeV] | $184(19)(4)$ |
| $\Gamma_{K_0^*(1950)\to\pi K}/\Gamma_{tot}$ | $0.70(7)(2)$ |

Results of Padé extractions at $N = 5$ for the $K_0^*(1950)$ including statistical (first bracket) and systematic uncertainties (second bracket).

# Tables V

| | |
|---|---|
| $\sqrt{s_R^{(6)}}$ [MeV] | $1408(4)(47) - i\ 180(7)(47)$ |
| $\Delta_{\text{sys}}^{(6)}$ [MeV] | 47 |
| $\sqrt{s_0^{(6)}}$ [GeV] | 1.36 |
| $\text{mod}\left(\tilde{g}_{\pi K}^{K_0^*(1430)}\right)$ [GeV] | 4.96(14)(78) |
| $\arg\left(\tilde{g}_{\pi K}^{K_0^*(1430)}\right)$ | 0.06(1)(4) |
| $\Gamma_{K_0^*(1430)\to\pi K}$ [MeV] | 304(8)(42) |
| $\Gamma_{K_0^*(1430)\to\pi K}/\Gamma_{\text{tot}}$ | 0.87(2)(11) |

Results of Padé extractions at $N = 6$ for the $K_0^*(1430)$ including statistical (first bracket) and systematic uncertainties (second bracket).

# Tables VI

| | $\left\lvert C^{us}_{K_0^*(1430)} \right\rvert$ [GeV] | $\mathrm{BR}(\tau \to K_0^*(1430)\nu_\tau)$ |
|---|---|---|
| Fit 1 | $0.23(4)$ | $0.31(11) \times 10^{-4}$ |
| Fit 2 | $0.37(8)$ | $0.78(38) \times 10^{-4}$ |
| Fit 3 | $0.11(2)$ | $0.07(3) \times 10^{-4}$ |
| Fit 4 | $0.31(5)$ | $0.55(20) \times 10^{-4}$ |
| Other theoretical determinations: | | |
| Ref. [Maltman, 1999] | $0.37$ | $0.79 \times 10^{-4}$ |
| Ref. [El-Bennich et al., 2009] | $0.28$ | $0.45 \times 10^{-4}$ |
| Experiment: | | |
| Ref. [Barate et al., 1999] | $< 0.93$ | $< 5 \times 10^{-4}$ |

# Formulas I

Further formulas from [Von Detten et al., 2021]:

$$T_0 = \begin{pmatrix} \frac{1}{\rho_1}\sin\delta_0 e^{i\delta_0} & 0 \\ 0 & 0 \end{pmatrix} . \tag{3}$$

$$\Omega = \mathbb{1} + T_0 G \tag{4}$$

$$\Omega = \begin{pmatrix} \Omega_{11} & 0 \\ 0 & 1 \end{pmatrix} , \ \Omega_{11} = \exp\left( \frac{s}{\pi} \int_{s_{\text{th}}}^{\infty} \mathrm{d}z \frac{\delta_0(z)}{z(z-s)} \right) . \tag{5}$$

$$\text{disc}\Omega_{if} = 2i(T_0^*)_{im}\rho_m \Omega_{mf} , \tag{6}$$

# Formulas II

$$T_R = \Omega t_R \Omega^T \tag{7}$$

$$t_R = V_\mathrm{R} + V_\mathrm{R} \Sigma t_R \,, \tag{8}$$

$$\Sigma = G\Omega \tag{9}$$

$$\Sigma_{ij}(s) = \frac{s}{2\pi i} \int_{s_\mathrm{th}}^{\infty} \mathrm{d}z \frac{\mathrm{disc}\Sigma_{ij}(z)}{z(z-s)} \,, \tag{10}$$

$$\mathrm{disc}\Sigma_{if} = \Omega^\dagger_{im} \mathrm{disc}\, G_{mm} \Omega_{mf} \,. \tag{11}$$

