Data-driven dispersive analysis of the ππ and πK scattering

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in coll. with Oleksandra Deineka and Marc Vanderhaeghen

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SFB 1044
THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

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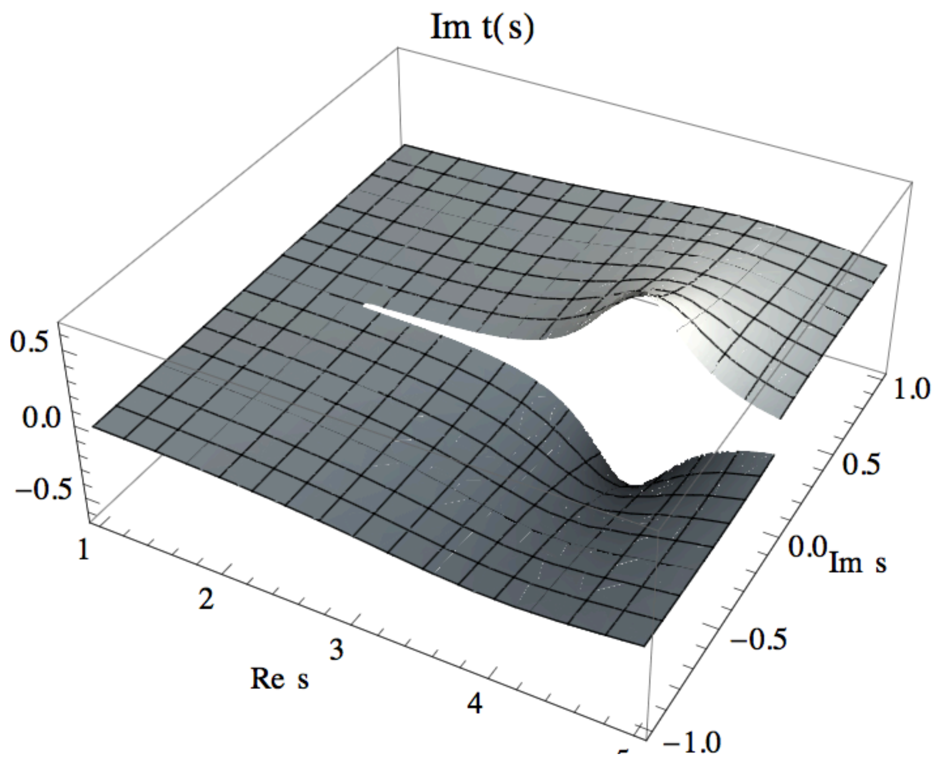
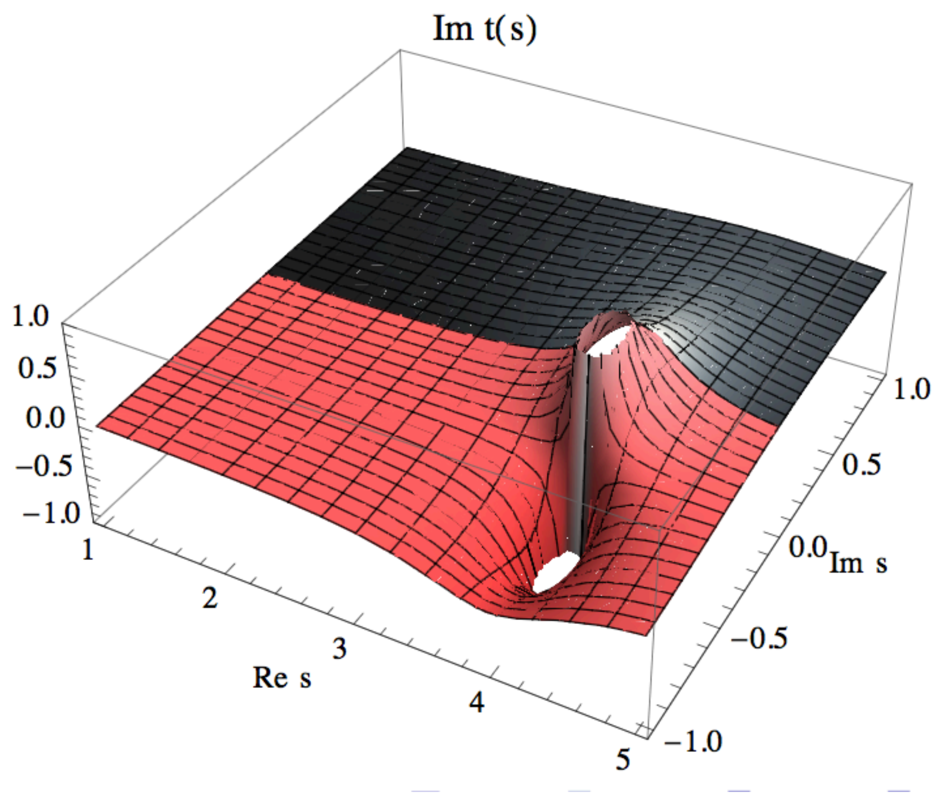
Introduction and Motivation

- New era of hadron spectroscopy, motivated by recent discoveries of unexpected exotic hadron resonances: LHCb, BESIII, COMPASS, Belle, …

- To correctly identify resonance parameters one has to search for poles in the complex plane





- Particularly important when
  - ➢ there is an interplay between several inelastic channels
  - ➢ the pole is lying very deep in the complex plane

Introduction and Motivation

- Call for a framework which complies with the main principles of the S-matrix theory:
  ➢ Unitarity
  ➢ Analyticity
  ➢ Crossing symmetry

Roy (Roy-Steiner) equations

- Practical application of **Roy-like equations** is **limited**:
  ➢ requires experimental knowledge of many partial waves in direct and crossed channels
  ➢ finite truncation limits results to a given kinematical region
  ➢ coupled-channel treatment is very complicated

$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4m_\pi^2}^{\infty} ds' \, K_{JJ'}^{II'}(s,s') \, \mathrm{Im} t_{J'}^{I'}(s')$$

subtraction polynomial

kernel functions known analytically

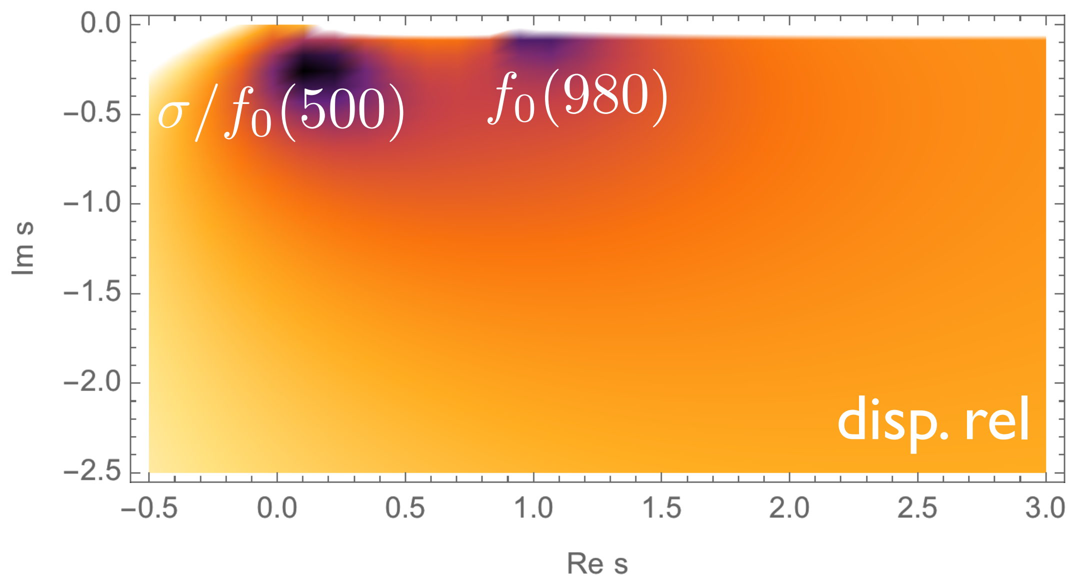
Roy (1971)
Colangelo et al. (2001)
Caprini et al. (2006)
Garcia-Martin et al. (2011)

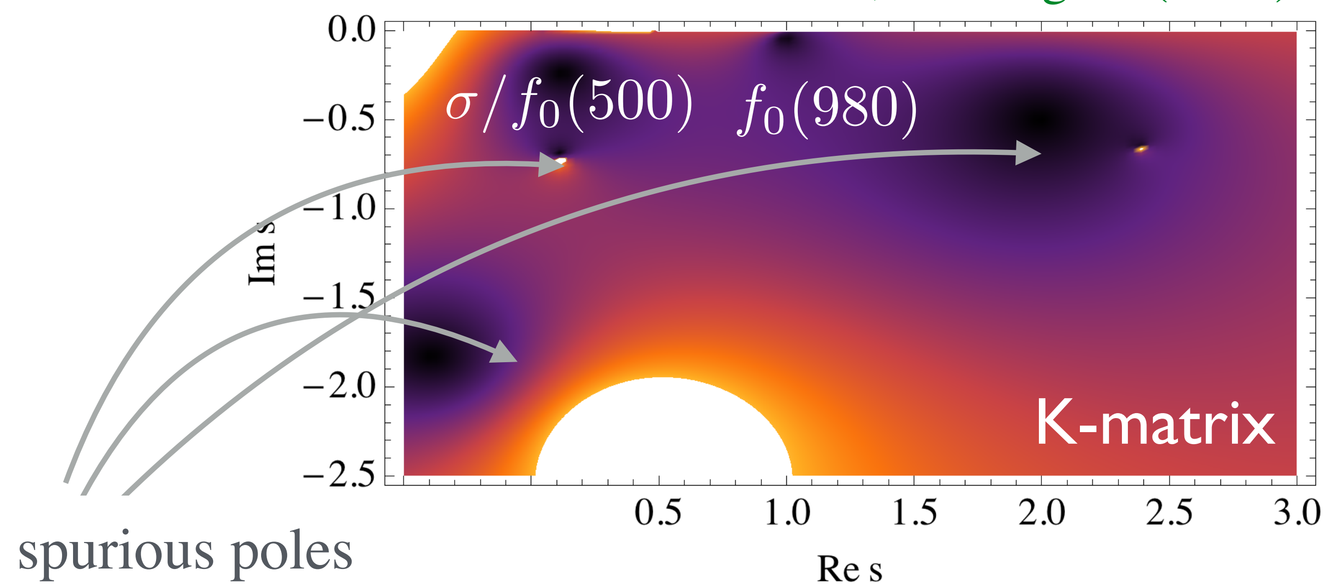
Introduction and Motivation

- It is a common practice to **ignore** all S-matrix constraints or implement just **unitarity**:

  ➢Sum of Breit-Wigner parameterisations
  ➢Bethe-Salpeter like equations
  ➢K-matrix
  ➢ ….

