Strong Interactions, (De)coherence and Quarkonia*

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Plan of the talk

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   (b) Quarkonia in QM-plane.
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1 Introduction

7. In this talk, we study geometric nature of the quark matter formation. Specifically, we shall illustrate that the components of the vacuum fluctuations define a set of local pair correlations against the vacuum parameters, eg. charge, mass and angular momentum.
8. Our consideration follows from the notion of the thermodynamic geometry [1, 2, 3]. Importantly, our framework provides a mathematical platform to exactly understand the nature of the pair local correlations and underlying geometric structures pertaining to the global phase transitions in quarkonium systems.
9. This perspective is well-known understanding for the phase structures of mixture of gases, black holes in string theory [4, 5, 6] and in other diverse contexts, as well.
10. The main purpose of the present talk is to determine the thermodynamic properties of the quarkonium configurations, in general.
11. Quantum chromodynamics(QCD), as the theory of strong interaction, celebrates physics [7] at both the high and low temperature domains. Our consideration thereby plays a crucial role in understanding the phases and stability of the of matter formation.

2 Quarkonium Bound States

12. A gluon is said to be soft, if the transverse momentum $k_\perp$ is kept small. In this limit, the behavior of abelian theory follows directly from the underlying Poisson distribution and their numerical counterparts.

13. In the case of non-abelian theory, the Sudhakov form factor turns out to be interesting tool for understanding statistical nature of the soft gluons.

14. Recall that the QCD coupling $\alpha_s(k_\perp)$ never lies near the limit $k_\perp \to 0$, and so the QCD effects are limited for the bound state thus formed after the resummation.

15. This consideration requires an integration over the $k_\perp$ and thus the necessity of Sudhakov form factor.

16. We illustrate the decays processes associated with the pions, viz., $\pi^0 \to \gamma\gamma$ and $\pi \to \mu\nu$.

17. Consider a virtual photon hitting a quark, then during this process soft gluons are emitted with $k_\perp \to 0$ and a non-zero total momentum $k^2 \neq 0$, satisfying the transverse momentum Poisson distribution $d^2P(k_\perp)$.

18. In this case, the Poisson summed vertex and associated limiting transverse momenta are respectively given by

$$\begin{aligned} \Gamma_{\pi\to\gamma\gamma} &\sim \frac{d^2P(k_\perp)}{dk_\perp^2}, \\ \Pi(0) &= \frac{d^2P(k_\perp)}{dk_\perp^2}|_{k_\perp=0} \end{aligned} \tag{1}$$

$$= \int d^2\overrightarrow{b}\,\exp(-h(b)), \tag{2}$$

where the exponent defined as

$$h(b) = \int d^3n_k(1 - \exp(-i\overrightarrow{b}.\overrightarrow{k_\perp})) \tag{3}$$

is known the Sudhakov form factor.

19. Recall that one is required to take the vanishing transverse momentum limit for a decay, which can be systematically with the help of the Sudhakov form factor [9].

20. Let us first illustrate the case of massless quarks. For the quark mass $m_q = 0$, in turns out that the Sudhakov form factor reduces to the following integral

$$h(b) = \int \frac{dk_l}{2k}\frac{dk_\perp}{k_\perp^2}(1 - \exp(-i\overrightarrow{k_\perp}.\overrightarrow{b})), \tag{4}$$

where $k_\perp \in (0, m_{P/2})$ is due to a physical reason that on average each soft gluon can take as much as half of the initial center of mass energy.

21. The consideration of Sudhakov form factor provides an interesting intrinsic geometric model for the limiting transverse momentum QCD coupling $\alpha_s(k_\perp)|_{k_\perp\to 0}$.

22. Inspired by the behavior of relativistic wave functions near the origin for a QCD potential [10, 11] and for the Rechardson potential for the quark bound states, one arrives at the transverse momentum dependent strong QCD coupling

$$\alpha_s(k_\perp^2) = \frac{12\pi}{(33 - 2N_f)} \frac{p}{\ln(1 + p(\frac{k_\perp}{\Lambda_{QCD}})^{2p})}, \tag{5}$$

in the limit of one loop gluon exchange potential.

23. Towards the determination of the index p, an interesting argument follows from the consideration of Polyakov [8] which we shall explore further from the perspective of thermodynamic geometry in the subsequent consideration.

24. Before doing so, let us consider the joint effects of the (i) confinement and (ii) rotation, and thus make a platform to describe the thermodynamic geometry of rotating quarkonia.

25. Such a simplest configuration is described by Regge trajectories with the leading order effective potential

$$V(r, J) = \frac{J(J+1)}{r^2} + Cr^{2p-1}. \tag{6}$$

26. In this case, the effective theory is inspired from the limiting QCD strong coupling

$$\alpha_s(Q^2) = b^{-1} \frac{p}{\ln(1 + p(\frac{Q^2}{\Lambda_{QCD}^2})^p)}, \tag{7}$$

where $b := (33 - 2N_f)/12\pi$.

27. In the momentum space, the net effective potential offers the right quarkonium bound states, after taking account of the one-loop exchange terms.

28. In the sense of one-loop quantum effects, we propose that the "quantum" nature of quark matters follows directly from the consideration of the thermodynamic intrinsic geometry of the Richardson type potentials.

29. As per the consideration of Polyakov, the index p defines the nature of the potential [8], i.e. one can examine whether the quarkonium configuration lies in the Coulomb phase ($p = 0$) or in the rising phase ($p > 1/2$).

30. Notice further that the Regge behavior may also be determined from interpolating values of the index p, which is one of the important matter of the present discussion.

31. For the future purpose, let us note that the minimization of the effective potential, containing both the confining and rotation effects, yields the value $p = 5/6$ for the index corresponding to the linear Regge regime.

32. We shall further exploit these facts, when we deal with the thermodynamic geometry of the rotating (massive) quarkonia.

3 Thermodynamic Geometry

33. Let us take a closer account of the essential features of the thermodynamic geometries from the perspective of the strongly coupled QCD.

34. Let us focus our attention on the thermodynamic geometry of quarkonium bound states with finitely many parameters of the effective field theory.

35. Let us consider the framework of the intrinsic Riemannian geometry whose covariant metric tensor is defined as the Hessian matrix of the QCD coupling, with respect to a finite number of arbitrary parameters carried by the soft gluons and quarks.

36. This consideration yields the space spanned by the n parameters of the strong QCD coupling α_s, which in the present consideration, exhibits a n-dimensional intrinsic Riemannian manifold M_n.

37. Following the definition of the thermodynamic geometry [1, 2, 3, 4, 5, 6], the components of the covariant metric tensor are given by

$$g_{ij} := \frac{\partial^2 \alpha_s(\vec{x})}{\partial x^j \partial x^i}, \tag{8}$$

where the vector $\vec{x} \in M_n$. In the quarkonium effective configuration, there are only a few in physical parameters which make the analysis fairly simple.

38. As mentioned in the foregoing section, the variables of the interest in the present study of quarkonium are the momentum scale parameter, $Q^2 := q$, the mass M and the angular momentum J, if any.

39. Thus, let us first illustrate the consideration of thermodynamic geometry for the non-rotating configurations, viz., $J = 0$.

40. In this case, we can take the QCD coupling $\alpha_s(q, p)$ as a function of q, p and thereby may explore the local and global thermodynamic properties towards the stability of the quarkonia on qp-surface.