# Formulas III

$$\hat{T}_{if} = \rho_i \Big( T^{\frac{1}{2}} + T^{\frac{3}{2}}/2 \Big)_{if} \,. \tag{12}$$

$$\delta_0(s) = L - \left(L - \delta_0(s_m)\right) \exp\left( - \frac{(s - s_m)\delta_0'(s_m)}{L - \delta_0(s_m)} \right) \tag{13}$$

$$\begin{aligned} \frac{d\Gamma}{d\sqrt{s}} = \frac{c_\Gamma}{s} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) q_{\pi K} \\ \times \left( q_{\pi K}^2 |\bar{f}_+|^2 + \frac{3\Delta_{\pi K}^2}{4s(1 + 2\frac{s}{m_\tau^2})} |\bar{f}_0|^2 \right) \,, \end{aligned} \tag{14}$$

$$c_\Gamma = \frac{G_F^2 m_\tau^3}{96\pi^3} S_{\mathrm{EW}}^\tau \left( |V_{us}| f_+(0) \right)^2 \left(1 + \delta_{\mathrm{EM}}^{K\tau}\right)^2 \,, \tag{15}$$

# Formulas IV

$$\langle \bar{K}^0(p_K)\pi^-(p_\pi)|\bar{s}\gamma^\mu u|0\rangle = (p_K - p_\pi)^\mu f_+(s) + (p_K + p_\pi)^\mu f_-(s)\,,$$

$$\langle \bar{K}^0(p_K)\pi^-(p_\pi)|\bar{s}u|0\rangle = \frac{\Delta_{\pi K}}{m_s - m_u} f_0(s)\,, \tag{16}$$

$$f_-(s) = \frac{\Delta_{\pi K}}{s}\big(f_0(s) - f_+(s)\big)\,. \tag{17}$$

$$\bar{f}_+(s) = \frac{f_+(s)}{f_+(0)}\,, \qquad \bar{f}_0(s) = \frac{f_0(s)}{f_+(0)}\,. \tag{18}$$

$$\bar{f}_0(\Delta_{\pi K}) = \frac{F_K}{F_\pi} + \Delta_{CT}\,, \tag{19}$$

# Formulas V

$$A_{CP}^{\tau,\mathrm{BSM}} = \frac{\mathrm{Im}\, c_T}{\Gamma_\tau \mathrm{BR}(\tau \to K_S \pi \nu_\tau)} \times \int_{s_{\pi K}}^{m_\tau^2} \mathrm{d}s' \kappa(s') |f_+(s')| |B_T(s')| \sin\left(\delta_+(s') - \delta_T(s')\right) , \tag{20}$$

$$\sqrt{s_\mathrm{R}} = M_\mathrm{R} - i \frac{\Gamma_\mathrm{R}}{2} . \tag{21}$$

$$\bar{f}_0(s) = \sqrt{\frac{2}{3}} \frac{\tilde{g}_{\pi K}^{\mathrm{R}} C_\mathrm{R}^{us}}{s_\mathrm{R} - s}, \quad \Gamma(\tau \to \mathrm{R}\nu_\tau) = \frac{6\pi^2 c_\Gamma \Delta_{\pi K}^2}{M_\mathrm{R}^4} \left(1 - \frac{M_\mathrm{R}^2}{m_\tau^2}\right)^2 \left|C_\mathrm{R}^{us}\right|^2 \tag{22}$$

# Analyticity and unitarity

- $S$-matrix describing scattering processes:

$$|\text{out}\rangle = S\,|\text{in}\rangle$$

- $T$-matrix is non-trivial part of $S$-matrix:

$$S = 1 + iT$$

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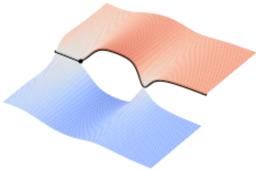
- analyticity up to cut(s) on the real axis
- Schwartz reflection principle:

$$T(s^*) = T(s)^* \;\Rightarrow\; \text{disc}\,T = 2i\,\text{Im}\,T$$



imag. part of $1^{\text{st}}$ sheet
(with right-hand cut (RHC))

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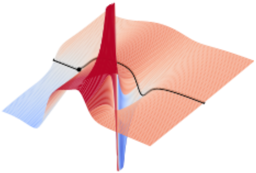

imag. part of 1st & 2nd sheet (with pole on 2nd sheet)