Dai, Pennington (2014)





- Alternatively, once can consider a coupled-channel **p.w. dispersion relation** which respects both **uniarity** and **analyticity**. It can be solved using **N/D ansazt**

Chew, Mandelstam (1960)
Luming (1964)
Johnson, Warnock (1981)

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} = \sum_c D^{-1}_{ac}(s)\, N_{cb}(s)$$

Introduction and Motivation

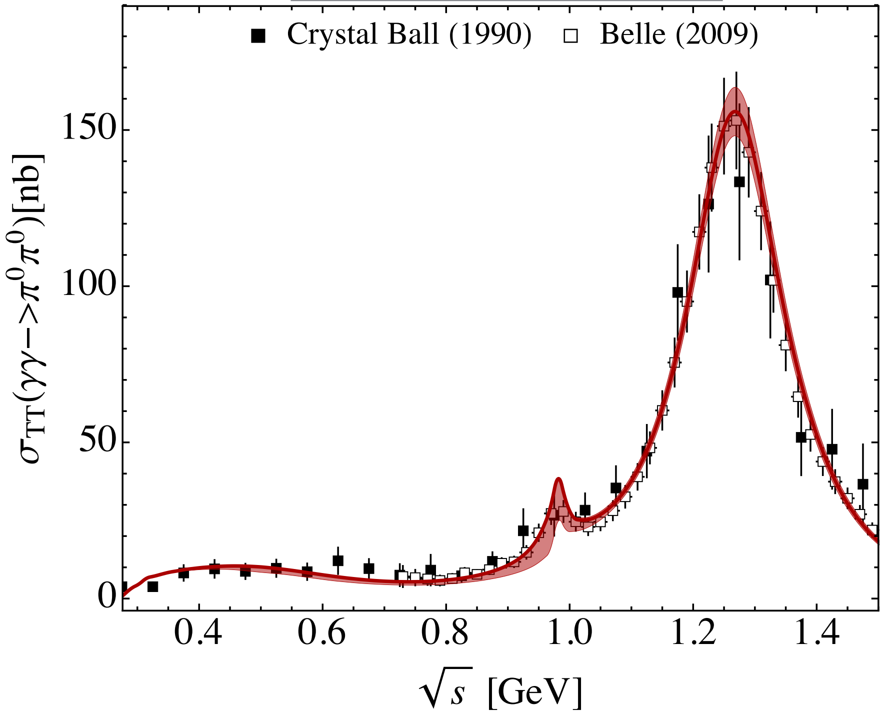
- Why do we solve **p.w. dispersion relation** for the ππ and πK scattering?

  - ➢ Well studied systems (amplitudes are known from Roy (Roy-Steiner) analyses) Caprini et al. (2006) Garcia-Martin et al. (2011)
  - ➢ Recent **lattice** studies of ππ (πK) scattering for m_π=236 (239) MeV Hadron Spectrum Coll.(2017, 2019)
  - ➢ The system of ππ (or πK) shows up very often as a part of FSI in many hadronic reactions:

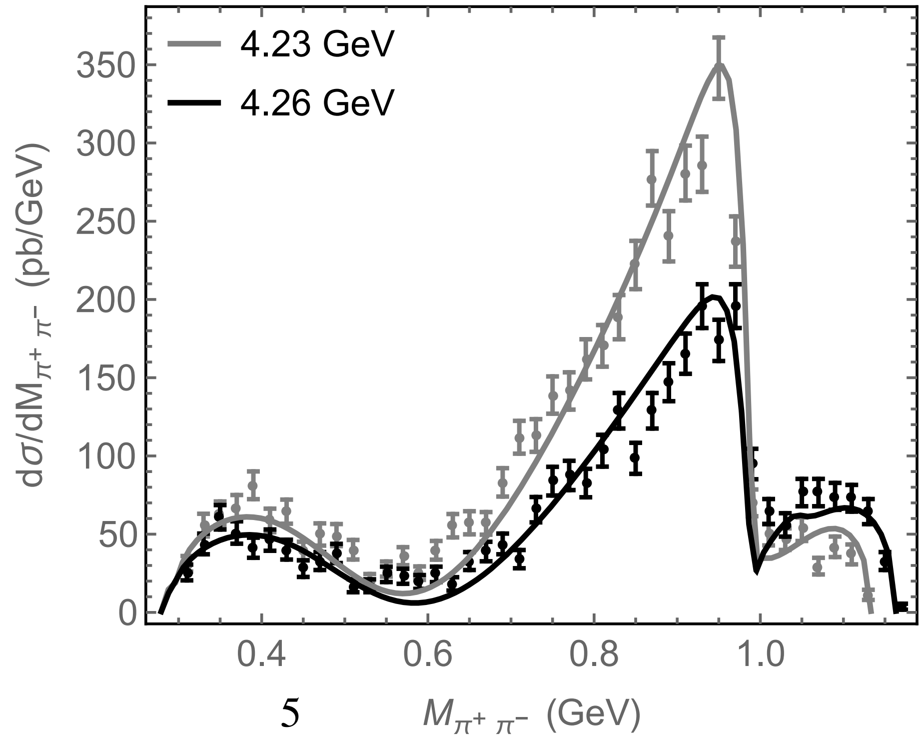
    η →3π, η' →ππη, γγ→ππ, e^+e^-→J/ψ(ψ')ππ, D→ππK, ….

- In practical applications, the FSI are implemented with the help of so-called **Omnes function**, which does not have left-hand cuts. It arises naturally from the *N*/*D* approach as an inverse of the *D*-function

γγ →$\pi^0\pi^0$



e^+e^- →J/ψππ

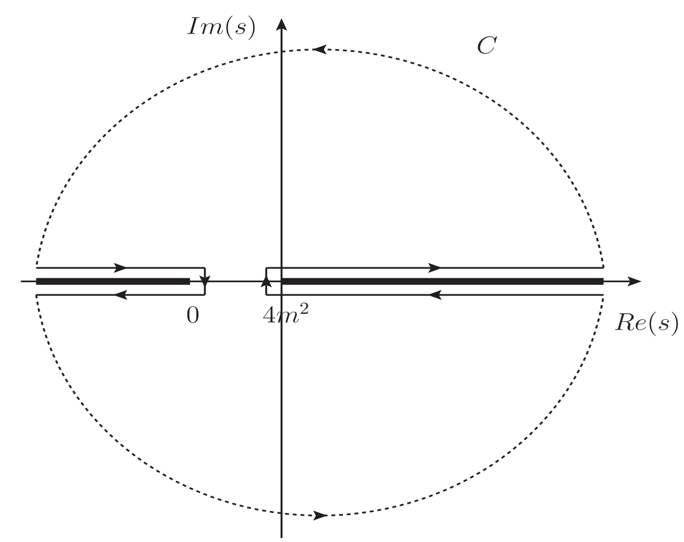


I.D, Deineka, Vanderhaeghen (2019, 2020)
I.D, Molnar, Vanderhaeghen (2019, 2020)
I.D, Hoferichter, Stoffer (2021)

f_0 to $(g-2)_\mu$

p.w. dispersion relation

- Based on maximal analyticity principle on can write **p.w. dispersion relation**



$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s}$$

- **Unitarity relation** for the p.w. amplitude
➢ guarantees that the p.w. amplitudes behave asymptotically no worse **than a constant**

$$\text{Disc}\, t_{ab}(s) = \sum_c t_{ac}(s)\, \rho_c(s)\, t^*_{cb}(s)$$

$$-\frac{1}{2\rho_1} \leq \text{Re}\, t_{11}(s) \leq \frac{1}{2\rho_1}, \quad 0 < \text{Im}\, t_{11}(s) \leq \frac{1}{\rho_1}$$

...

- In accordance with **unitarity bound** we subtract once the dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc } t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s')\, \rho_c(s')\, t^*_{cb}(s')}{s'-s}$$

N/D method

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc } t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

can be solved using N/D method with input from $U_{ab}(s)$ **above threshold**

Chew, Mandelstam (1960)
Luming (1964)
Johnson, Warnock (1981)

$$t_{ab}(s) = \sum_c D^{-1}_{ac}(s)\,N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s') - U_{cb}(s))}{s'-s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ab}(s')\,\rho_b(s')}{s'-s}$$

the obtained N/D solution can be checked that it **fulfils** the p.w. dispersion relation