41. For a given QCD coupling $\alpha_s(q, p) := A$, an equilibrium quarkonium configuration is achieved at the points, where the first derivatives

$$\frac{\partial A(q,p)}{\partial q} = 0, \quad \frac{\partial A(q,p)}{\partial p} = 0 \tag{9}$$

vanish identically, implying the existence of the equilibrium data $\{q_0, p_0\}$ on an intrinsic Riemannian surface $(M_2(R), g)$.

42. In order to verify whether the quarkonium configuration is minimally coupled at $\{q_0, p_0\}$, we may follow the set-up of thermodynamic geometry and define the components of the thermodynamic metric tensor as

$$g_{qq} = \frac{\partial^2 A}{\partial q^2}, \quad g_{qp} = \frac{\partial^2 A}{\partial q \partial p}, \quad g_{pp} = \frac{\partial^2 A}{\partial p^2}. \tag{10}$$

43. In the present case of the $(M_2(R), g)$, it follows that the determinant of the metric tensor is

$$\|g(q,p)\| \;=\; A_{qq}A_{pp} - A_{qp}^2. \tag{11}$$

44. Explicitly, we can calculate Γ_{ijk}, R_{ijkl}, R_{ij} and R for the above two dimensional thermodynamic geometry (M_2, g) of the non-rotating quarkonia and may easily see that the scalar curvature is given by

$$\begin{aligned} R(q,p) \;=\; & -\frac{1}{2\|g(q,p)\|^2}(A_{pp}A_{qqq}A_{qpp} - A_{qq}A_{qpp}^2 - A_{pp}A_{qqp}^2 \\ & + A_{qp}S_{qqp}A_{qpp} + A_{qq}A_{qqp}A_{ppp} + A_{qp}A_{qqq}A_{ppp}). \end{aligned} \tag{12}$$

45. As a global intrinsic geometric invariant, the scalar curvature accompanies information of the correlation volume of underlying quark fluctuations.

46. The scalar curvature further explicates the nature of long range global correlations and phase transitions, if any, deriving from a given phase.

47. In this sense, we anticipate that the set of particles corresponding to the specific decay, are statistically interacting, if the underlying quarkonium configuration has a non-zero thermodynamic scalar curvature.

48. Incrementally, we notice that the configurations under present consideration are allowed to be effectively attractive or repulsive, and weakly interacting in general.

49. For the two dimensional thermodynamic geometry [4] defined as the intrinsic Riemannian surface $(M_2(R), g)$, the relation of the thermodynamic scalar curvature to the thermodynamic curvature tensor is given as

$$R(q,p) = \frac{2}{\|g\|}R_{qpqp}. \tag{13}$$

4 Massless Quarkonia

50. Let us first examine the massless non-rotating quarkonia and study the thermodynamic stability properties as emphasized earlier, and subsequently include the rotation.

4.1 Non-rotating Quarkonia

51. Considering Eqn.(7), one obtains the following expression

$$A(q,p) := \frac{p}{b\ln(1 + p(q/L)^p)}. \tag{14}$$

for the strong QCD coupling, where $L := \Lambda_{QCD}^2$.

52. To compute the thermodynamic metric tensor in the parameter space, we employ the Eqn.(10), which leads to the following expression for the components of the metric tensor

$$g_{qq} = \frac{p^3}{bq^2}\frac{n_{11}^Q}{r_{11}^Q}, \quad g_{qp} = \frac{p^2}{bq}\frac{n_{12}^Q}{r_{12}^Q}, \quad g_{pp} = \frac{1}{b}\frac{n_{22}^Q}{r_{22}^Q}. \tag{15}$$

53. In this framework, we observe that the geometric nature of parametric pair correlations offers the notion of fluctuating quarkonia.

54. In order to simplify the subsequent notations, let us define the logarithmic factor as

$$l(p) := \ln(1 + p(q/L)^p). \tag{16}$$

55. In this case, we find that the factors in the numerator of the local pair correlation functions are expressed as follows

$$\begin{aligned} n_{11}^Q &:= 2p^2(q/L)^{2p} + l(p)((q/L)^p - p(q/L)^p + (q/L)^{2p}p), \\ n_{12}^Q &:= 2(q/L)^{2p}p + l(p)(2(q/L)^{2p}p^2\ln(q/L) - 3(q/L)^p \\ &\quad -2(q/L)^{2p}p - (q/L)^p p\ln(q/L)), \\ n_{22}^Q &:= 2(q/L)^{2p}(p + 2p^2\ln(q/L) + p^3\ln(q/L)^2) \\ &\quad -l(p)((q/L)^p + (q/L)^{2p}p + 4(q/L)^p p\ln(q/L) \\ &\quad +2(q/L)^{2p}p^2\ln(q/L) + (q/L)^p p^2\ln(q/L)^2). \end{aligned} \tag{17}$$

56. In case of the denominator of the local pair correlation functions, of the massless non-rotating configuration, we find that the factors $\{r_{ii}^Q | \ i = 1, 2, 3\}$ take uniform value $l(p)^3\exp(2l(p))$ for all the local pair correlation functions.

57. Thus, the fluctuating quarkonium configuration may be easily analyzed in terms of the parameters of the underlying effective theory.

58. Moreover, it is evident that the principle components of the metric tensor, signifying self pair correlations remain positive definite functions.

59. In a given QCD phase, this happens when the parameters $\{q, p\}$ are confines in the domain

$$\{\mathcal{D} := (q, p) \in \ M_2 | \ n_{11}^Q > 0, \ n_{22}^Q > 0\}. \tag{18}$$

60. Over this domain of $\{q, p\}$, it is worth mentioning that the massless non-rotating quarkonia is well-behaved and locally stable.

61. The global stability is offered by computing the determinant of the metric tensor, and requiring it to be positive definite.

62. Following the Eqn.(11), we find further for the generic value of the parameters that the Gaussian fluctuations form a stable set of correlations over $\{q, p\}$, if the determinant of the metric tensor

$$\|g\| = \frac{p^3}{b^2q^2l(p)^5\exp(3l(p))}n_g^Q \tag{19}$$

remains a positive function on the intrinsic qp-surface $(M_2(R), g)$.

63. Explicitly, we obtain that the numerator of the determinant of the metric tensor can be expressed as

$$\begin{aligned} n_g^Q(q,p) : \;=\; & 2(q/L)^{3p}(p+3p^2+2p^2 ln(q/L)+2p^3 ln(q/L)+p^3 ln(q/L)^2) \\ & -l(p)(2(q/L)^{2p}+p(q/L)^{3p}+2p^2(q/L)^{3p} ln(q/L)+4p^2(q/L)^{3p} \\ & +4p(q/L)^{2p} ln(q/L)+p^2(q/L)^{2p} ln(q/L)^2+2p^2(q/L)^{2p} ln(q/L) \\ & +7p(q/L)^{2p}). \end{aligned} \tag{20}$$

64. The behavior of the determinant of the metric tensor shows that such massless non-rotating quarkonium becomes unstable for opposite values of q and p for positive $n_g^Q(q,p)$.

65. For generic q and p, the nature of the determinant of the metric tensor is depicted in the Eqn.(19), showing that the quarkonia become unstable in the vanishing limit of q or p.

66. It is worth mentioning for a common sigh of the index parameter p and b, that the non-rotating massless quarkonia is stable in the regions of the $(M_2(R),g)$, where the function $n_g^Q(q,p)$ picks up the positive sign.