- resonance structures from poles on the second sheet

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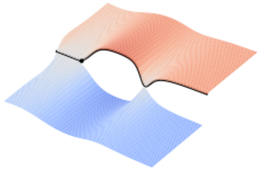


imag. part of 1st sheet
(with right-hand cut (RHC))

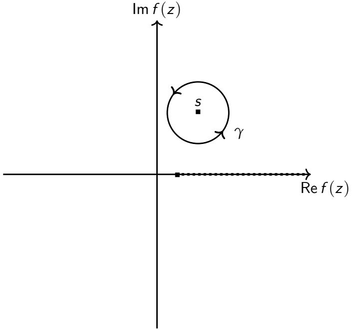
- resonance structures from poles on the second sheet
- unitarity of the $S$-Matrix (above threshold):

$$S^\dagger S = 1 \;\Rightarrow\; \text{Im}(T^{-1}) = -\sigma = -\frac{q_{\pi K}}{\sqrt{s}} \quad \text{(phase space)}$$

# Dispersion theory

- Cauchy's integral formula:

$$f(s) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-s} \, \mathrm{d}z$$



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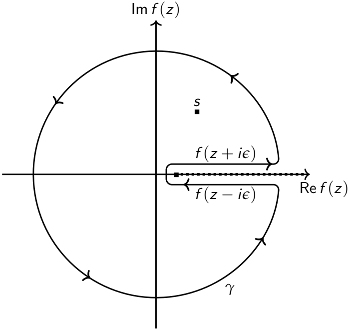
- discontinuity:

$$\mathrm{disc} f(z) := \lim_{\epsilon \to 0^+} \left(f(z+i\epsilon) - f(z-i\epsilon)\right)$$



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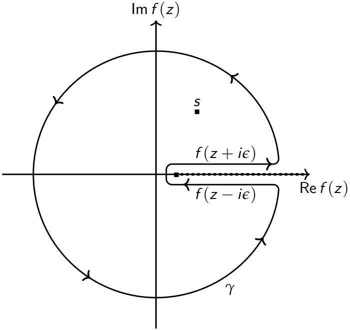
  $\mathrm{disc} f(z) := \lim_{\epsilon\to 0^+}\left(f(z+i\epsilon) - f(z-i\epsilon)\right)$

- if $f(z) \xrightarrow{|z|\to\infty} 0$



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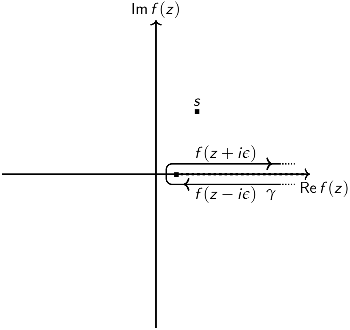
- dispersion relation:

$$f(s) = \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{\mathrm{disc} f(z)}{z-s}\,\mathrm{d}z$$



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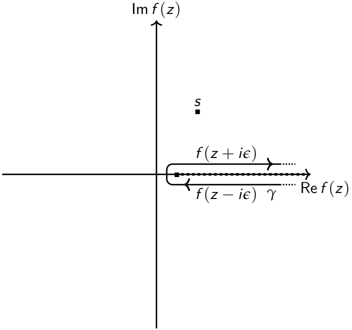
- dispersion relation:

$$f(s) = \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{\mathrm{disc} f(z)}{z-s}\,\mathrm{d}z$$

- if $\tilde{f}(z) = \frac{f(z)-f(\tilde{s})}{z-\tilde{s}} \xrightarrow{|z|\to\infty} 0$ with $\tilde{s} < s_0$



# Dispersion theory

- Cauchy's integral formula:

$$f(s) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-s}\,\mathrm{d}z$$

- discontinuity:

$$\mathrm{disc} f(z) := \lim_{\epsilon \to 0^+} \left(f(z+i\epsilon) - f(z-i\epsilon)\right)$$

- dispersion relation:

$$f(s) = \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{\mathrm{disc} f(z)}{z-s}\,\mathrm{d}z$$

- once subtracted dispersion relation:

$$f(s) = f(\tilde{s}) + \frac{s-\tilde{s}}{2\pi i} \int_{s_0}^{\infty} \frac{\mathrm{disc} f(z)}{(z-s)(z-\tilde{s})}\,\mathrm{d}z$$