- Bound state case

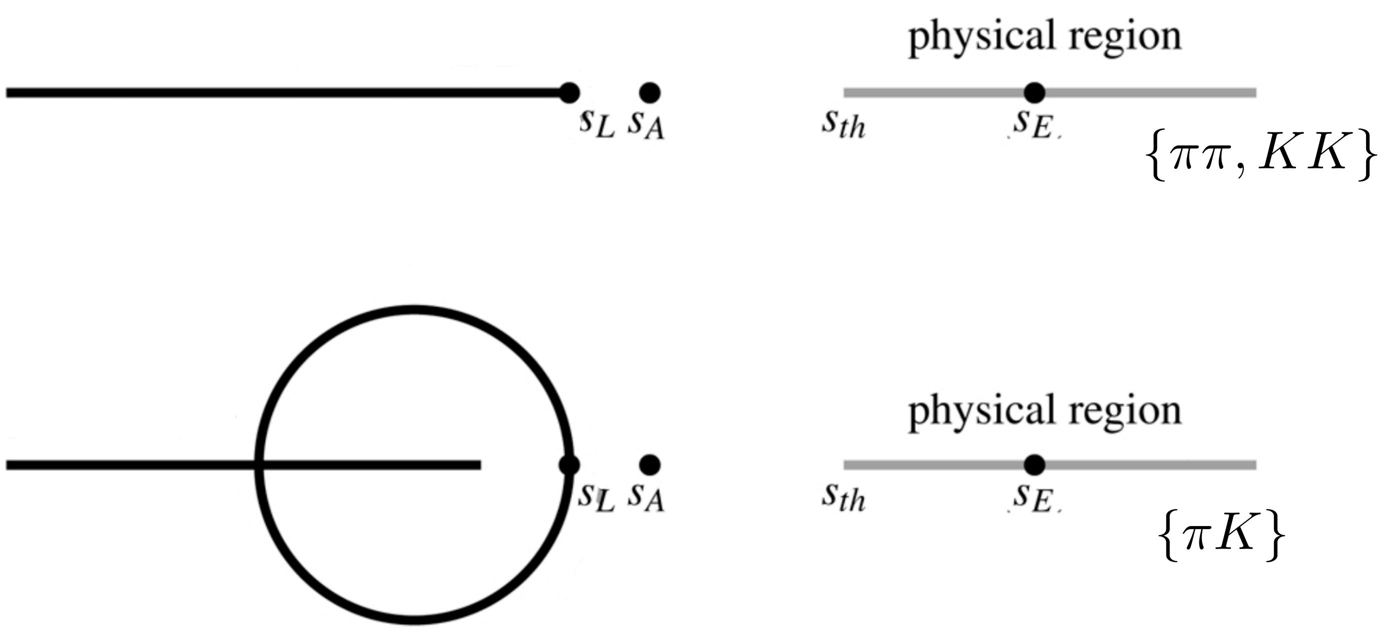
$$\det(D_{ab}(s_B)) = 0$$

$$t_{ab}(s) = U_{ab}(s) + \frac{s}{s_B}\frac{g_a g_b}{s_B - s} + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

Left-hand cuts

- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane.

$$U_{ab}(s) = t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc } t_{ab}(s')}{s'-s}$$



Left-hand cuts

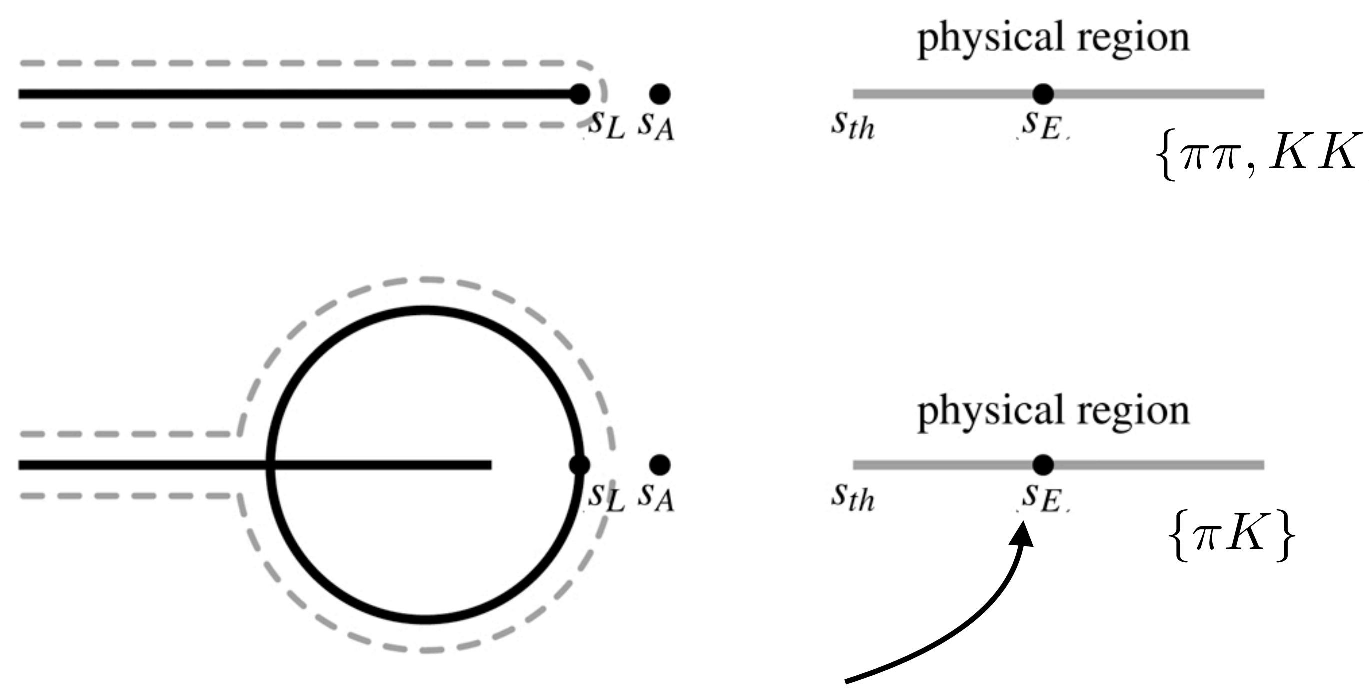
- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane. We approximate them as an expansion in a **conformal mapping variable** $\xi(s)$

Gasparyan, Lutz (2010)

$$U_{ab}(s) = t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc }t_{ab}(s')}{s'-s}$$

$$\simeq \sum_{n=0}^{\infty} C_{ab,n}\,(\xi_{ab}(s))^n$$

unknown coefficients fitted to data or/and EFT



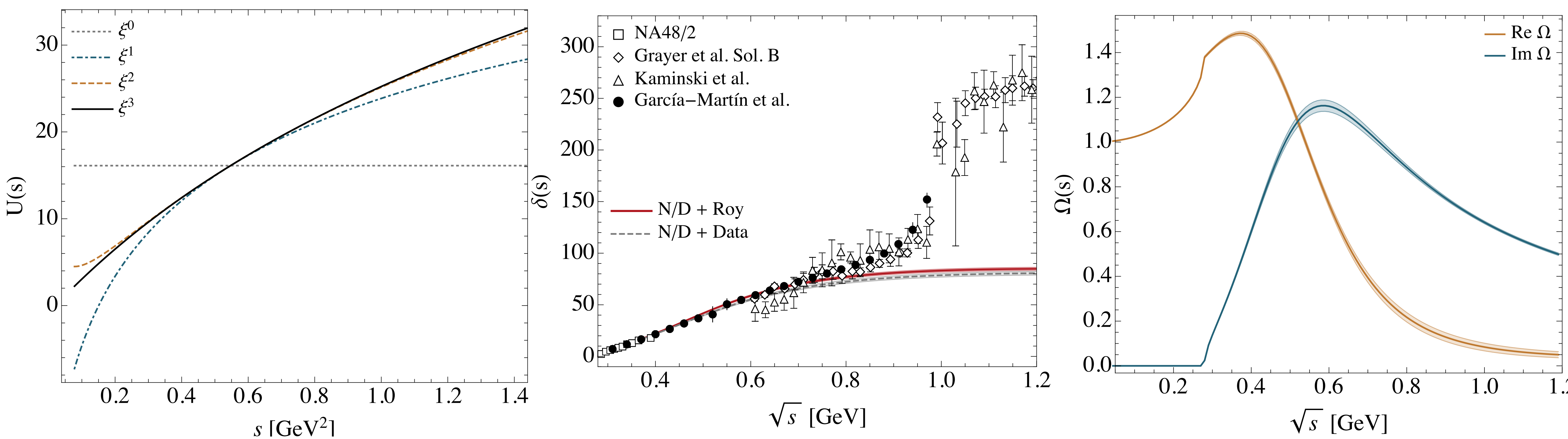
$$\xi(s_E) = 0$$
$$\xi(s_L) = -1$$

$$\sqrt{s_E} = \frac{1}{2}(\sqrt{s_{th}} + \sqrt{s_{max}})$$

source of the systematic uncertainties

- **Adler zero** add additional constraint on the unknown coefficients and improve the convergence close to the left-hand cut