67. In this case, there is only one non-trivial component of the Riemann Christoffel tensor R_{qpqp}. As per the definition of the Eqn.(12), our computation shows that the scalar curvature reduce to the following specific form

$$R(q,p) = \frac{bl(p)}{2p^2(n_g^Q)^2}(n_R^{(0)Q}+n_R^{(1)Q}l(p)+n_R^{(2)Q}l(p)^2+n_R^{(3)Q}l(p)^3). \tag{21}$$

68. The factors of the numerator of the scalar curvature take the following expressions

$$\begin{aligned} n_R^{(0)Q} \;=\; & \ln(q/L)^4-16(q/L)^{2p}p^4\ln(q/L)-16(q/L)^{2p}p^7\ln(q/L)^3 \\ & -56(q/L)^{2p}p^6\ln(q/L)^2-48(q/L)^{2p}p^6\ln(q/L)-4(q/L)^{2p}p^7\ln(q/L)^4 \\ & -16(q/L)^{2p}p^7\ln(q/L)^2-16(q/L)^{2p}p^6\ln(q/L)^3-64(q/L)^{2p}p^5\ln(q/L) \\ & -24(q/L)^{2p}p^5\ln(q/L)^2-36(q/L)^{2p}p^5-4(q/L)^{2p}p^3, \\ n_R^{(1)Q} \;=\; & 28p^6(q/L)^{2p}\ln(q/L)^2+14p^4(q/L)^{2p}+20p^5(q/L)^{2p}\ln(q/L)^2 \\ & +50p^3(q/L)^p+40p^3(q/L)^p\ln(q/L)+146p^4\ln(q/L)(q/L)^p \\ & +56p^4(q/L)^p\ln(q/L)^2+8p^6(q/L)^{2p}\ln(q/L)^3+4p^3(q/L)^{2p} \\ & +10p^2(q/L)^p+48(q/L)^{2p}p^5+102(q/L)^pp^4+112p^5(q/L)^p\ln(q/L)^2 \\ & +32(q/L)^{2p}p^6\ln(q/L)+24(q/L)^pp^6\ln(q/L)^2+32p^5(q/L)^p\ln(q/L)^3 \\ & +104(q/L)^pp^5\ln(q/L)+24(q/L)^pp^6\ln(q/L)^3+6(q/L)^pp^6\ln(q/L)^4 \\ & +16p^4(q/L)^{2p}\ln(q/L)+50p^5\ln(q/L)(q/L)^{2p}, \\ n_R^{(2)Q} \;=\; & -6p^3(q/L)^{2p}-8p^5(q/L)^{2p}\ln(q/L)-6p^4\ln(q/L)(q/L)^{2p} \\ & +6p^4(q/L)^{2p}+10p^3(q/L)^p-14p^2(q/L)^p-20p^2-12p \\ & -30p^2\ln(q/L)+8(q/L)^pp^6\ln(q/L)^2l(p)^2-20p^3(q/L)^p\ln(q/L) \\ & -4p^5(q/L)^{2p}\ln(q/L)^2l(p)^2-90p^3\ln(q/L)-36p^3\ln(q/L)^2 \\ & -62(q/L)^pp^4l(p)^2-16(q/L)^{2p}p^5-16p^4\ln(q/L)^3 \\ & -8p^5\ln(q/L)^2-56p^4\ln(q/L)-56p^4\ln(q/L)^2 \end{aligned}$$

$$
\begin{aligned}
& -8p^5\ln(q/L)^3 - 2p^5\ln(q/L)^4 - 8(q/L)^p p^5\ln(q/L) \\
& +8(q/L)^p p^6\ln(q/L)^3 + 2(q/L)^p p^6\ln(q/L)^4 - 82p^3 \\
& -16p^4(q/L)^p\ln(q/L)^2 - 34p^4\ln(q/L)(q/L)^p, \\
n_R^{(3)Q} \;=\; & -15p^3\ln(q/L) + 2p^3(q/L)^{2p} - 4p^4(q/L)^{2p} - 8p^5\ln(q/L)^2(q/L)^p \\
& -9p^4\ln(q/L)(q/L)^p - 2p^4\ln(q/L)^2(q/L)^p - 2p^5\ln(q/L)^3(q/L)^p \\
& +p^3\ln(q/L)(q/L)^p - 8p^5\ln(q/L)(q/L)^p - 20p^3(q/L)^p + 9p^2(q/L)^p \\
& -5p^4\ln(q/L)^3 - 4p^5\ln(q/L)^2 - 16p^4\ln(q/L) - 19p^4\ln(q/L)^2 \\
& -4p^5\ln(q/L)^3 - p^5 + 16(q/L)^{-p}p^2 - 6(q/L)^{-p}p + 5(q/L)^{-p}p^2\ln(q/L)^2 \\
& +(q/L)^{-p}p^3\ln(q/L)^3 + 10(q/L)^{-p}p^2\ln(q/L) + 3(q/L)^{-p}p^3\ln(q/L)^2 \\
& +6(q/L)^{-p}p\ln(q/L) + 4(q/L)^{-p}p^3\ln(q/L) + 6(q/L)^{-p} - 28p^2 \\
& -24(q/L)^{2p}p^4 + 7p^3 - 3p^3\ln(q/L)^2 + 3p^2\ln(q/L) + 12p. \qquad (22)
\end{aligned}
$$

69. In the case when the Ricci scalar curvature $R(q,p)$ vanishes, the underlying quarkonium system is found to be in equilibrium. Such a state of the configuration can arise with $\{n^{(i)Q} = 0, i = 0,1,2,3\}$, if the other factors of the scalar curvature remain non-zero.

70. In the other case, when the $R(q,p)$ diverges, the configuration goes over a transition. Such an extreme behavior of the quarkonia is expected to happen, when either the index p or the the numerator of the determinant of the metric tensor n_g^Q vanish.

71. Geometrically, the intrinsic qp-surface becomes the flat Euclidean plane in the first case, while it gets infinitely curved in the second one.

72. For a range of QCD parameters, $\{L, b\}$, the determinant of the metric tensor explicates the stability of the non-rotating configurations in the limit of massless quarkonia.

73. For the generic massless non-rotating quarkonia, we notice that the limiting scalar curvature interestingly simplifies to the slice shape.

74. Physically, the presence of peaks in the determinant of the metric tensor shows a non-trivial interaction in the system.

75. For the regime of the Coulambic potential, the respective surface of the determinant of the metric tensor and scalar curvature respectively shown the global stability properties, for a given p.

76. We observe that the stability of massless non-rotating quarkonia exists in certain bands. A closer view shows that the quarkonia are decaying and interacting particles in the Coulambic limit.

77. For the regime of the rising potential, the respective determinant of the metric tensor and scalar curvature shows that the limiting rising potential quarkonia are stable and non-interacting particles.

78. This follows from the fact that the determinant instability is present only for specific q.

4.2 Regge Rotating Quarkonia

79. In the present case, the nature of massless rotating quarkonia can be analyzed as the function of the parameters q, p and angular momentum J.

80. Following Eqn.(6), we find that the modified strong QCD coupling takes the form

$$A(q,p,J) = \frac{1}{b}\frac{p}{\ln(1+p(q/L)^p)} + \frac{1}{b_1}J(J+1). \tag{23}$$

81. To focus on the intrinsic geometry of the present case, we chose the parameters q, p, J as the variables for the QCD coupling.