single-channel {ππ}

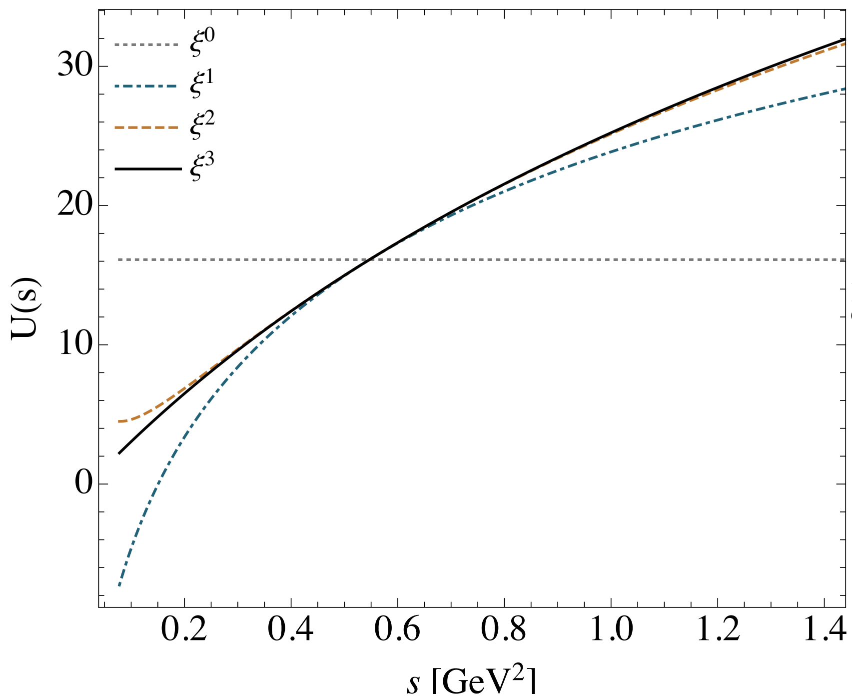


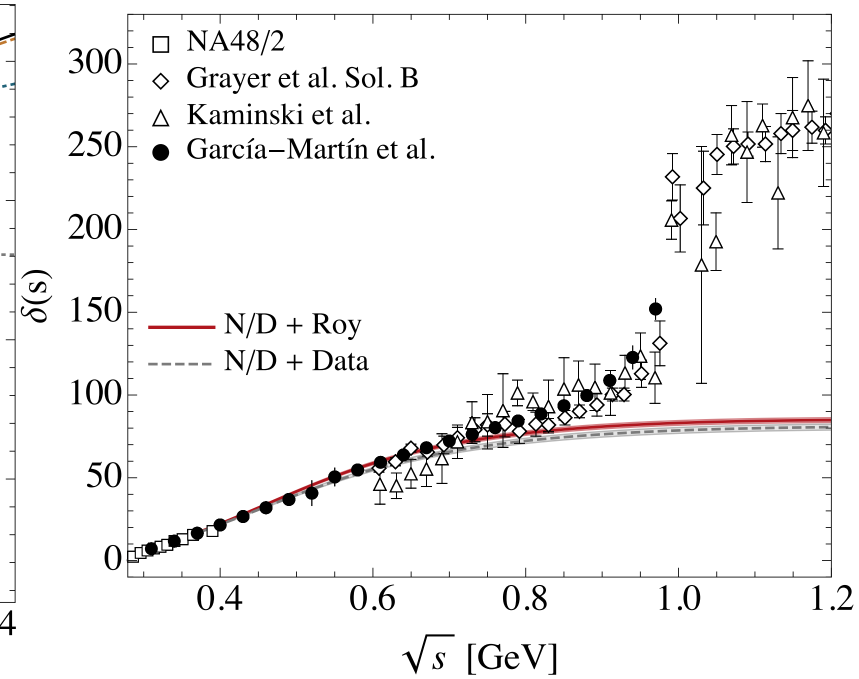
Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

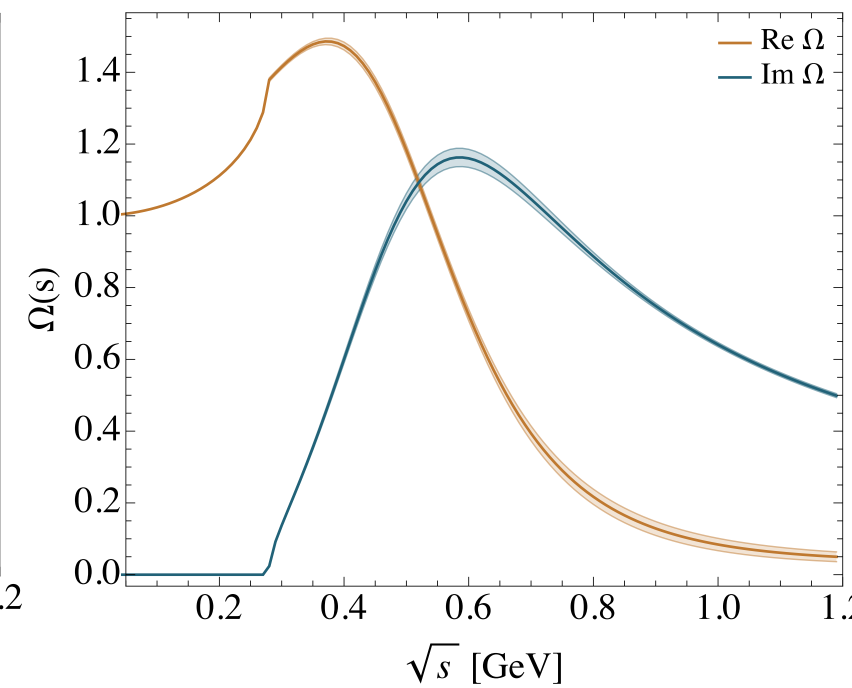
	Our results		Roy-like analyses	
	pole position, MeV	coupling, GeV	pole position, MeV	coupling, GeV
$\sigma/f_0(500)$	$458(7)^{+4}_{-10} - i\,245(6)^{+7}_{-10}$	$\pi\pi : 3.15(5)^{+0.11}_{-0.20}$	$449^{+22}_{-16} - i\,275(15)$	$\pi\pi : 3.45^{+0.25}_{-0.29}$
fit to Exp	$435(7)^{+6}_{-8} - i\,250(5)^{+6}_{-8}$			

Caprini et al. (2006)
Garcia-Martin et al. (2011)

$$\Omega(s) = D^{-1}(s)$$







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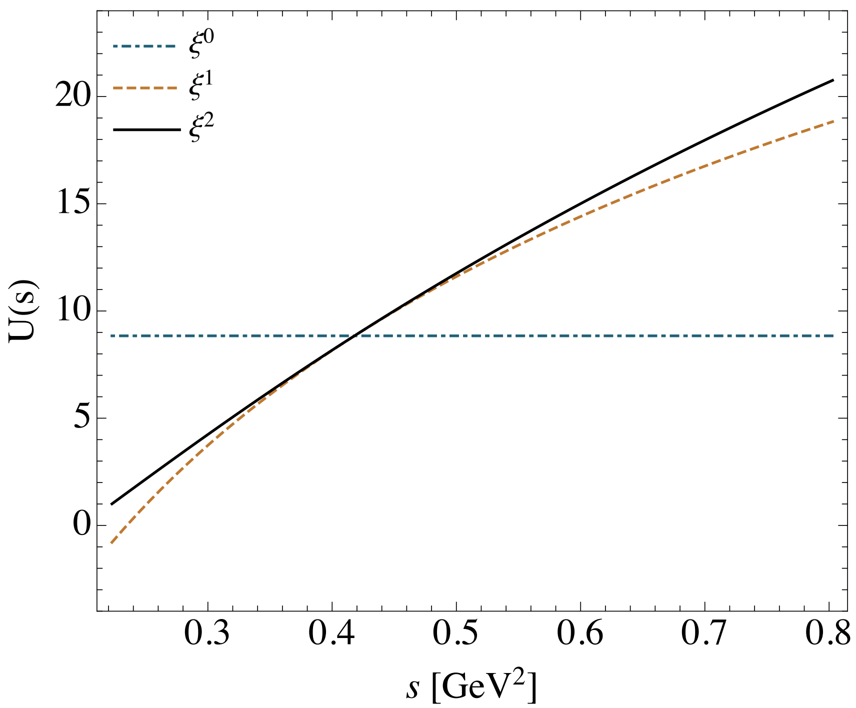
Caprini et al. (2006)
Garcia-Martin et al. (2011)

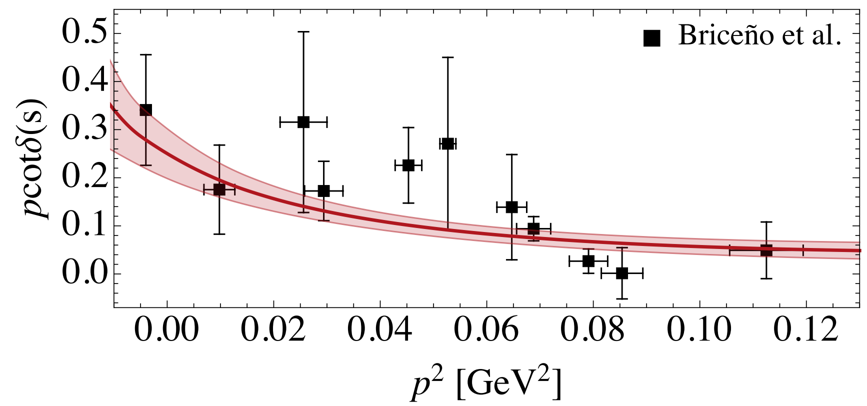
$$\Omega(s) = D^{-1}(s) = \exp\left(\frac{s}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right)$$

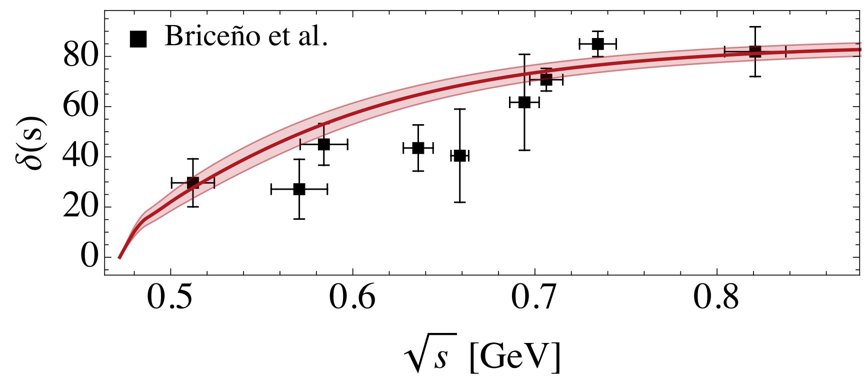
- Similar results for single-channel ππ phase-shift and Omnes function can obtained by using mIAM with the ChPT input for the left-hand cuts and subtraction constants

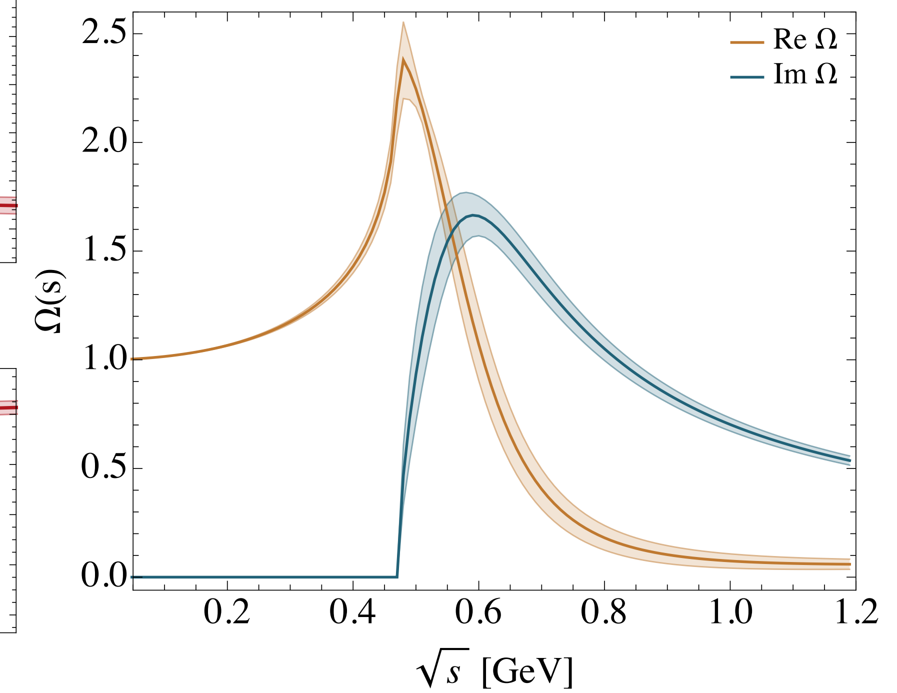
Gomez Nicola et al. (2008), Hanhart et al. (2008), Nebreda et al. (2010) Pelaez et al. (2010)

single-channel {ππ} for m_π=236 MeV









Input: lattice data + Adler zero NLO

	lattice		UχPT predictions	
	pole position, MeV	Method	pole position, MeV	Method
$\sigma/f_0(500)$	$498(21)^{+12}_{-19} - i\,138(13)^{+5}_{-10}$	N/D	$510 - i\,175$	$\mathrm{mIAM_{NNLO}}$, fit D
$(m_\pi = 236$ MeV$)$	$(550 - 780) - i\,(115 - 285)$	K-matrix	$490(15) - i\,180(10)$	$\mathrm{BSE_{NLO}}$

Briceno et al. (2017)

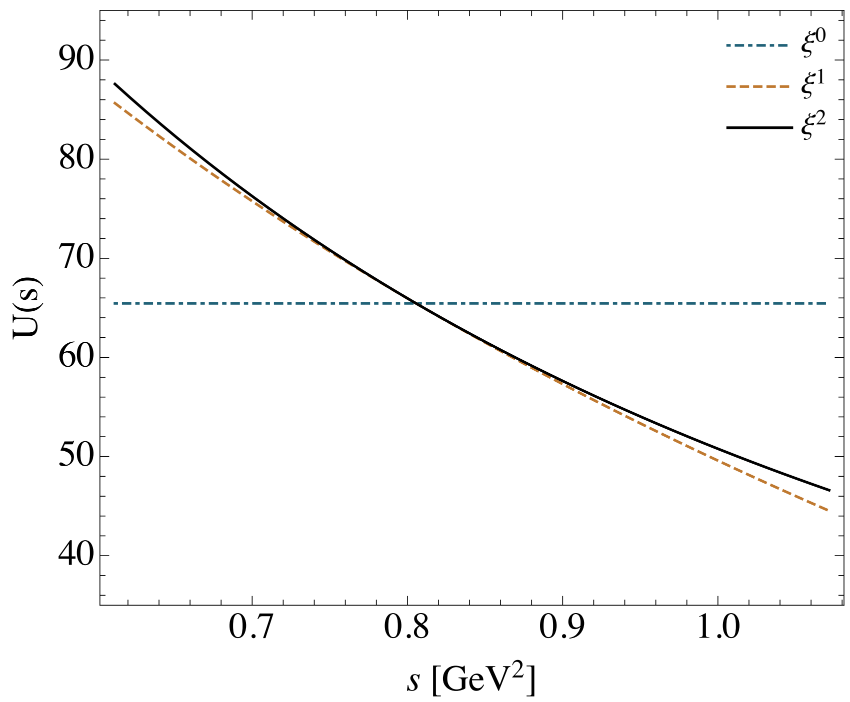
Pelaez, Rios (2010)
Albaladejo, Oller (2012)

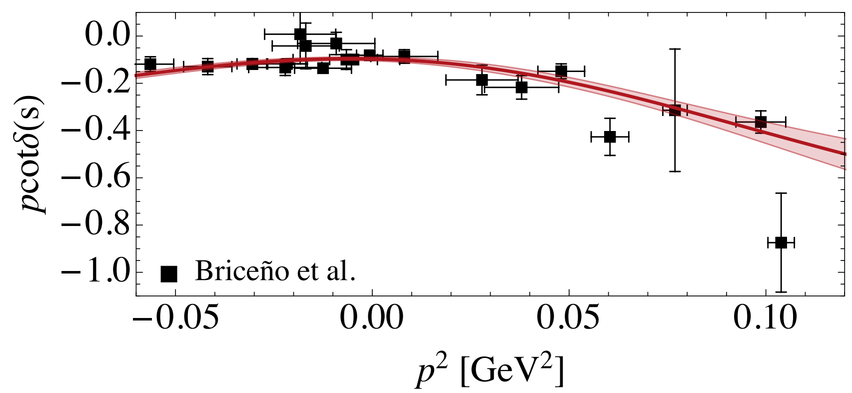
$m_\pi\, a(N/D\text{+lattice}) = 0.98(19)$

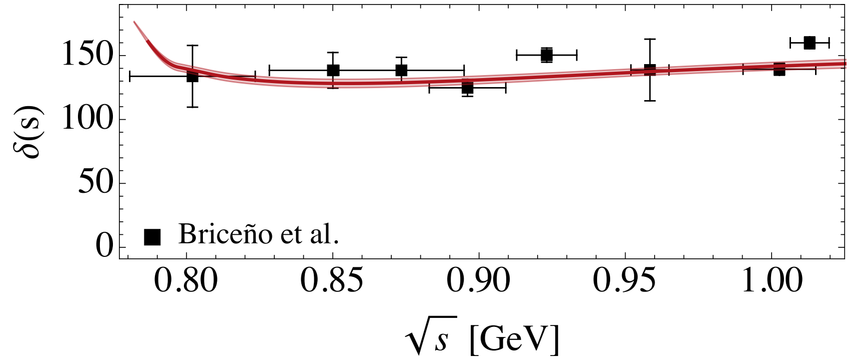
$m_\pi\, a(\chi\mathrm{PT_{NNLO}}) = 0.75 - 0.87$

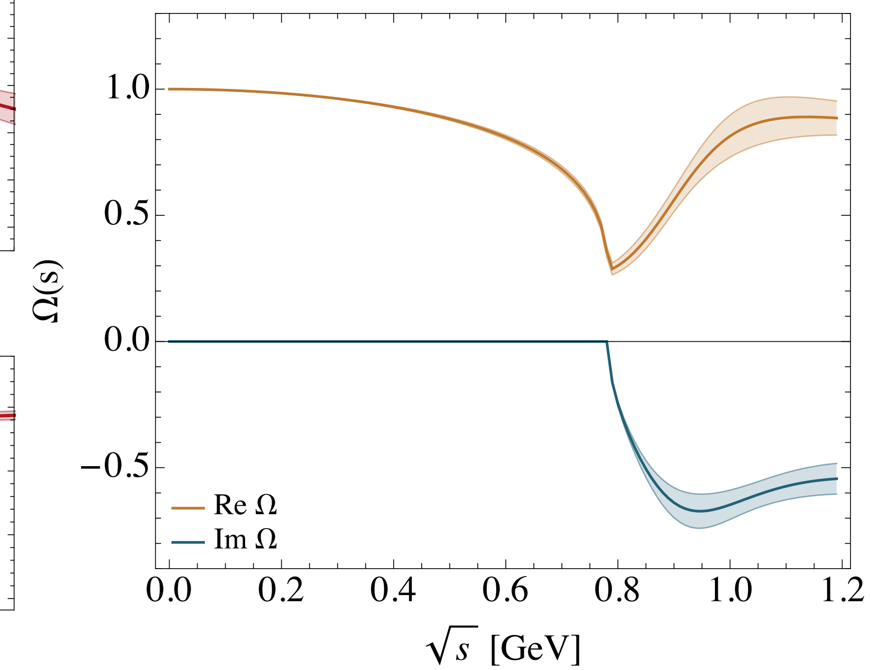
Colangelo et al. (2001)

single-channel {ππ} for m_π=391 MeV









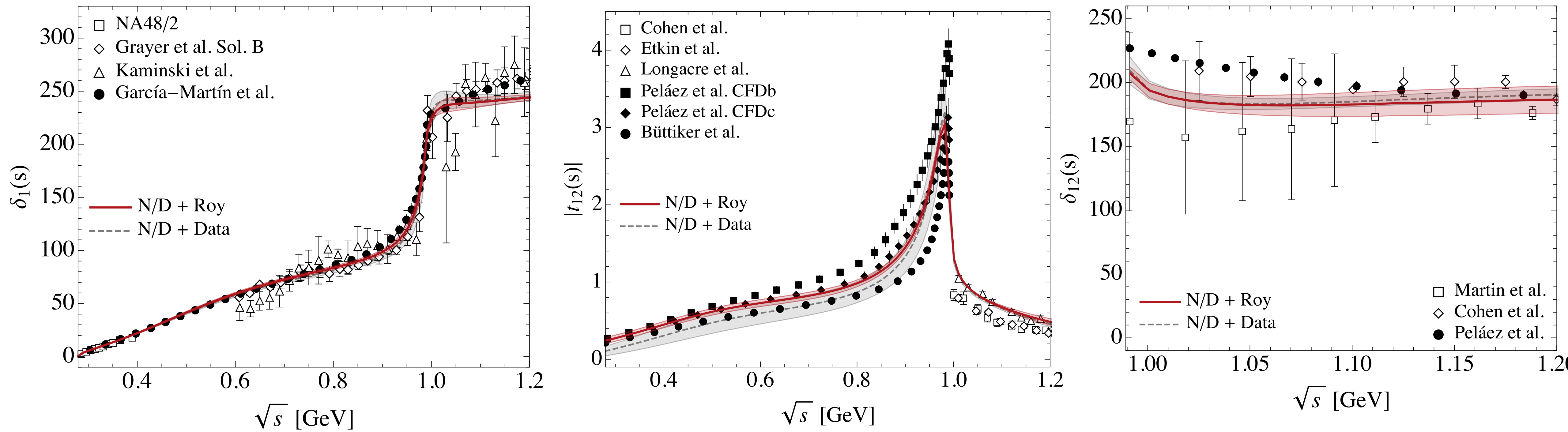
Input: lattice data

	lattice		UχPT predictions	
	pole position, MeV	Method	pole position, MeV	Method
$\sigma/f_0(500)$	758(5)(0)	N/D	765	mIAM$_{\rm NNLO}$, fit D
($m_\pi = 391$ MeV)	758(4)(0)	K-matrix		Pelaez, Rios (2010)