82. As in the previous subsection, we may again exploit the definition of the Hessian function $Hess(A(q,p,J))$ of the QCD coupling.

83. Considering the analysis of the Regge model, we find that the components of the metric tensor, in the present framework, have the same characterizations for the g_{qq}, g_{qp} and g_{pp}, as obtained for the corresponding non-rotating massless quarkonia.

84. Further, the rotation component of the metric tensor turns out to be $g_{JJ} = \frac{2}{b_1}$.

85. Whilst, Eqn.(23) shows that the remaining components of the metric tensor, involving J and either the q or p, vanish identically.

86. It follows that the pure pair correlations $\{g_{qq}, g_{pp}, g_{JJ}\}$ between the parameters $\{q, p, J\}$ remain positive same as in the case of the non-rotating quarkonia.

87. Further, our computation demonstrates the over-all nature of the parametric fluctuations. In fact, we find that the determinant of the metric tensor reduces to the following simple expression

$$\|g\| = \frac{2p^3}{b^2 b_1 q^2 l(p)^5 \exp\left(3l(p)\right)} n_g^Q, \tag{24}$$

where the $n_g^Q(q,p)$ remains the same as for the non-rotating case.

88. It is worth mentioning that the Regge rotating massless quarkonia is well-behaved, as long as the corresponding rotating massless quarkonia remains so.

89. Over the domain of the parameters $\{q, p, J\}$, we thus observe that the Gaussian fluctuations have the same set of thermodynamic metric stability structures, as long as $b_1 > 0$.

90. The observation of the metric structure shows that the Regge rotating massless quarkonia is stable on the qp-surface if the index parameter p and the function $n_g^Q(q,p)$, appearing in the numerator of the determinant of the metric tensor have the same sign.

91. Furthermore, we may easily analyze the underlying important conclusions for the specific considerations of Regge rotating massless quarkonia.

92. The Regge rotating massless quarkonia shows a similar thermodynamic geometric structure as its non-rotating case.

93. In precise, in the Regge trajectory model, we thus find interestingly that all possible local and global thermodynamic stability behavior of the massless quarkonia remains the same up to the sign of b_1, as if there were no rotation in the underlying configuration.

94. The fact that the efficiency of the rotation induces a mass to the quarkonium is analyzed by considering the Bloch-Nordsieck resummation of the angular phases.

95. Let's now move to the strongly coupled quarkonia with nonzero mass and rotation.

5 Massive Quarkonia

96. Let us firstly illustrate the cases for the two parameter configurations with either $\{q, J\}$ or $\{q, m\}$ fluctuating and then systematically extend the consideration for the generic three parameter quarkonia.

5.1 Quarkonia in QJ-plane

97. As per the consideration of massive quarkonia, the resummed strong QCD coupling takes the following form

$$A(q,J) := \frac{1}{b}\frac{p}{\ln(1+p(q/L)^p)}\ln(\frac{\sqrt{J}+\sqrt{J-q}}{\sqrt{J}-\sqrt{J-q}})(1-J(0,a\sqrt{q})), \tag{25}$$

where $J(\nu, x)$ is the Bessel function of the first kind of the order ν.

98. As stated earlier, the thermodynamic metric in the parameter space is given by the Hessian matrix $Hess(A(q, J))$ of the strong QCD coupling with respect to the variables defining the thermodynamic manifolds.

99. In order to simplify the subsequent expressions, it is worth defining the logarithmic factor of the concerned Bloch-Nordsieck rotation as

$$f(q,J) := \ln(\frac{\sqrt{J}+\sqrt{J-q}}{\sqrt{J}-\sqrt{J-q}}). \tag{26}$$

100. In this framework, it turns out that the fluctuation nature of parametric pair correlations may be easily divulged in terms of the momentum transfer and rotation parameter of the underlying quarkonia.

101. Following Eqn.(10), we find, under the Gaussian fluctuations of $\{q, J\}$, that the components of the metric tensor are

$$\begin{aligned}
g_{qq} &= -\frac{p}{4bl(p)^3\exp(2l(p))q^{5/2}(J-q)^{3/2}}(n_{11}^{(0)J}+n_{11}^{(1)J}l(p)+n_{11}^{(2)J}l(p)^2),\\
g_{qp} &= \frac{p}{2bl(p)^2\exp(l(p))q^{3/2}\sqrt{J}(J-q)^{3/2}}(n_{12}^{(0)J}+n_{12}^{(1)J}l(p)),\\
g_{pp} &= -\frac{p}{2bl(p)J^{3/2}(J-q)^{3/2}}(2J-q)(1-J(0,a\sqrt{q})).
\end{aligned} \tag{27}$$

102. As a result, we find without any approximation that the factors in the numerator of the local pair correlation associated with the qq-components are expressed as the linear combinations of the scaling $(q/L)^{np})$, where $n \in Z$.

103. Similarly, the factors in the numerator of the associated qJ-component are given as the linear combinations of the scaling $(q/L)^{np})$, where $n \in Z$.

104. Herewith, we see that the geometric nature of the parametric pair correlation functions turns out to be remarkably interesting, viz., the fluctuating quarkonia may be easily described in terms of the q and J.

105. For the configurations with the same sign of the index p and constant b, it is evident that the principle components of the metric tensor, signifying self pair correlations, are positive definite functions in a non-trivial range q.

106. The local stability requires that (i) qq- fluctuations satisfy the constraint

$$n_{11}^{(0)J} + n_{11}^{(1)J} l(p) + n_{11}^{(2)J} l(p)^2 < 0 \tag{28}$$

and (ii) JJ- fluctuations be constrained to the following limiting values of the Bessel function

$$\begin{aligned} J(0, a\sqrt{q}) &< 1, \quad q > 2J, \\ &> 1, \quad q < 2J. \end{aligned} \tag{29}$$

107. A straightforward computation demonstrates the over-all nature of the parametric fluctuations. In this case, we find that the determinant of the metric tensor reduces to the following expression

$$g = -\frac{p^2}{8b^2 l(p)^4 \exp(2l(p)) q^{5/2} J^{3/2} (J-q)^{3/2}} (n_g^{(0)J} + n_g^{(1)J} l(p) + n_g^{(2)J} l(p)^2), \tag{30}$$

where the coefficients $\{n_g^{(1)J}, n_g^{(2)J}\}$ appearing in the determinant of the metric tensor factorize as follows

$$\begin{aligned} n_g^{(1)J} &= 4p^2 (q/L)^p (n_g^{(12)J} + n_g^{(13)J} p(q/L)^p), \\ n_g^{(2)J} &= n_g^{(20)J} + 2n_g^{(21)J} p(q/L)^p + n_g^{(23)J} ((q/L)^p)^2 p^2. \end{aligned} \tag{31}$$

108. Explicitly, we find that all factors appearing in the numerator of the determinant of the metric tensor, viz. $\{n_g^{(0)J}, n_g^{(12)J}, n_g^{(13)J}, n_g^{(20)J}, n_g^{(21)J}, n_g^{(23)J}\}$, can also be presented as linear linear combinations of the scaling $(q/L)^{np})$, where $n \in Z$.

109. Hereby, we see how the pure components of the metric tensor behave, when they are considered as the function of the parameter q, index p and the angular momentum J.

110. For the choice $L = 22500$, $p = 5/6$, $b = 1$ and $a = 100000$, the qq-heat capacity shows a large variation of the minima and maxima, which respectively take in an order of the amplitude $+10^{10}$ and -10^{11} and occur for $q, J \in (0, 4)$.

111. In this case, we further observe for all $J \in (0,4)$ that the underlying quarkonia become highly unstable as q positively approaches the origin.

item The JJ-heat capacity shows only a negative variation of the order -10^5, which occurs when q, J tend towards the origin.

112. In this case, we notice that all local interactions are present in the region $q \in (0,4)$ and $J \in (-4,4)$. The strength of the local interactions depends on the domain chosen in the (q, J) space.

113. These fluctuations signify that the thermodynamical interactions are dominantly present in the strongly coupled quarkonia.

item The determinant of the metric tensor shows the global nature of the stability, and the corresponding diagonal components of the metric tensor shows the local nature of the stability.

114. Our exact formulae explicate the thermodynamically (un)stability regions for the underlying massive quarkonia and the allowed QCD phases for the matter formation.

115. Combining the effects of all fluctuations of the $\{q, J\}$, we observe that the quarkonia are stable for $q := Q^2 \in (1,4)$.

116. In general, the global stability requires that the determinant of the metric tensor must be positive definite, which in the present case transform as

$$n_g^J := n_g^{(0)J} + n_g^{(1)J} l(p) + n_g^{(2)J} l(p)^2 \quad < \quad 0. \tag{32}$$

117. In this case, it turns out that the thermodynamic curvature may be written as the series of the charmonium logarithmic factor $l(p)$ and Bloch-Nordsieck logarithmic factor $f(q, J)$ of rotation as the coefficient of the expansion.

118. Systematically, the exact expression for the scalar curvature takes the form

$$R(q, J) = \frac{bl(p)}{2p^2 (n_g^J)^2} \sum_n B_n \times (l(p))^n, \tag{33}$$

where the B_n in the numerator of the scalar curvature are polynomials in p, whose coefficients are the functions of the Bloch-Nordsieck logarithmic factor $f(q, J)$.

119. We find that the denominator of the scalar curvature precisely takes the numerator of the determinant of the metric tensor as its square.

120. The quantitative properties of the scalar curvature and the Riemann curvature tensor remain similar, as we shall discuss in the sequel.

5.2 Quarkonia in QM-plane

121. To focus on the general case, we choose the variable Q as the transverse momentum with the understanding that $k = k_\perp$ and the mass considered as an arbitrary real parameter of the system.

122. Following the convention of Bloch-Nordsieck resummation, the strong QCD coupling can be expressed as

$$A(k,m) = \frac{1}{b}\frac{p}{\ln(1+p(k^2/L)^p)}\ln(\frac{m+\sqrt{m^2-k^2}}{m-\sqrt{m^2-k^2}})(1-J(0,ak)). \tag{34}$$

123. When the Bloch-Nordsieck resummed strong QCD coupling $A(k,m)$ is allowed to fluctuate as a function of the $\{k,m\}$, we may again exploit the definition of the Hessian function $Hess(A(k,m))$.

124. Herewith, we find that the components of the metric tensor are given by

$$\begin{aligned} g_{kk} &= \frac{p}{bl(p)^3\exp\left(2(l(p))k^6(m^2-k^2)^{3/2}\right.}(n_{11}^{(0)M}+n_{11}^{(1)M}l(p)+n_{11}^{(2)M}l(p)^2), \\ g_{km} &= \frac{2p}{bl(p)^2\exp\left(l(p))k(m^2-k^2)^{3/2}\right.}(n_{12}^{(0)M}+n_{12}^{(1)M}l(p)), \\ g_{km} &= \frac{2mp}{bl(p)(m^2-k^2)^{3/2}}(-1+J(0,ak)), \end{aligned} \tag{35}$$

where the coefficients $\{n_g^{(1)M}, n_g^{(2)M}\}$ appearing in the determinant of the metric tensor factorize as

$$\begin{aligned} n_{11}^{(1)M} &= 2(k^2/L)^p p^2(n_{11}^{(12)M}+2n_{11}^{(13)M}p(k^2/L)^p), \\ n_{11}^{(2)M} &= n_{11}^{(20)M}+2pn_{11}^{(21)M}(k^2/L)^p+p^2n_{11}^{(22)M}(k^2/L)^{2p}. \end{aligned} \tag{36}$$

125. Without any approximation, we find that the factors appearing in the numerator of the local pair correlation pertaining to the kk-components can, as before, be expressed as the linear combinations of the scaling $(k/L)^{np})$, where $n \in Z$.

126. In fact, the factors of the numerator of the km-component of the metric tensor can be expressed as the linear combinations of the scaling $(k/L)^{np})$, where $n \in Z$.

127. When the fluctuations of the quarkonia are described in terms of the transverse momentum k and mass m, we observe that the principle components of the metric tensor, signifying self pair correlations, are positive definite functions in a non-trivial range of k.

128. For the configurations with the same sign of the index p and constant b, it turns out that the local stability requires that (i) qq- fluctuations satisfy the constraint

$$n_{11}^{(0)M}+n_{11}^{(1)M}l(p)+n_{11}^{(2)M}l(p)^2 > 0, \tag{37}$$

and (ii) mm- fluctuations pose a constraint on the Bessel function that the underlying transverse momentum of the configuration must be limited as $J(0,ak) > 1$.

129. The determinant of the metric tensor turns out to be in a similar rational form, as it was obtained in the previous subsection. In fact, the simplifications are relatively straightforward, and we find the following compact expression for the determinant of the metric tensor

$$g = -\frac{2p^2}{b^2l(p)^4\exp\left(l(p)^2)k^2(m^2-k^2)^{3/2}\right.}(n_g^{(0)M}+n_g^{(1)M}l(p)+n_g^{(2)M}l(p)^2), \tag{38}$$

with the following factorization

$$
\begin{aligned}
n_g^{(1)M} &= 2p^2(k^2/L)^p((n_g^{(12)M} + p((k^2/L)^p)n_g^{(13)M}), \\
n_g^{(2)M} &= n_g^{(20)M} + 2n_g^{(21)M}p(k^2/L)^p + n_g^{(22)M}((k^2/L)^p)^2p^2. \qquad (39)
\end{aligned}
$$

130. In this case, the coefficients $n_g^{(0)M}$ and $\{n_g^{(ij)M}|i,j=1,2\}$ are expressible as the linear combinations of the powers of the scaling $(k/L)^{np})$, where $n \in Z$.

131. As per the requirement of the positive definiteness of the determinant of the metric tensor, the global stability of underlying quarkonia with fluctuating $\{k,m\}$ leads to the following constraint

$$
n_g^{(0)M} + n_g^{(1)M}l(p) + n_g^{(2)M}l(p)^2 \quad < \quad 0. \qquad (40)
$$

132. The globally covariant Riemann curvature tensor R_{kmkm} offers a similar property, as we have described in the foregoing consideration.

133. Specifically, we find that the R_{kmkm} has various factors of $\{(m^2-k^2)^{n/2}|n \in Z\}$ and various powers of the logarithmic coupling $l(p)$.

134. Some of the interesting terms of R_{kmkm} are $-600m^6(m^2-k^2)^{11/2}l(p)^4k^{18}$, $+712m^{18}(m^2-k^2)^{9/2}l(p)^4k^8$, ... and $-1244m^{28}(m^2-k^2)^{7/2}l(p)^4k^8$.