Briceno et al. (2017)

$$\Omega(s) = \left(\frac{s_B - s}{s_B}\right) D^{-1}(s) = \exp\left(\frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right)$$

coupled-channel {ππ, KK}



Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

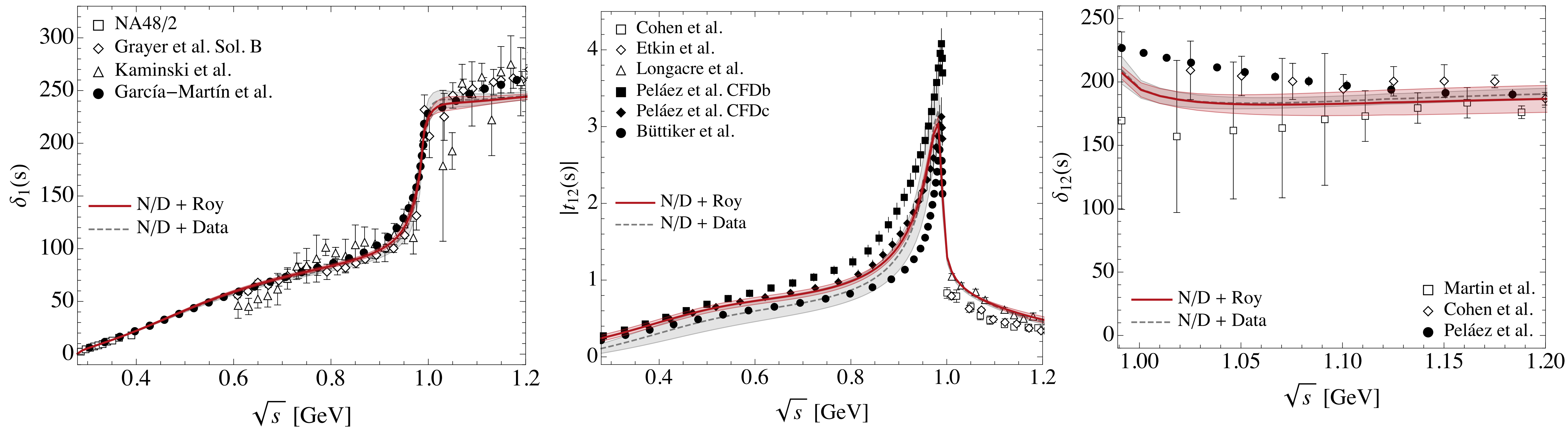
$$t_{ab}(s) = \begin{pmatrix} \frac{\eta(s)\, e^{2\,i\,\delta_1(s)}-1}{2\,i\,\rho_1(s)} & |t_{12}(s)|\, e^{\delta_{12}(s)} \\ |t_{12}(s)|\, e^{\delta_{12}(s)} & \frac{\eta(s)\, e^{2\,i\,\delta_2(s)}-1}{2\,i\,\rho_2(s)} \end{pmatrix}_{ab}$$

$$\eta(s) = \sqrt{1 - 4\,\rho_1(s)\,\rho_2(s)\,|t_{12}(s)|^2}$$

$$\delta_{12}(s) = \delta_1(s) + \delta_2(s)\,\theta(s > 4m_K^2)$$

In the **two-channel approximation** one needs to make the choice of which experimental data/Roy analysis include in the fit

coupled-channel {ππ, KK}

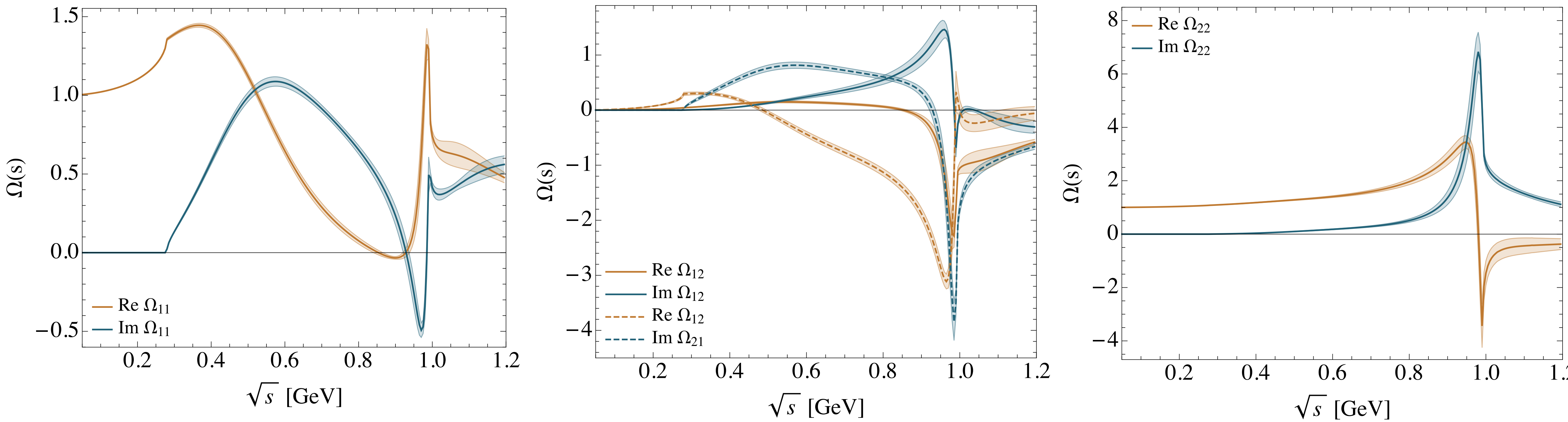


Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

	Our results		Roy-like analyses	
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i\,256(9)^{+5}_{-8}$	$\pi\pi : 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K} : 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i\,275(15)$	$\pi\pi : 3.45^{+0.25}_{-0.29}$ $K\bar{K}$: –
fit to Exp	$454(12)^{+6}_{-7} - i\,262(12)^{+8}_{-12}$			
$f_0(980)$	$993(2)^{+2}_{-1} - i\,21(3)^{+2}_{-4}$	$\pi\pi : 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K} : 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i\,25^{+11}_{-6}$	$\pi\pi : 2.3(2)$ $K\bar{K}$: –
fit to Exp	$990(7)^{+2}_{-4} - i\,17(7)^{+4}_{-1}$			

Caprini et al. (2006)
Garcia-Martin et al. (2011)
Moussallam (2011)

Omnes function {ππ, KK}



Omnes function fulfils the unitarity relation on the right-hand cut and analytic everywhere else.
For the case of no bound states or CDD poles:

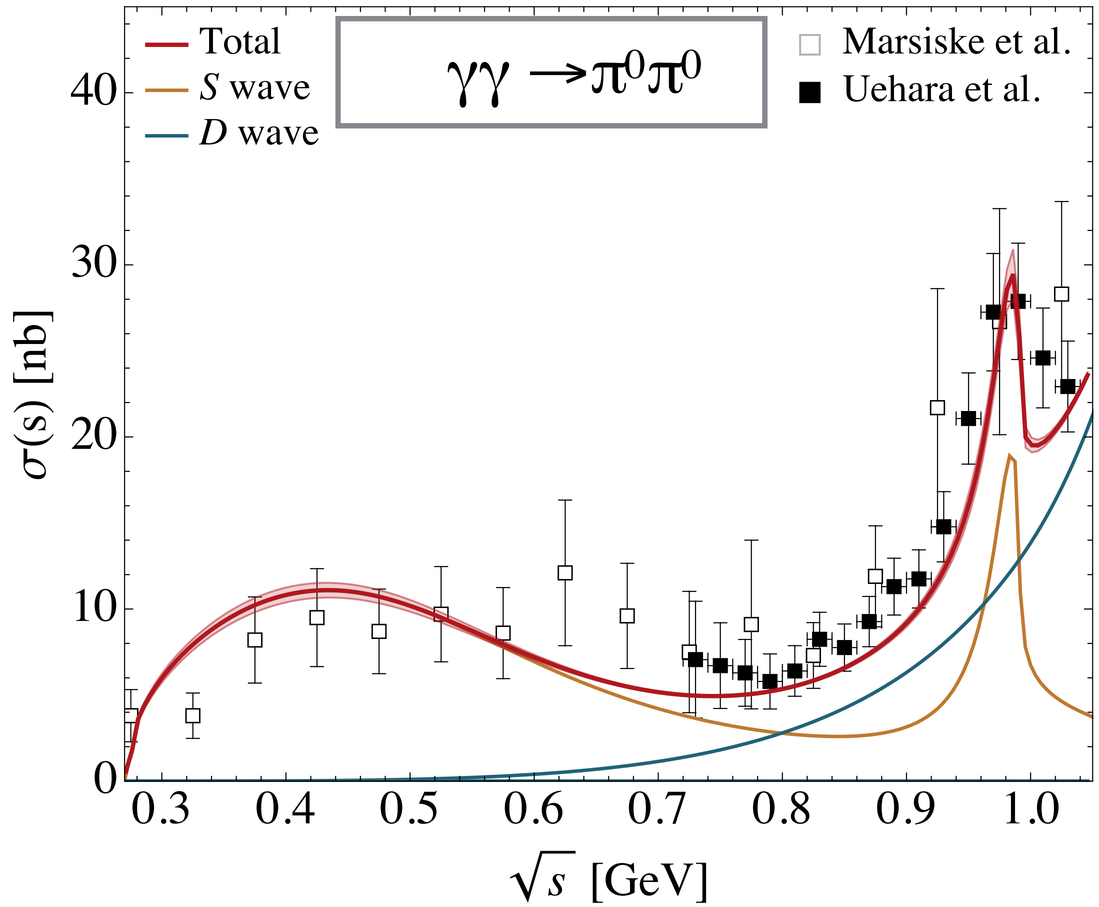
$$\Omega_{ab}(s) = D^{-1}_{ab}(s)$$

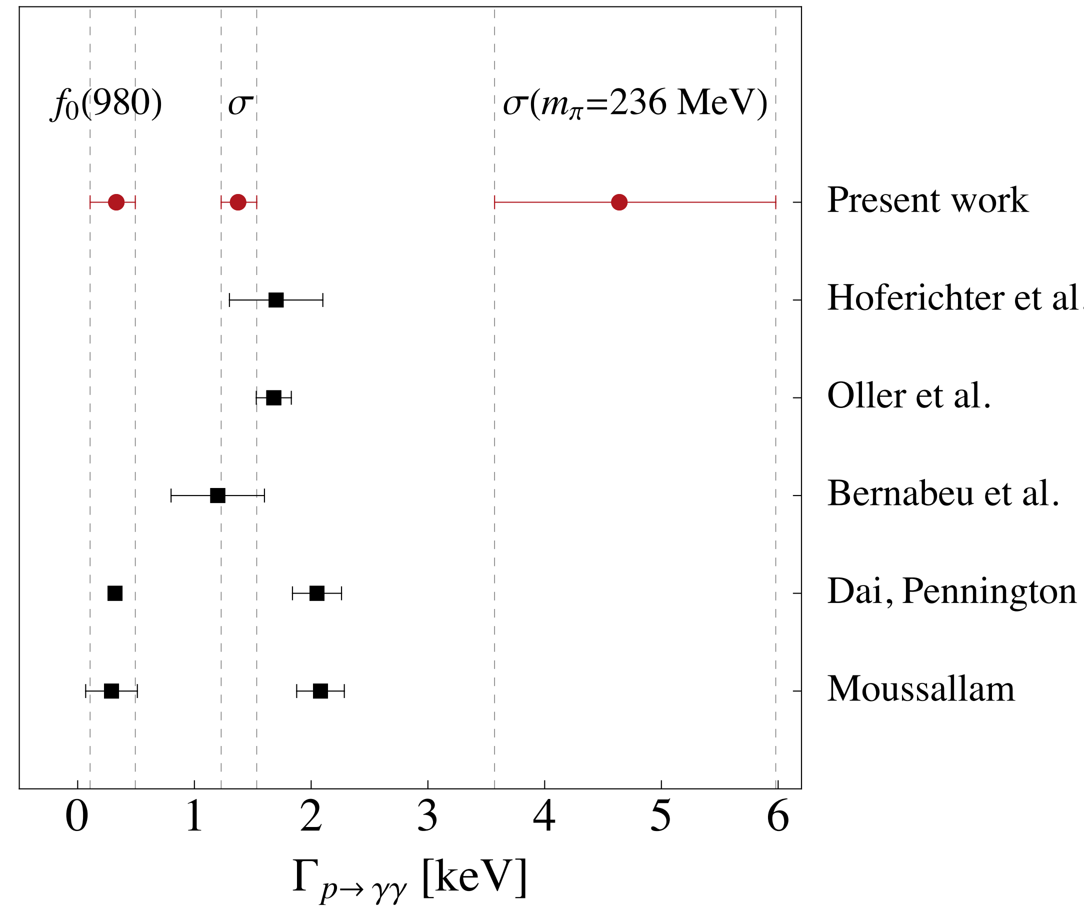
which automatically satisfies a once-subtracted dispersion relation (i.e. *Ω(s)* is asymptotically bounded)

$$\Omega_{ab}(s) = \delta_{ab} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t^*_{ac}(s')\, \rho_c(s')\, \Omega_{cb}(s')}{s' - s}$$

different from
Donoghue et al. (1990)
Moussallam (2000)

applications of Omnes function





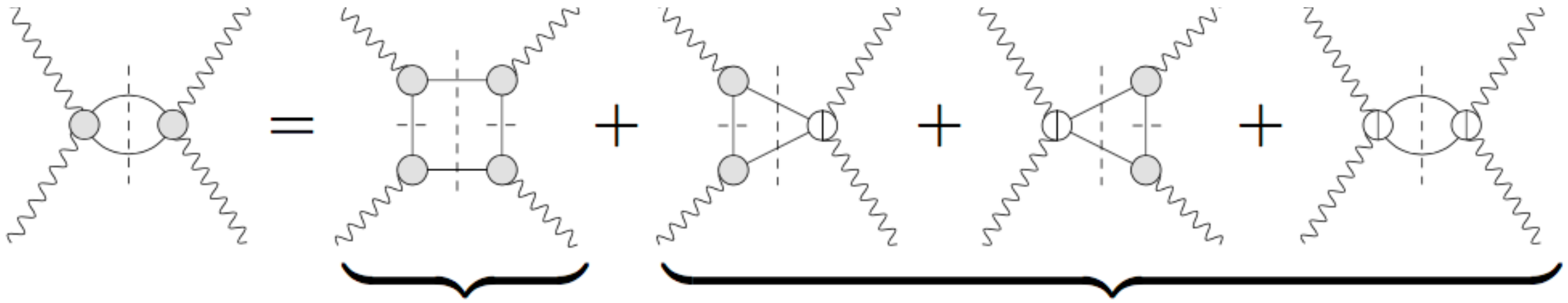
If in the coupled-channel system {γγ, ππ, KK} we neglect γ γ intermediate states in the unitarity relation: (3 × 3) dispersion relation reduces down to (2×1) which requires **hadronic Omnes (2×2) matrix** as input

$$\begin{pmatrix} t_{12}(s) \\ t_{13}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} U_{12}(s) \\ U_{13}(s) \end{pmatrix}}_{\text{Born}} + D^{-1}(s)\left[-\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\text{Disc}\,D(s')}{s'-s}\begin{pmatrix} U_{12}(s') \\ U_{13}(s') \end{pmatrix}\right]$$

Alternatively, this equation can be obtained by writing a dispersion relation for $(t(s) - U(s))D^{-1}(s)$

Muskhelishvili (1953) Omnès (1958)

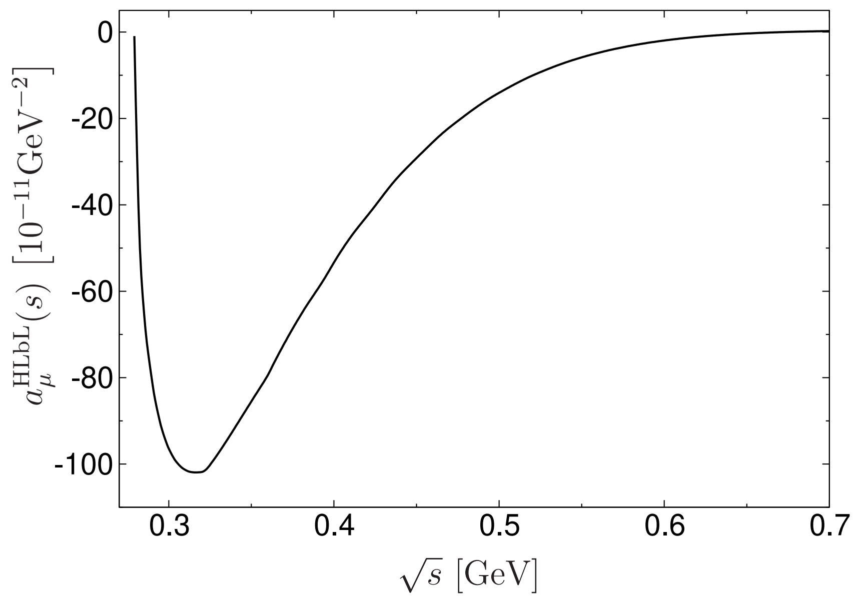
applications of Omnes function

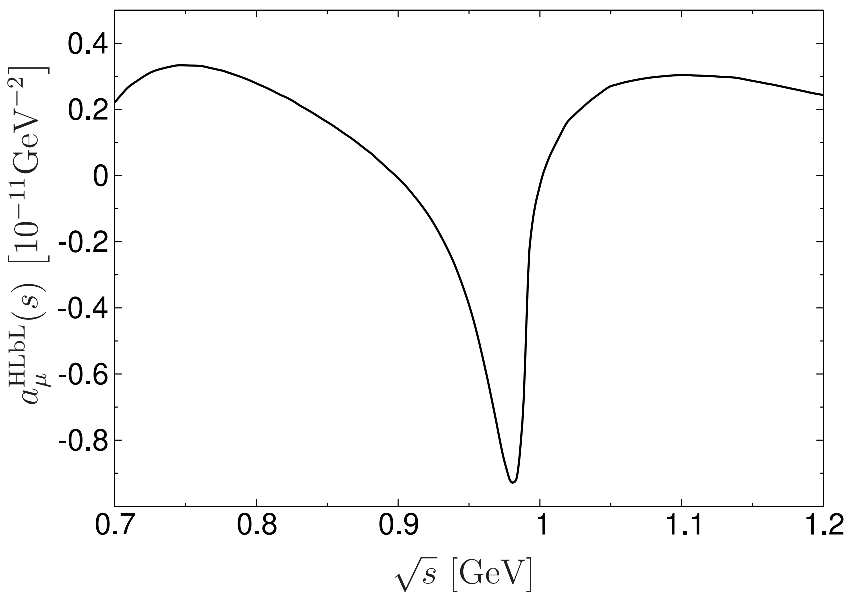


Colangelo, Hoferichter, Procura, Stoffer (2014-2017)