135. It is worth mentioning that R_{kmkm} contains approximately 50,000 terms whose presentation is quite lengthy for the present discussion.

136. Nevertheless, the globally invariant scalar curvature and globally covariant Riemann curvature tensor can straightforwardly be obtained.

137. As described in the previous configuration, the globally covariant physical properties of two parameter quarkonia follow directly from the product of two geometrically invariant quantities, viz. R and $\|g\|$.

138. The kk-heat capacity shows a large variation of minima, when going from a negative k for a given positive mass $m \in (0,5)$. We find that the order of instability turns out to be as high as -10^{16}.

139. In this case, we further notic in the other quarqants that the underlying quarkonia remains nearly stable in the other three distinct pathes.

140. Furthermore, we find that the mm-heat capacity indicates a similar behavior of the negative variation.

141. In contrast to the foregoing case, in this case we find that the local interactions are present only in four disjoint regions, all of which exclude the origin.

142. As expected, the strength of the interactions depends on the domain chosen in the (k,m) plane, and thus the nature of the stability of the quarkonia for fluctuating $\{k,m\}$.

143. The present consideration explicates the of regions of the thermodynamic (un)stability for the massive quarkonia and the (un)stable phases of the one loop QCD.

144. Specifically, the significance of the fluctuations of $\{k, m\}$ is that they show thermodynamical interactions in the strongly coupled massive quarkonia.

145. We find that the quarkonia can be highly unstable, even in the linear Regge regime.

146. As the non-linear effects become stronger and stronger, it turns out that the thermodynamic instability and correlations grow further. This motivates us to extend our analysis to the case of more general quarkonium configurations.

5.3 Generic Quarkonia

147. Finally, we analyze the properties of general massive rotating quarkonia, when all parameters of the theory are allowed to fluctuate.

148. To do so, let us consider the scale q, index of effective potential p and angular momentum J as the parameters of the present interest.

149. In the framework of the Bloch-Nordsieck resummation, the strong QCD coupling takes the following form

$$A(q,p,J) = \frac{1}{b}\frac{p}{\ln(1+p(q/L)^p)} \ln(\frac{\sqrt{J}+\sqrt{J-q}}{\sqrt{J}+\sqrt{J-q}})(1-J(0,a\sqrt{q})). \tag{41}$$

150. After some simplification, we find that the components of the metric tensor are

$$\begin{aligned}
g_{qq} &= -\frac{p}{4bl(p)^3\exp(2l(p))q^{5/2}(J-q)^{3/2}}(n_{11}^{(0)G}+n_{11}^{(1)G}l(p)+n_{11}^{(2)G}l(p)^2),\\
g_{qp} &= \frac{1}{2bl(p)^3\exp(2l(p))q^{3/2}(J-q)^{1/2}}(n_{12}^{(0)G}+n_{12}^{(1)G}l(p)+n_{12}^{(2)G}l(p)^2),\\
g_{qJ} &= \frac{p}{2bl(p)^2\exp(l(p))q^{3/2}(J-q)^{3/2}J^{1/2}}(n_{13}^{(0)G}+n_{13}^{(1)G}l(p)),\\
g_{pp} &= \frac{f}{bl(p)^3\exp(2l(p))}(-1+J(0,a\sqrt{q}))(n_{22}^{(0)G}+n_{22}^{(1)G}l(p)),\\
g_{pJ} &= -\frac{1}{bl(p)^2\exp(l(p))(J-q)^{1/2}J^{1/2}}(-1+J(0,a\sqrt{q}))(n_{23}^{(0)G}+(l(p)n_{23}^{(1)G}),\\
g_{JJ} &= \frac{p}{2bl(p)(J-q)^{3/2}J^{3/2}}(-1+J(0,a\sqrt{(}q)))(2J-q),
\end{aligned} \tag{42}$$

where the coefficients $\{n_{11}^{(1)G}, n_{11}^{(2)G}, n_{12}^{(1)G}, n_{12}^{(2)G}\}$ appearing in the components of the metric tensor are shown to factorize as before.

151. In the powers of p, the exact factorizations are given as follows: (i) the qq-component

$$\begin{aligned}
n_{11}^{(1)G} &= 4p^2(q/L)^p n_{11}^{(12)G} + 4p^3((q/L)^p)^2 n_{11}^{(12)G},\\
n_{11}^{(2)G} &= n_{11}^{(20)G} + 2p(q/L)^p n_{11}^{(21)G} + p^2((q/L)^p)^2,
\end{aligned} \tag{43}$$

(ii) the qp-component

$$\begin{aligned} n_{12}^{(1)G} &= p(q/L)^p n_{12}^{(11)G} + p^2((q/L)^p)^2 n_{12}^{(12)G}, \\ n_{12}^{(2)G} &= n_{12}^{(20)G} + 2p(q/L)^p n_{11}^{(21)G} + p^2((q/L)^p)^2 n_{12}^{(22)G}. \end{aligned} \tag{44}$$

152. Interestingly, the factors in the numerator of the qq-components can be cascaded as the linear combination over the powers $((q/L)^p)^n$, where $n \in Z$.

153. In a parallel way, the factors in the numerator of the qp-components turn out to be the linear combination over the powers $((q/L)^p)^n$, where $n \in Z$.

154. Also, the factors in the numerator of the qJ-components satisfy a similar linearity in the powers $((q/L)^p)^n$, where $n \in Z$.

155. Surprisingly, the corresponding factors in the numerator of the pp-components are very simple, in fact we find that they are given by

$$\begin{aligned} n_{22}^{(0)G} &= -2p(q/L)^{2p}(1 + 2p\ln(q/L) + p^2\ln(q/L)^2), \\ n_{22}^{(1)G} &= (q/L)^p(2 + 4p\ln(q/L) + p^2\ln(q/L)^2) \\ &\quad + p(q/L)^{2p}(1 + 2p\ln(q/L)). \end{aligned} \tag{45}$$

156. Finally, the factors in the numerator of the pJ-components are

$$\begin{aligned} n_{23}^{(0)G} &= -p(q/L)^p - p^2\ln(q/L)(q/L)^p, \\ n_{23}^{(1)G} &= 1 + p(q/L)^p. \end{aligned} \tag{46}$$

157. The local stability of the configuration requires the following three simultaneously constraints. For the same sign of $\{p, b\}$, the thermodynamic stability enforces:

(i) the qq- fluctuations satisfy

$$n_{11}^{(0)G} + n_{11}^{(1)G} l(p) + n_{11}^{(2)G} l(p)^2 < 0, \tag{47}$$

(ii) the JJ- fluctuations remain within the limiting values of the Bessel function

$$\begin{aligned} J(0, a\sqrt{q}) &> 1, \ \ 2J < q, \\ &< 1, \ \ 2J > q, \end{aligned} \tag{48}$$

and

(iii) the pp- fluctuations satisfy

$$\begin{aligned} n_{22}^{(0)G} + n_{22}^{(1)G} l(p) &> 0, \ \ J(0, a\sqrt{q}) > 1, \\ &< 0, \ \ J(0, a\sqrt{q}) < 1 \end{aligned} \tag{49}$$

for the same sign of $\{f, b\}$.

158. The pp-heat capacity, even for a small q, shows a large minimum of depth 10^6, in the regime of the angular momentum $J \in (0, 5)$.

159. The silent feature of the JJ-fluctuations is that we find the two distinct local behaviors for $J > 0$ and $J < 0$.

160. In the limit of vanishing q and J, the local interactions reach an order of 10^4. For the fluctuating $\{q, p, J\}$, the strength of the thermodynamic interactions depends on the domain chosen in the (q, p, J) plane, and thus so does the thermodynamic stability of the quarkonia.

161. Physically, the heat capacities under the fluctuations, which are defined as the self-pair correlations, remain positive quantities in the domain of the parameters and thus form the well-defined basis of the manifold (M_3, g).

162. Subsequently, we find that the qp-surface is stable, if there exits a positive surface minor

$$p_S^G := -\frac{1}{4b^2 l(p)^5 \exp(3l(p)) q^{5/2}(J-q)^{3/2}}(n_S^{(0)G} + n_g^{(1)G} l(p) + n_S^{(2)G} l(p)^2 + n_S^{(3)G} l(p)^3), \quad (50)$$

where the $n_S^{(0)G}$ has the following illuminating factorization

$$n_S^{(0)G} = 8f^2((q/L)^p)^3(p^4 n_S^{(01)G} + p^5 n_S^{(02)G} + p^6 n_S^{(03)G}). \quad (51)$$

163. Interestingly, the factors $\{n_S^{(01)G}, n_S^{(02)G}, n_S^{(03)G}\}$ can be expressed as the linear combination over the $\ln(q/L)^n$, where $n \in Z$.

164. It is worth mentioning that the factors of the $l(p)$-terms can further be expressed as

$$n_S^{(1)G} = (q2/L)^{2p} n_S^{(12)G} + ((k^2/L)^{3p} n_S^{(13)G}) \quad (52)$$

with the following additional structures

$$n_S^{(1i)G} = n_S^{(1i0)G} + n_S^{(1i1)G} f + n_S^{(122)G} f^2, i = 2, 3. \quad (53)$$

165. In particular, we find that the 120, 121 and 122 factorize as per the following explicit expressions

$$\begin{aligned} n_S^{(12i)G} &= p^2 n_S^{(12i2)G} + 2p^3 n_S^{(21i3)G} + p^4 n_S^{(12i4)G}, i = 0, 1 \\ n_S^{(122)G} &= p^2 n_S^{(2122)G} + 2p^3 n_S^{(2123)G} + p^4 n_S^{(2124)G} + 4p^5 n_S^{(2125)G} \end{aligned} \quad (54)$$

166. As mentioned before, the sub-factors $\{n_S^{(120i)G}, n_S^{(121i)G}, n_S^{(122i)G}\}$ can be expressed as linear combinations in the powers of the scaling $\ln(q/L)^n$, where $n \in Z$.

167. It follows that the 122-factors take five powers of p, whose precise expressions can be given as the linear combination of powers of the scaling $\ln(q/L)^n$, where $n \in Z$.

168. Further, we take advantage of the following factorizations for the 131 and 132 expressions

$$n_S^{(13i)G} = 4p^3 n_S^{(13i3)G} + 8p^4 n_S^{(13i4)G} + 4p^5 n_S^{(13i5)G}, i = 1, 2. \quad (55)$$

169. Subsequently, the factors of 130 and 131 appear in the powers of the p, and are given as the linear combination of powers of the scaling $\ln(q/L)^n$, where $n \in Z$.

170. The factors of 132-powers, appearing as the powers of the p, can be expressed as the foregoing coefficients.

171. It turns out that the factor of the $l(p)^2$ terms can be expressed as

$$n_S^{(2)G} = (q/L)^p n_S^{(21)G} + (q/L)^{2p} n_S^{(22)G} + ((q/L)^{3p} n_S^{(23)G}) \quad (56)$$

with the following factorizations

$$n_S^{(2i)G} = n_S^{(2i0)G} + n_S^{(2i1)G} f + n_S^{(2i2)G} f^2, i = 1,2,3 \quad (57)$$

and sub-factorizations of the following specific forms

$$n_S^{(21i)G} = 2pn_S^{(21i1)G} + 2p^2 n_S^{(21i2)G} + p^3 n_S^{(21i3)G}, i = 1,2. \quad (58)$$

172. The precise sub-factors of the 21-factorizations can again be expressed as the linear combination of powers of the scaling $\ln(q/L)^n$, where $n \in Z$.

173. Further, the 212-components have similar factorization.

174. It follows that the 22- factors obey the following particular relations

$$n_S^{(22i)G} = p^2 n_S^{(22i1)G} + 2p^3 n_S^{(22i2)G} + p^4 n_S^{(22i3)G}, i = 1,2. \quad (59)$$

175. In the above relations, the factors of the 220 and 221-components are given in the same manner as the linear combination in the scaling $\ln(q/L)^n$, where $n \in Z$.

176. In this case, we can further obtain the same trend for the 222-factors.

177. After applying a similar technique, the 23 factors are

$$n_S^{(23i)G} = p^3 n_S^{(23i3)G} + 2p^4 n_S^{(23i4)G}, i = 1,2. \quad (60)$$

178. The 230 and 231 sub-factors can be expressed as the linear combination in the scaling $\ln(q/L)^n$, where $n \in Z$.

179. Further, the 232 factors come as per the expected identification.

180. We notice that the factors of the $l(p)^3$-terms can be expressed as

$$n_S^{(3)G} = n_S^{(30)G} + (q/L)^p n_S^{(31)G} + (q/L)^{2p} n_S^{(32)G} + ((q/L)^{3p} n_S^{(33)G}). \quad (61)$$

181. In this identifications, we have the following quadratic expressions in the scalar factor f

$$n_S^{(3i)G} = n_S^{(3i0)G} + n_S^{(3i1)G} f + n_S^{(3i2)G} f^2, i = 0,1,2,3 \quad (62)$$

182. As per the expectation, we find that the 30-factors appear nicely as the a two variable polynomial in J, q.

183. For the general quarkonia with $\{q, p, J\}$ fluctuating, the stability of the qp-surface requires that the principle minor p_S^G remains positive on (M_3, g).

184. Correspondingly, this leads to the constraint that the $\{q, p, J\}$ satisfy

$$n_S^{(0)G} + n_g^{(1)G} l(p) + n_S^{(2)G} l(p)^2 + n_S^{(3)G} l(p)^3 \ < \ 0. \tag{63}$$

185. For the limiting small scale $q \to 0^+$, the surface minor p_S^G shows a large negative minima of the order of height -10^{17}, which occurs in the angular momentum regime $J \to 5^-$.

186. After considering the contributions of the fluctuations to the strongly coupled massive rotating quarkonia, we obtain the following general expressions for the determinant of the metric tensor

$$\|g\| = -\frac{pf(-1 + J(0, a\sqrt{q}))}{8b^3 l(p)^6 \exp(3l(p)) q^{5/2} J^{3/2} (J-q)^{3/2}} (n_g^{(0)G} + n_g^{(1)G} l(p) + n_g^{(2)G} l(p)^2 + n_g^{(3)G} l(p)^3), \tag{64}$$

where the $n_g^{(0)G}$-term factorizes as

$$n_g^{(0)G} \ = \ ((q/L)^p)^3 (8p^4 n_g^{(04)G} + 8p^5 n_g^{(05)G} + 8p^6 n_g^{(06)G}). \tag{65}$$

187. After simplification, we obtain the sub-factorizations

$$n_g^{(0i)G} \ = \ n_g^{(0i0)G} + f n_g^{(0i1)G}, i = 4, 5, 6. \tag{66}$$

As per our computation, the $\{n_g^{(040)G}, n_g^{(041)G}, n_g^{(050)G}, n_g^{(051)G}, n_g^{(060)G}, n_g^{(061)G}\}$ are expressed as the linear combination over integral powers of the Bessel function and $\ln{(q/L)^n}$.