- Using S-wave helicity amplitudes on $\gamma^*\gamma^* \to \pi\pi$, $f_0(500)$ contribution was calculated previously
- Recently, we extended to KK channel, which gave access to the $f_0(980)$ contribution

$$a_\mu^{\rm HLbL} = \int_{4m_\pi^2}^{\infty} ds' a_\mu^{\rm HLbL}(s')$$

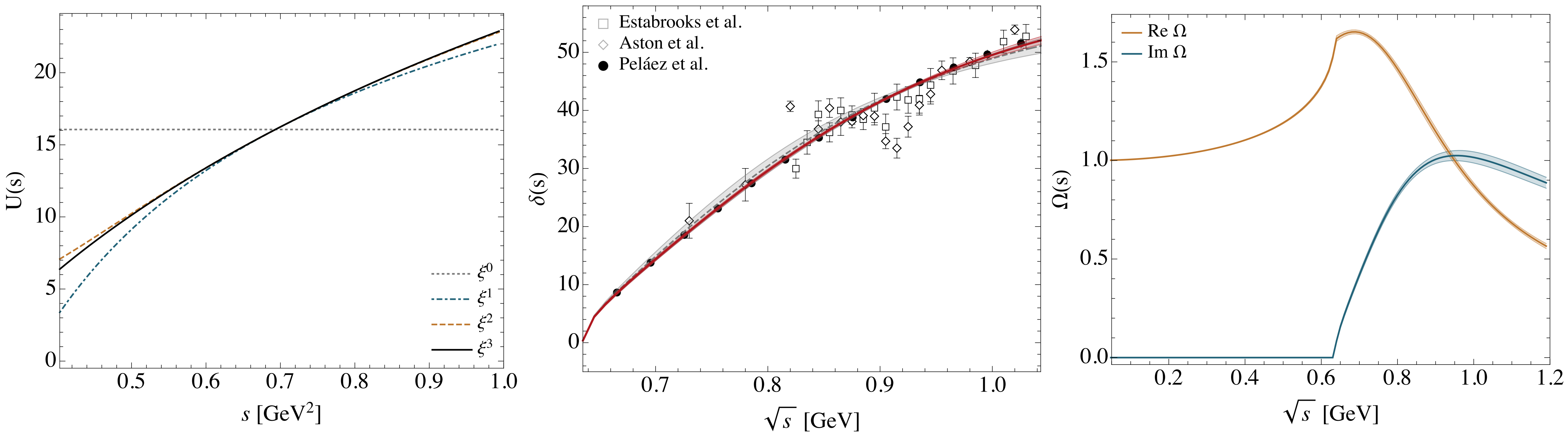




Danilkin, Hoferichter, Stoffer (2021)

$$a_\mu^{\rm HLbL}[\text{S-wave}, I=0]_{\rm rescattering} = -9.8(1) \times 10^{-11}$$
$$a_\mu^{\rm HLbL}[f_0(980)]_{\rm rescattering} = -0.2(1) \times 10^{-11}$$

single-channel {πK}

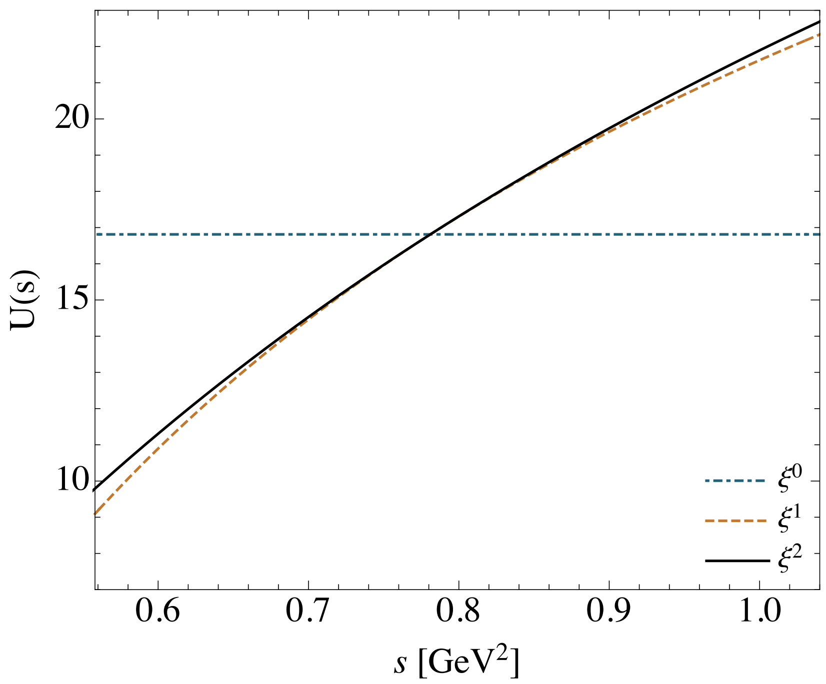


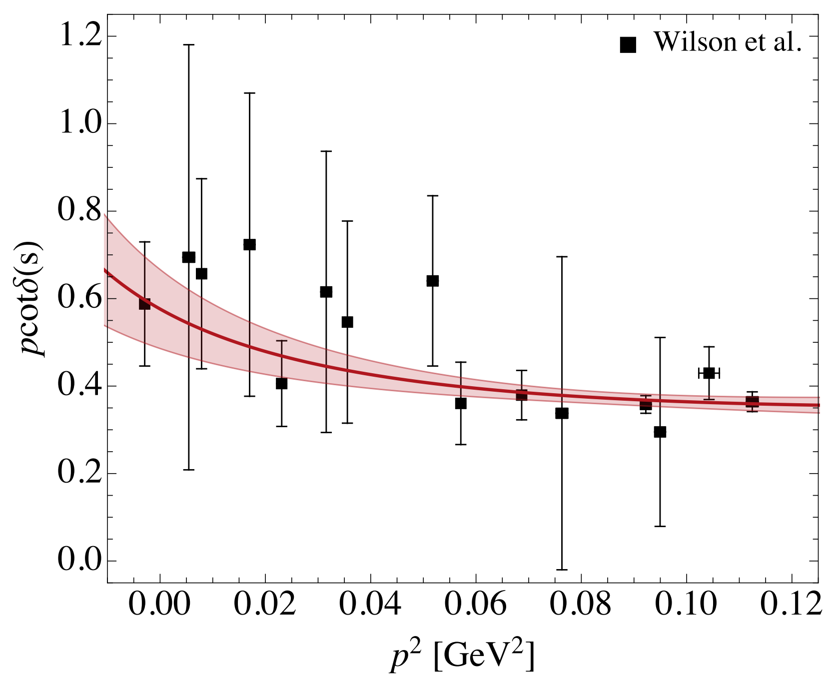
Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

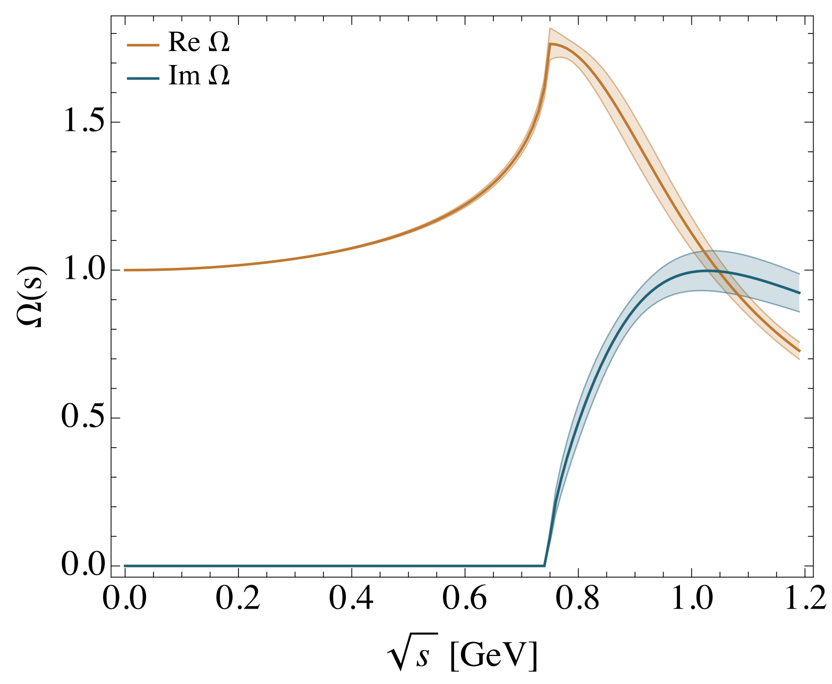
	Our results		Roy-Steiner analyses	
	pole position, MeV	coupling, GeV	pole position, MeV	coupling, GeV
$\kappa/K_0^*(700)$	$702(12)^{+4}_{-5} - i\,285(16)^{+8}_{-13}$	$\pi K: 4.12(14)^{+13}_{-18}$	$653^{+18}_{-12} - i\,280(16)$	$\pi K: 3.81(9)$
fit to Exp	$689(24)^{+3}_{-2} - i\,263(33)^{+5}_{-8}$			

Pelaez, Rodas (2020)

single-channel {πK} for m_π=239 MeV







Input: lattice data + Adler zero NLO

	N/D+lattice pole position, MeV	UχPT predictions pole position, MeV	Methods
$\kappa/K_0^*(700)$, $m_\pi = 239$ MeV	$747(39)^{+2}_{-0} - i\,265(16)^{+7}_{-6}$	$m_\kappa/m_\kappa^{\text{physical}} = 1.04$	mIAM_{NLO}
m_π =physical	$702(12)^{+4}_{-5} - i\,285(16)^{+8}_{-13}$	$\Gamma_\kappa/\Gamma_\kappa^{\text{physical}} = 0.83$	

Nebreda, Pelaez (2010)

Summary and outlook

- We presented a data driven analysis of $\pi\pi$ and πK reactions using the once-subtracted **p.w. dispersion relation**, in which left-hand cuts were accounted for using conformal expansion that converges uniformly in the resonance region

  ➢ for $f_0(500)$, $f_0(980)$, $K_0(700)$ resonances we obtained consistent results with Roy-like analyses, therefore one can apply it for processes, where no Roy analysis is available

  ➢ Obtained coupled-channel $\{\pi\pi, KK\}$ Omnes matrix has already been implemented in the analysis of
  $e^+e^- \to J/\psi\pi\pi$, $J/\psi KK$ and $f_0(980)$ to $(g-2)_\mu$

- **Good alternative** to widely used unitarization techniques like K-matrix, Bethe-Salpeter equations, …

  ➢ Can be applied to a vast **experimental** or **lattice data** which possesses a broad (or coupled-channel) resonance that does not have a genuine QCD nature ($a_0(980)$, X(6900), T_{cc}^+, …)

  ➢ can be matched to EFT

Outlook:
➢ anomalous thresholds
➢ CDD poles
➢ higher p.w.

coupled-channel {ππ, KK}