188. We find that the $l(p)$ terms factorize as follows

$$n_g^{(1)G} \ = \ ((q/L)^p)^2 n_g^{(12)G} + ((q/L)^p)^3 n_g^{(13)G} \tag{67}$$

with the following sub-factorizations

$$n_g^{(1i)G} \ = \ n_g^{(1i0)G} + f n_g^{(1i1)G}, i = 2, 3. \tag{68}$$

189. The factors of the 12-components are given by

$$\begin{aligned} n_g^{(120)G} &= n_g^{(1202)G} p^2 + n_g^{(1203)G} p^3 + n_g^{(1204)G} p^4, \\ n_g^{(121)G} &= n_g^{(1212)G} p^2 + n_g^{(1213)G} p^3 + n_g^{(1214)G} p^4 + n_g^{(1215)G} p^5 \end{aligned} \tag{69}$$

with the sub-factorizations $\{n_g^{(120i)G}, n_g^{(121j)G}\}$ as the linear combination over integral powers of the Bessel function and $\ln{(q/L)^n}$.

190. Similarly, the factors of 13-components are given as

$$\begin{aligned} n_g^{(130)G} &= n_g^{(1303)G} p^3 + n_g^{(1304)G} p^4 + n_g^{(1305)G} p^5, \\ n_g^{(131)G} &= n_g^{(1313)G} p^3 + n_g^{(1314)G} p^4 + n_g^{(1315)G} p^5 + n_g^{(1316)G} p^6. \end{aligned} \tag{70}$$

In this case also, the subfactors $\{n_g^{(130i)G}, n_g^{(130j)G}\}$ can further be expressed as the linear combination over integral powers of the Bessel function and $\ln{(q/L)^n}$.

191. It follows that the $l(p)^2$ terms factorize as

$$n_g^{(2)G} = (q/L)^p n_g^{(21)G} + ((q/L)^p)^2 n_g^{(22)G} + ((q/L)^p)^3 n_g^{(23)G}, \tag{71}$$

where the sub-factorizations take the following forms

$$n_g^{(2i)G} = n_g^{(2i0)G} + f n_g^{(2i1)G}, i = 1, 2, 3. \tag{72}$$

192. Explicitly, we find that the individual factors appear as

$$n_g^{(21i)G} = n_g^{(21i1)G} p + n_g^{(21i2)G} p^2 + n_g^{(21i3)G} p^3, i = 0, 1. \tag{73}$$

193. Both the 210-factors and 211-factors can be expressed as a polynomial in the Bessel function and $\ln (q/L)^n$.

194. The 22-factors can be cascaded into the following form

$$n_g^{(22i)G} = n_g^{(22i2)G} p^2 + n_g^{(22i3)G} p^3 + n_g^{(22i4)G} p^4, i = 0, 1. \tag{74}$$

195. As before, the individual sub-factors of both the 220- and 221-factors reduce to polynomial expressions in the Bessel function and $\ln (q/L)^n$.

196. Similarly, the 23-factors turn out to be

$$n_g^{(23i)G} = n_g^{(23i3)G} p^3 + n_g^{(23i4)G} p^4, i = 0, 1, \tag{75}$$

where the sub-factors $\{n_g^{(2303)G}, n_g^{(2304)G}, n_g^{(2313)G}, n_g^{(2314)G}\}$ are expressible as polynomial expressions in the Bessel function and $\ln (q/L)^n$.

197. Finally, the $l(p)^3$ terms are given by

$$n_g^{(3)G} = n_g^{(30)G} + (q/L)^p n_g^{(31)G} + ((q/L)^p)^2 n_g^{(32)G} + ((q/L)^p)^3 n_g^{(33)G}, \tag{76}$$

where the factors are expressed as

$$n_g^{(3i)G} = n_g^{(3i0)G} + f n_g^{(3i1)G}, i = 0, 1, 2, 3. \tag{77}$$

198. In this case, we find that the sub-factors pertaining to 30-, 31- and 32-factors are expressible as polynomial expressions in the Bessel function only.

199. For instance, for small values of the momentum scale and angular momentum, viz $q, J \to 0^+$, we observe that the determinant of the metric tensor picks up a large amplitude of order 10^{21}.

200. From this observation, we predict that the regions of the thermodynamic stability are present for $q, J \in (1, 4)$.

201. Globally, the stability of (M_3, g) constraint the principle minors $\{g_{ii}, p_S^G, \|g\|\}$ to remain positive.

202. Specifically, for the same sign of $\{b, p, f\}$, the volume stability of the (M_3, g) imposes the following constraint

$$n_g^{(0)G} + n_g^{(1)G} l(p) + n_g^{(2)G} l(p)^2 + n_g^{(3)G} l(p)^3 \begin{array}{l} > 0, \ J(0, a\sqrt{q}) < 1, \\ < 0, \ J(0, a\sqrt{q}) < 1. \end{array} \tag{78}$$

203. Importantly, it is worth mentioning that both the limiting configurations with $J = q$ and $J(0, a\sqrt{q}) = 1$ are abided from the thermodynamic stability constraints.

204. To summarize the phases of generic quarkonia, the exact formula for the scalar curvature may analogously be deduced as the one we have offered for the qJ-plane.

205. With some modification of the pre-factors, we find that the summation over $l(p)$ naturally arises with the B_n as the polynomials in p, whose coefficients can be expressed as the functions of the Bloch-Nordsieck logarithmic factor $f(q, J)$.

206. Up to a phase of QCD, we observe that the global properties of the three parameter quarkonia remain the same as we have exactly indicated for the QJ- and QM-planes.

6 Conclusion and Outlook

207. We have examined the role of the thermodynamic intrinsic geometry for a class of quarkonium configurations.

208. We have offered geometric perspective to the confinement- deconfinement phase of (heavy) quarkonia in a hot QCD medium and thereby described the statistical nature of the inter-quark forces.

209. In the sense of one-loop quantum effects, the "quantum" nature of quark matters is shown to follows directly from the thermodynamic consideration of Richardson potential.

210. Our analysis provides an understanding of the formation of hot and dense states of quark gluon plasma matter both in the heavy ion collisions and in the early universe.

211. In the case of the non-abelian theory, we find that the Sudhakov form factor is an efficient tool for understanding statistical nature of the soft gluons.

212. In the limit of the Block-Nordsieck resummation, we observe that the strong coupling, obtained from the Sudhakov form factor, yields the statistical nature of hadronic bound states, e.g. kaons and D_s particles.

213. Our study of the quarkonia could further be explore towards other configurations concerning the non-perturbative and non-abelian nature of the gauge theories.

214. Such a consideration provides a unified description, encompassing all the regimes of QCD at finite temperature, i.e. the Coulombic, the linear rising and the Regge rotating regimes, for both massless and massive quarkonia.

215. Our results can be thus used to investigate the statistical nature of soft gluons and associated phenomenon at the LHC.

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